OSCILLATION OF SECOND ORDER NEUTRAL DELAY
DIFFERENTIAL EQUATIONS

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Abstract. We establish some new oscillation criteria for the second order neutral delay
differential equation

$$[r(t)[x(t) + p(t)x[\tau(t)]]]^\alpha - 1 [x(t) + p(t)x[\tau(t)]]' + q(t)f(x[\sigma(t)]) = 0.$$ 

The obtained results supplement those of Dzurina and Stavroulakis, Sun and Meng, Xu
and Meng, Baculíková and Lacková. We also make a slight improvement of one assumption
in the paper of Xu and Meng.

Keywords: differential equation, oscillation, second order, delay, neutral type, integral
averaging method

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1. Introduction

In this paper we deal with the oscillation of the second order neutral delay differential equation

$$(E^+) \quad [r(t)[x(t) + p(t)x[\tau(t)]]]^\alpha - 1 [x(t) + p(t)x[\tau(t)]]' + q(t)f(x[\sigma(t)]) = 0,$$

where $\alpha > 0$ is a constant, $p, q \in C[t_0, \infty)$, $f \in C(\mathbb{R}, \mathbb{R})$.

We suppose throughout the paper that the following hypotheses hold:

(H1) $q(t) \geq 0$, $q(t) = 0$ only at isolated points, $0 \leq p(t) \leq 1$, $p(t) \not\equiv 1$ on any $(T, \infty)$;

(H2) $r(t) \in C^1[t_0, \infty)$, $r(t) > 0$, $R(t) := \int_{t_0}^{t} r^{-1/\alpha}(s) \, ds \to \infty$ as $t \to \infty$;

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By a solution of Eq. (\(E^+\)) we mean a function \(x(t) \in C^1[T_x, \infty), \quad T_x \geq t_0,\) such that \(z(t) = x(t) + p(t)x[\tau(t)]\) has the property \(r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1[T_x, \infty)\) and \(x(t)\) satisfies \((E^+)\) on \([T_x, \infty).\) We consider only those solutions \(x(t)\) of \((E^+)\) which satisfy \(\sup\{|x(t)|: \quad t \leq T\} > 0\) for all \(T \geq T_x.\) We assume that \((E^+)\) possesses such a solution. A nontrivial solution of \((E^+)\) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation \((E^+)\) is oscillatory if all of its solutions are oscillatory.

The oscillatory properties of the corresponding linear equation

\[
(r(t)y')' + q(t)y[\tau(t)] = 0
\]

have been extended to \((E^+)\) with \(p(t) \equiv 0\) and \(f(x) = x\) by Mirzov \([11], [12], [13]\), Elbert \([5], [6]\), Kusano et al. \([8], [9]\), Chern et al. \([3]\), Agarwal et al. \([1]\).

Dzurina and Stavroulakis \([4]\) generalized these oscillatory criteria to a particular case of \((E^+)\) when \(p(t) \equiv 0, \quad f(x) = |x|^{\alpha-1}x,\) namely

\[
(r(t)|u'(t)|^{\alpha-1}u'(t))' + q(t)|u[\tau(t)]|^{\alpha-1}u[\tau(t)] = 0. \tag{*}
\]

In \([4]\), Eq. (*) was studied in two separate cases under the assumptions \(0 < \alpha < 1\) and \(\alpha \geq 1,\) respectively. Sun and Meng in \([14]\) presented a technique that offers a perfect result for all \(\alpha > 0.\)

Baculíková and Lacková \([2]\) have studied a particular case of \((E^+)\) of the form

\[
[r(t)[x(t) + p(t)x(\tau(t))]'|^{\alpha-1}[x(t) + p(t)x(\tau(t))]'| + q(t)[x[\sigma(t)]|^{\alpha-1}x[\sigma(t)] = 0.
\]

Their oscillatory condition obtained by using the integral averaging method requires the restriction \(\alpha \geq 1.\) The technique presented in this paper allows us to drop this restriction.

The main aim of this paper is to extend the integral averaging technique to \((E^+)\) in order to obtain new oscillatory criteria for the general equation \((E^+).\)
2. Main results

We need the following lemma.

**Lemma 2.1** (See [7]). If $A$ and $B$ are nonnegative constants, then

$$F(A, B) = A^\lambda - \lambda AB^{\lambda-1} + (\lambda - 1)B^\lambda \geq 0, \quad \lambda > 1$$

and the equality holds if and only if $A = B$.

