SOME NEW CLASSES OF GRACEFUL LOBSTERS OBTAINED FROM DIAMETER FOUR TREES

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Abstract. We observe that a lobster with diameter at least five has a unique path \( H = x_0, x_1, \ldots, x_m \) with the property that besides the adjacencies in \( H \) both \( x_0 \) and \( x_m \) are adjacent to the centers of at least one \( K_{1,s} \), where \( s > 0 \), and each \( x_i \), \( 1 \leq i \leq m - 1 \), is adjacent at most to the centers of some \( K_{1,s} \), where \( s \geq 0 \). This path \( H \) is called the central path of the lobster. We call \( K_{1,s} \) an even branch if \( s \) is nonzero even, an odd branch if \( s \) is odd and a pendant branch if \( s = 0 \). In the existing literature only some specific classes of lobsters have been found to have graceful labelings. Lobsters to which we give graceful labelings in this paper share one common property with the graceful lobsters (in our earlier works) that each vertex \( x_i \), \( 0 \leq i \leq m - 1 \), is even, the degree of \( x_m \) may be odd or even. However, we are able to attach any combination of all three types of branches to a vertex \( x_i \), \( 1 \leq i \leq m \), with total number of branches even. Furthermore, in the lobsters here the vertices \( x_i \), \( 1 \leq i \leq m \), on the central path are attached up to six different combinations of branches, which is at least one more than what we find in graceful lobsters in the earlier works.

Keywords: graceful labeling, lobster, odd branch, even branch, inverse transformation, component moving transformation

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1. Introduction

Recall that a graceful labeling of a tree \( T \) with \( q \) edges is a bijection \( f : V(T) \to \{0, 1, 2, \ldots, q\} \) such that \( \{|f(u) - f(v)| : \{u, v\} \text{ is an edge of } T\} = \{1, 2, \ldots, q\} \). A tree which has a graceful labeling is called a graceful tree. A lobster is a tree having a path from which every vertex has distance at most two. If \( L \) is a lobster with diameter at least five and \( P \) is a path of maximum length in \( L \) then we obtain the path \( H = x_0, x_1, \ldots, x_m \) from \( P \) by deleting two vertices from both the ends. \( H \) is independent of \( P \), i.e. \( H \) is unique, and it is called the central path of \( L \). Throughout
the paper we use $H$ to denote the central path of a lobster with diameter at least five. It follows directly from the definition of a lobster that besides the adjacencies in $H$ each $x_i$ is adjacent at most to the centers of some stars $K_{1,s}$ where $s \geq 0$. For $x_i \in V(H)$, if $x_i$ is adjacent to the center of $K_{1,s}$ where $s \geq 0$ then we call $K_{1,s}$ an even branch if $s$ is nonzero even, an odd branch if $s$ is odd, and a pendant branch if $s = 0$. Furthermore, whenever we say $x_i$, for some $0 \leq i \leq m$, is attached to an even number of branches we mean a “non zero” even number of branches unless otherwise stated.

In 1979, Bermond [1] conjectured that all lobsters are graceful, which is a special case of the famous and unsolved “graceful tree conjecture” of Ringel and Kotzig (1964) [11], [12], which states that all trees are graceful. Bermond’s conjecture is also open and very few classes of lobsters are known to be graceful. Ng [9], Wang et al. [13], Chen et al. [2], Morgan [8] (see [3]), and Mishra and Panigrahi [5], [6], [7], [10] have given graceful labeling to some classes of lobsters. In the graceful lobsters due to Ng [9] and Chen et al. [2], the vertices of the central path are attached to the isomorphic copies of at most two non isomorphic branches. Morgan [8] has proved that all lobsters with perfect matching are graceful. The graceful lobsters of this paper share one common feature with the graceful lobsters in [5], [6], [7], [10], [13] that the degree of each $x_i$, $0 \leq i \leq m - 1$, is even and the degree of $x_m$ is odd. However, the graceful lobsters of this paper possess simultaneously the following features, which we do not find in the graceful lobsters appearing in the earlier works mentioned above.

1. The vertices $x_i$, $1 \leq i \leq m$, on the central path are attached up to six different combinations of branches, which is at least one more than what we find in graceful lobsters in the earlier works [5], [6], [7], [10], [13].