**Proof.** Note that if $A = 0$ then $F(A, B) = (\lambda - 1)B^\lambda \geq 0$. For $A > 0$ we have

$$F(A, B) = A^\lambda[1 - \lambda C^{\lambda-1} + (\lambda - 1)C^\lambda],$$

where $C = B/A$. Using standard methods of Calculus one can easily verify that

$$f(C) = 1 - \lambda C^{\lambda-1} + (\lambda - 1)C^\lambda \geq 0.$$

The proof is complete. □

We will use a “modified” integral averaging method. Let us consider a function $H(t, s)$ satisfying the following conditions:

1. $H(t, s) > 0$ for $t > s \geq t_0$,
2. $H(t, t) = 0$ and $\partial H(t, s)/\partial s < 0$.

Denote for $t > s \geq t_0$

$$Q(t, s) = H^{-\alpha}(t, s)\left(\alpha \sigma'(s)H(t, s) + R[\sigma(s)]r^{1/\alpha}[\sigma(s)] \frac{\partial H(t, s)}{\partial s}\right)^{\frac{\alpha + 1}{\alpha}}.$$

**Theorem 2.1.** If

$$\limsup_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ H(t, s)R[\sigma(s)]\beta q(s)(1 - p[\sigma(s)])^\alpha - \frac{Q(t, s)}{(\alpha + 1)^{\alpha + 1}R[\sigma(s)]^{1/\alpha}[\sigma(s)]^{1/\alpha} [\sigma'(s)]^\alpha} \right] ds = \infty,$$

then Eq. $(E^+)$ is oscillatory.

**Proof.** Assume to the contrary that $x(t)$ is a nonoscillatory solution of Eq. $(E^+)$. We may assume that $x(t) > 0$. The case of $x(t) < 0$ can be proved by the same arguments. Set

$$z(t) = x(t) + p(t)x[\tau(t)].$$

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Then \( z(t) \geq x(t) > 0 \) and
\[
[r(t)|z'(t)|^{\alpha-1}z'(t)]' = -q(t)f(x[\sigma(t)]) \leq 0.
\]
There are two possibilities for \( z'(t) \):
(i) \( z'(t) > 0 \),
(ii) \( z'(t) < 0 \) for \( t \geq t_1 \geq t_0 \).

The condition (ii) implies that for some positive constant \( M \) and for all \( t \geq t_1 \geq t_0 \)
\[
r(t)|z'(t)|^{\alpha-1}z'(t) \leq -M < 0.
\]
Thus
\[
-z'(t) \geq \left( \frac{M}{r(t)} \right)^{1/\alpha}.
\]
Integrating the above inequality from \( t_1 \) to \( t \), we obtain
\[
z(t) \leq z(t_1) - M^{1/\alpha}(R(t) - R(t_1)).
\]
Letting \( t \to \infty \) in the above inequality and using \((H_2)\), we get \( z(t) \to -\infty \). This contradiction proves that (i) holds.

For the case (i), we obtain
\[
(2) \quad x(t) = z(t) - p(t)x[\tau(t)] \geq z(t) - p(t)z[\tau(t)] \geq (1 - p(t))z(t).
\]
Combining the above inequality and \((H_3)\) with Eq. \((E^+)\), we have
\[
(3) \quad [r(t)(z'(t))^{\alpha}]' + \beta q(t)(1 - p[\sigma(t)])^\alpha z[\sigma(t)] \leq 0
\]
and
\[
[r(t)(z'(t))^{\alpha}]' \leq 0.
\]
Therefore
\[
r(t)(z'(t))^{\alpha} \leq r[\sigma(t)](z'[\sigma(t)])^{\alpha},
\]
which implies that
\[
(4) \quad \frac{z'[\sigma(t)]}{z'(t)} \geq \left( \frac{r(t)}{r[\sigma(t)]} \right)^{1/\alpha}.
\]
Define
\[
(5) \quad w(t) = R^\alpha[\sigma(t)]\frac{r(t)(z'(t))^{\alpha}}{z[\sigma(t)]} > 0
\]
for \( t \geq t_1 \).
Differentiating $w(t)$, we obtain

\begin{equation}
\begin{aligned}
w'(t) &= \alpha R^{\alpha-1}[\sigma(t)]\frac{\sigma'(t)r(t)(z'(t))^{\alpha}}{r^{1/\alpha}[\sigma(t)]z^{\alpha}[\sigma(t)]} + R^\alpha[\sigma(t)]\frac{[r(t)(z'(t))^{\alpha}]'}{z^{\alpha}[\sigma(t)]} \\
&\quad - \alpha R^\alpha[\sigma(t)]\frac{r(t)(z'(t))^{\alpha}z'[\sigma(t)]\sigma'(t)}{z^{\alpha+1}[\sigma(t)]}.
\end{aligned}
\end{equation}