2. The central path contains a vertex that may be attached to only one type, any combination of two types, or any combination of all three types, of branches with total number of branches even.

3. In this paper we find graceful lobsters with vertices on the central path attached to combination(s) containing all three types of branches preceded by the vertices attached to combination(s) containing two types of branches. Only in [10], some lobsters satisfy this property with some restrictions on the number of odd, even, and pendant branches. The graceful lobsters appearing in [7], [10], [13] are particular cases of the graceful lobsters of this paper in which one or more combinations are absent.

The lobsters of this paper have one of the following properties.

1. The vertex $x_0$ is attached to $(e, 0, o)$. For some $t_1$, $0 \leq t_1 < m$, if $t_1 \geq 1$ then each $x_i$, $1 \leq i \leq t_1$, is attached to $(o, 0, o)$. For integers $t_2$, $t_3$, $t_4$, and $t_5$ with
1 \leq t_1 < t_2 < t_3 < t_4 \leq t_5 \leq m$, each $x_i$, $t_1 + 1 \leq i \leq t_2$, is attached to $(o, e, o)$ and we have either (I) or (II) below.

(I) Each $x_i$, $t_2 + 1 \leq i \leq t_3$, is attached to $(e, o, o)$ and we have (a) or (b) below.

(a) Each $x_i$, $t_3 + 1 \leq i \leq t_4$, is attached to $(0, o, o)$, each $x_i$, $t_4 + 1 \leq i \leq t_5$, is attached to $(0, e, e)$, and each of the rest of the $x_i$ is attached to $(0, e, 0)$.

(b) Each $x_i$, $t_3 + 1 \leq i \leq t_4$, is attached to $(e, e, e)$ and we have either (i) or (ii) below.

(i) Each $x_i$, $t_3 + 1 \leq i \leq t_4$, is attached to $(e, e, 0)$ and each of the rest of the $x_i$ is attached to $(e, 0, 0)$ (or $(0, e, 0)$).

(ii) Each $x_i$, $t_4 + 1 \leq i \leq t_5$, is attached to $(e, 0, e)$ (($0, e, e)$) and each of the rest of the $x_i$ is attached to $(e, 0, 0)$ (respectively, $(0, e, 0)$).

(II) Each $x_i$, $t_2 + 1 \leq i \leq t_3$, is attached to $(o, o, e)$ and we have one of the following.

(a) Each $x_i$, $t_3 + 1 \leq i \leq t_4$, is attached to $(o, o, 0)$, each $x_i$, $t_4 + 1 \leq i \leq t_5$, is attached to $(e, e, 0)$, and each of the rest of the $x_i$ is attached to $(e, 0, 0)$ or $(0, e, 0)$.

(b) Same as (I)(b).

2. Lobsters obtained from those in (1) above by putting $t_1 = 0$.

3. The vertex $x_0$ is attached to one of the combinations of $(0, o, 0)$, $(e, o, o)$, $(0, o, e)$, $(0, e, o)$, $(e, o, e)$, and $(e, e, o)$. For integers $t_1, t_2$ with $1 \leq t_1 < t_2 \leq m$, each $x_i$, $1 \leq i \leq t_1$, is attached to $(o, o, 0)$, each $x_i$, $t_1 + 1 \leq i \leq t_2$, is attached to $(e, e, 0)$, and each of the rest of the $x_i$, if any, is attached to $(e, 0, 0)$ or $(0, e, 0)$.

4. The vertex $x_0$ is attached to one of the combinations of $(0, e, o)$, $(e, e, o)$, and $(o, o, o)$. For integers $t_1, t_2$ with $1 \leq t_1 < t_2 \leq m$, each $x_i$, $1 \leq i \leq t_1$, is attached to $(0, o, o)$, each $x_i$, $t_1 + 1 \leq i \leq t_2$, is attached to $(0, e, e)$, and each of the rest of the $x_i$, if any, is attached to $(0, e, 0)$.

2. Preliminaries

To prove our results we need some definitions, terminology and existing results which are described below.

**Lemma 2.1** [4], [13]. *If $f$ is a graceful labeling of a tree $T$ with $n$ edges then the inverse transformation of $f$, defined as $f_n(v) = n - f(v)$ for all $v \in V(T)$, is also a graceful labeling of $T$.*

**Definition 2.2.** For an edge $e = \{u, v\}$ of a tree $T$, we define $u(T)$ as that connected component of $T - e$ which contains the vertex $u$. Here we say $u(T)$ is a component incident on the vertex $v$. If $a$ and $b$ are vertices of a tree $T$, $u(T)$ is a component incident on $a$ and the component $u(T)$ does not contain the vertex $b$, then
deleting the edge \{a, u\} from T and making b and u adjacent is called the component moving transformation. Here we say the component \(u(T)\) has been moved from a to b.

Throughout the paper we write “the component u” instead of writing “the component \(u(T)\)” . Therefore, whenever we wish to refer to u as a vertex, we write “the vertex u”. By the label of the component “\(u(T)\)” we mean the label of the vertex u. Moreover, we will not distinguish between a vertex and its label.

**Lemma 2.3** [4]. Let \(f\) be a graceful labeling of a tree T; let a and b be two vertices of T; let \(u(T)\) and \(v(T)\) be two components incident on a where \(u(T) \cup v(T) \neq b\). Then the following assertions hold:

(i) if \(f(u) + f(v) = f(a) + f(b)\) then the tree \(T^*\) obtained from T by moving the components \(u(T)\) and \(v(T)\) from a to b is also graceful.

(ii) if \(2f(u) = f(a) + f(b)\) then the tree \(T^{**}\) obtained from T by moving the component \(u(T)\) from a to b is also graceful.

**Lemma 2.4** [4]. Let T be a diameter four tree with q edges. If \(a_0\) is the center vertex and the degree of \(a_0\) is \(2k + 1\) then there exists a graceful labeling \(f\) of T such that

(a) \(f(a_0) = 0\) and the labelings of the neighbours of \(a_0\) are \(1, 2, \ldots, k, q, q - 1, \ldots, q - k\);

(b) from the sequence \(S = (q, 1, q - 1, 2, q - 2, 3, \ldots, q - k + 1, k, q - k)\) of vertex labels, the centers of the odd branches get labels consecutively from the beginning, then the centers of the even branches get labels consecutively and finally the centers of the pendant branches get labels.

3. Results

We begin this section with a theorem (Theorem 3.3) which describes a technique by which one can generate graceful trees from a given graceful tree of a certain type. Subsequently, we apply this technique to a diameter four tree whose center has odd degree to construct graceful lobsters. The lemma given below is used in proving Theorem 3.3.

**Lemma 3.1.** Let \(S_0 = (t_1, t_2, \ldots, t_{2p})\) be a finite sequence of natural numbers in which the sums of consecutive terms are alternately \(l + 1\) and \(l\), beginning (and ending) with the sum \(l + 1\). For an integer \(r \geq 1\), let \(S_r = \varphi_{l+r}(S'_r)\), where \(S'_r\) is the sequence obtained from \(S_{r-1}\) by deleting any odd number of terms from the
beginning as well as from the end and \( \varphi_{l+r}(S'_r) = (l+r-x)_{x \in S'_r} \). Then the sums of consecutive terms in the sequence \( S_r \) are alternately \( l + r + 1 \) and \( l + r \), beginning (and ending) with the sum \( l + r + 1 \).

Proof. We first consider the case when \( r = 1 \). Let the sequence \( S'_1 \) be obtained from \( S_0 \) by deleting \( 2k + 1 \) terms from the beginning and \( 2k_1 + 1 \) terms from the end. For \( 2k + 2 \leq i \leq 2p - 2k_1 - 1 \) we have

\[
\varphi_{l+1}(t_i) + \varphi_{l+1}(t_{i+1}) = 2(l + 1) - (t_i + t_{i+1})
= \begin{cases} 
2(l + 1) - (l + 1) & \text{if } (t_i + t_{i+1}) = l + 1 \\
2(l + 1) - l & \text{if } (t_i + t_{i+1}) = l \\
l + 1 & \text{if } (t_i + t_{i+1}) = l + 1. \\
l + 2 & \text{if } (t_i + t_{i+1}) = l.
\end{cases}
\]