Using (3), (4) and (5), we have

\begin{equation}
\begin{aligned}
w'(t) &\leq \frac{\alpha \sigma'(t)}{R[\sigma(t)]r^{1/\alpha}[\sigma(t)]}w(t) - R^\alpha[\sigma(t)]\beta q(t)(1 - p[\sigma(t)])^{\alpha} \\
&\quad - \frac{\alpha \sigma'(t)}{R[\sigma(t)]r^{1/\alpha}[\sigma(t)]}R^{\alpha + 1}[\sigma(t)]r^{(\alpha + 1)/\alpha}(t)(z'(t))^{\alpha + 1} \\
&\quad - R^\alpha[\sigma(t)]\beta q(t)(1 - p[\sigma(t)])^{\alpha}.
\end{aligned}
\end{equation}

Multiplying this inequality with $H(t, s) > 0$ and then integrating from $t_1$ to $t$ we have

\begin{equation}
\begin{aligned}
\int_{t_1}^{t} H(t, s)R^\alpha[\sigma(s)]\beta q(s)(1 - p[\sigma(s)])^{\alpha} \, ds \leq \int_{t_1}^{t} H(t, s) \frac{\alpha \sigma'(s)}{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}w(s) \, ds \\
&\quad - \int_{t_1}^{t} H(t, s) \frac{\alpha \sigma'(s)}{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}w^{(\alpha + 1)/\alpha}(s) \, ds - \int_{t_1}^{t} H(t, s) w'(s) \, ds.
\end{aligned}
\end{equation}

Now integrating (by parts) from $t_1$ to $t$ we arrive at

\begin{equation}
\begin{aligned}
\int_{t_1}^{t} H(t, s)R^\alpha[\sigma(s)]\beta q(s)(1 - p[\sigma(s)])^{\alpha} \, ds \\
&\leq H(t, t_1)w(t_1) + \int_{t_1}^{t} \frac{\alpha \sigma'(s)H(t, s)}{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]} \\
&\quad \times \left[w(s)\left(1 + \frac{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}{\alpha \sigma'(s)H(t, s)} \cdot \frac{\partial H(t, s)}{\partial s}\right) - w^{(\alpha + 1)/\alpha}(s)\right] \, ds.
\end{aligned}
\end{equation}

Set $A = w(s)$ and

\begin{equation}
\begin{aligned}
B &= \left[\frac{1}{\lambda} \left(1 + \frac{R[\sigma(s)]r^{1/\alpha}[\sigma(s)]}{\alpha \sigma'(s)H(t, s)} \cdot \frac{\partial H(t, s)}{\partial s}\right)\right]^{1/(\lambda - 1)},
\end{aligned}
\end{equation}

where $\lambda = (\alpha + 1)/\alpha > 1$. Then

\begin{equation}
\begin{aligned}
(\lambda - 1)B^\lambda &= \frac{(\alpha \sigma'(s)H(t, s) + R[\sigma(s)]r^{1/\alpha}[\sigma(s)]\partial H(t, s)/\partial s)^{\alpha + 1}}{\alpha(\alpha + 1)^{\alpha + 1}H^{\alpha + 1}(t, s)[\sigma'(s)]^{\alpha + 1}}.
\end{aligned}
\end{equation}
Applying Lemma 2.1 to (7) and using (8) and the definition of the function $Q(t, s)$, we conclude that

$$
\frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ H(t, s) R^\alpha[\sigma(s)] \beta q(s)(1 - p[\sigma(s)])^\alpha 
- \frac{Q(t, s)}{(\alpha + 1)^{\alpha+1} R[\sigma(s)] r^{1/\alpha}[\sigma(s)][\sigma'(s)]^\alpha} \right] ds \leq w(t_1).
$$

Letting $t \to \infty$ we get a contradiction with (1), since the left hand side of the previous inequality tends to $\infty$. This completes the proof of Theorem 2.1. $\square$

3. Concluding remarks

Remark 1. Note that if $p(t) \equiv 1$ then (1) is never fulfilled. This is due to the fact that (2) gives in this case only $x(t) \geq 0$ and our arguments of the proof of Theorem 2.1 fail. So condition $(H_1)$ must hold and this assumption has to be added also to Theorem 1 in [15].