Therefore, the sums of consecutive terms of the sequence \( S_1 \) are \( l + 1 \) and \( l + 2 \) alternately. Moreover, the sum of the first two terms, i.e. \( \varphi_{l+1}(t_{2k+2}) + \varphi_{l+1}(t_{2k+3}) \), is \( l + 2 \) as \( t_{2k+2} + t_{2k+3} = l \). Since the total number of terms in \( S_1 \) is even the sum of the last two terms is \( l + 2 \). Thus, the lemma holds if we take \( r = 1 \). For \( r > 1 \) the proof follows if we repeat the above procedure \( r \) times. \( \square \)

Construction 3.2. Let \( T \) be a graceful tree with \( q \) edges. Let \( a_0 \) be a non-leaf vertex of \( T \) with degree \( 2k + 1 \). Suppose there exists a graceful labeling \( f \) of \( T \) in which \( a_0 \) gets the label 0 and the labels of the neighbours of \( a_0 \) are 1, 2, ..., \( k \), \( q \), \( q - 1 \), \( q - 2 \), ..., \( q - k \) (see Figure 1).

![Figure 1](image-url)

Figure 1. The tree \( T \) with vertex \( a_0 \) and its neighbours. The circles around the neighbouring vertices of represent the respective components incident on \( a_0 \).

We consider the sequence \( A = (q, 1, q-1, 2, q-2, 3, \ldots, k, q-k) \) of vertices adjacent to \( a_0 \) (as we do not distinguish between a vertex and its label). We construct a tree \( T_1 \) (see Figure 2) from \( T \) by identifying the vertex \( y_0 \) of a path \( H' = y_0, y_1, \ldots, y_m \) with \( a_0 \) and moving the components (incident on the vertex \( a_0 \)) in \( A \) to \( y_i \) in the following way:
(1) At \( y_0 \) we retain \( 2\lambda_0 + 1 \) components, where \( \lambda_0 \geq 0 \). In particular, we retain \( 2p_0 \) components, \( 0 \leq p_0 \leq \lambda_0 \), whose labels are from the beginning of \( A \), namely \( q, 1, q-1, 2, q-2, 3, \ldots, q-p_0 + 1, p_0 \), and \( 2\lambda_0 + 1 - 2p_0 \) components whose labels are from the end of \( A \), namely \( q - k, \ldots, k - \lambda_0 + p_0 + 1, q - k + \lambda_0 - p_0 \). Then we delete the components from \( A \) retained at \( y_0 \) and denote the sequence of the remaining terms of \( A \) by \( A^{(1)} \).

(2) Let \( l, 1 \leq l < m \), be a fixed integer. For \( i = 1, 2, \ldots, l \), we move \( 2\lambda_i \) components from \( A^{(i)} \) to \( y_i \), where \( \lambda_i \geq 1 \). In particular, we move \( 2p_i + 1 \), \( 0 \leq p_i < \lambda_i \), components whose labels are from the beginning of \( A^{(i)} \) and \( 2\lambda_i - 2p_i - 1 \) components whose labels are from the end of \( A^{(i)} \), where, for \( i \geq 2 \), \( A^{(i)} \) is obtained from \( A^{(i-1)} \) by deleting the components which are moved to \( y_{i-1} \).

(3) Let \( 2p_0 + \sum_{i=1}^{l} (2p_i + 1) = k_1 \) and \( 2(\lambda_0 - p_0) + 1 + \sum_{i=1}^{l} (2\lambda_i - 2p_i - 1) = k_2 \). Here we notice that if \( l \) is odd (even) then \( k_1 \) is odd (even) and \( k_2 \) is even (respectively, odd). Let \( A^{(l+1)} \) be the sequence obtained from \( A^{(l)} \) by deleting the components which are moved to \( y_l \). Then one finds that \( A^{(l+1)} = \left( \frac{1}{2} (k_1 - 1) + 1, q - \frac{1}{2} (k_1 - 1) - 1, \ldots, \right) \).
\[ k - \frac{1}{2}k_2, q - k + \frac{1}{2}k_2 \] if \( l \) is odd and \( A^{(l+1)} = (q - \frac{1}{2}k_1, \frac{1}{2}k_1, \ldots, q - k + \frac{1}{2}(k_2 - 1) + 2, k - \frac{1}{2}(k_2 - 1) - 1 \) if \( l \) is even.