Setting $H(t, s) = (t - s)^n$, $n$ being a positive integer, Theorem 2.1 reduces to

**Theorem 3.1.** If

$$
\limsup_{t \to \infty} \frac{1}{(t - t_1)^n} \int_{t_1}^{t} \left[ (t - s)^n R^\alpha[\sigma(s)] \beta q(s)(1 - p[\sigma(s)])^\alpha 
- \frac{Q(t, s)}{(\alpha + 1)^{\alpha+1} R[\sigma(s)] r^{1/\alpha}[\sigma(s)][\sigma'(s)]^\alpha} \right] ds = \infty,
$$

where

$$Q(t, s) = (t - s)^n \left( \alpha \sigma'(s) - \frac{n R[\sigma(s)] r^{1/\alpha}[\sigma(s)]}{t - s} \right)^{\alpha+1},$$

then Eq. $(E^+) \text{ is oscillatory.}$

For the particular case of $(E^+)$, namely for

$$[|x'(t)|^{\alpha-1} x'(t)]' + q(t) |x[\sigma(t)]|^{\alpha-1} x[\sigma(t)] = 0,$$

we have

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Corollary 3.1. If
\[
\limsup_{t \to \infty} \frac{1}{(t-t_1)^n} \int_{t_1}^{t} (t-s)^n \left[ \sigma(s)^{\alpha} q(s) - \left( \frac{\alpha}{\alpha + 1} \right)^{\alpha+1} \sigma'(s) \left( 1 - \frac{n \sigma(s)}{\alpha(t-s) \sigma'(s)} \right)^{\alpha+1} \right] ds = \infty
\]
then the equation (9) is oscillatory.

Recently, W. T. Li [Theorem 2.2 in [10]] presented the following oscillatory criterion for
\[
[r(t)x'(t)]^{\alpha-1} x'(t) + q(t)x[\sigma(t)]^{\alpha-1} x[\sigma(t)] = 0.
\]

Denote
\[
\frac{\partial H}{\partial s} = -h_2(t, s) \sqrt{H(t, s)}.
\]

Theorem 3.2. If there exists a positive nondecreasing function \( q(t) \in C^1[t_0, \infty) \) such that
\[
\limsup_{t \to \infty} \int_{t_1}^{t} \left[ H(s, t_1) q(s) - \frac{r[\sigma(s)] q(s) (h_2(s, t_1) + \frac{\sigma'(s)}{\sigma(s)} \sqrt{H(s, t_1)})}{(\alpha + 1)^{\alpha+1} (\sigma'(s))^{\alpha} [H(s, t_1)]^{(\alpha-1)/2}} \right] ds > 0 \tag{11}
\]
and
\[
\limsup_{t \to \infty} \int_{t_1}^{t} \left[ H(t, s) q(s) - \frac{r[\sigma(s)] q(s) (h_2(t, s) + \frac{\sigma'(s)}{\sigma(s)} \sqrt{H(t, s)})}{(\alpha + 1)^{\alpha+1} (\sigma'(s))^{\alpha} [H(t, s)]^{(\alpha-1)/2}} \right] ds > 0, \tag{12}
\]
then the equation (10) is oscillatory.

On the other hand, Theorem 2.1 for (10) reduces to

Corollary 3.2. If
\[
\limsup_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} H(t, s) R^\alpha[\sigma(s)] q(s) \left[ \frac{(\alpha \sigma'(s) H(t, s) + R[\sigma(s)] r^{1/\alpha}[\sigma(s)] \cdot \partial H(t, s)/\partial s)^{\alpha+1}}{(\alpha + 1)^{\alpha+1} H^\alpha(t, s) R[\sigma(s)] r^{1/\alpha}[\sigma(s)] [\sigma'(s)]^{\alpha}} \right] ds = \infty,
\]
then the equation (10) is oscillatory.

Corollary 3.2 supplements Theorem 3.2 and reduces the conditions (11) and (12) to one condition (13).
References


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