(4) For any \( n \in \mathbb{N} \), if possible, partition \( A^{(l+1)} \) into \( n \) parts, say \( A^{(l+1)} = B_1 \cup B_2 \cup \ldots \cup B_n \), where \( |B_i| = 2r_i \) (say), in such a way that the first \( 2r_1 \) terms of \( A^{(l+1)} \) are in \( B_1 \), the next \( 2r_2 \) terms of \( A^{(l+1)} \) are in \( B_2 \) and so on.

Now the components in \( B_j \), \( 1 \leq j \leq n \), are distributed to the vertices \( y_i \), \( l + 1 \leq i \leq s_j \), \( l + 1 \leq s_j \leq m \), of \( H' \) in the following way:

For \( l + 1 \leq i \leq s_j \), we move \( 2\alpha_i^{(j)} \) components from \( B_j^{(i)} \) to \( y_i \), where \( \alpha_i^{(j)} \geq 1 \). In particular, we move \( 2q_i^{(j)} + 1, 0 \leq q_i^{(j)} < \alpha_i^{(j)} \), components whose labels are from the beginning of \( B_j^{(i)} \), and \( 2\alpha_i^{(j)} - 2q_i^{(j)} - 1 \) components whose labels are from the end of \( B_j^{(i)} \), where for \( i \geq l + 2 \) (if \( s_j \geq l + 2 \)), \( B_j^{(i)} \) is obtained from \( B_j^{(i-1)} \) by deleting the components which are moved to \( y_{i-1} \) and \( B_j^{(l+1)} = B_j \).

The positive integers \( \alpha_i^{(j)} \), \( i = l + 1, l + 2, \ldots, s_j \), are chosen in such a way that
\[
\sum_{i=l+1}^{s_j} \alpha_i^{(j)} = r_j. \quad \text{Therefore, } 2k + 1 = k_1 + k_2 + 2 \sum_{j=1}^{n} r_j.
\]

In the following theorem, for a graceful tree \( R \) with \( n \) edges and a graceful labeling \( g \) of \( R \) we use the notation “\( g(R) \)” to denote the tree \( R \) with the graceful labeling \( g \).

Also, for any sequence \( F = (a_1, a_2, \ldots, a_r) \), \( g_n(F) \) is the sequence \( (n - a_1, n - a_2, \ldots, n - a_r) \).

**Theorem 3.3.** The tree \( T_1 \) in Construction 3.2 is graceful.

**Proof.** Recall that we denote an edge with end points \( x \) and \( y \) by \( \{x, y\} \). We first consider the tree \( T \cup \{y_0, y_1\} \), where the vertices \( a_0 \) and \( y_0 \) are identified. We give the label \( q + 1 \) to \( y_1 \). Clearly \( T \cup \{y_0, y_1\} \) is graceful with the graceful labeling \( f^{(1)} \), where \( f^{(1)} \) is the same as \( f \) on \( T \) and gives the label \( q + 1 \) to \( y_1 \). Then we move all the components in \( A^{(1)} \) to \( y_1 \) and let the resulting tree be \( T^{(1)} \). One can notice that \( A^{(1)} \) can be partitioned into pairs of labels whose sum is \( q + 1 \) (consecutive terms). By Lemma 2.3(i), \( T^{(1)} \) is a graceful tree with the graceful labeling \( f^{(1)} \).

Next, we consider the inverse transformation \( f_{q+1}^{(1)} \) of \( f^{(1)} \) of \( T^{(1)} \). By Lemma 2.1, \( f_{q+1}^{(1)} \) is a graceful labeling of \( T^{(1)} \) and the label of \( y_1 \) in \( f_{q+1}^{(1)}(T^{(1)}) \) is 0. Next, we make \( y_2 \) adjacent to \( y_1 \) and give the label \( q + 2 \) to \( y_2 \). Obviously, the tree \( T^{(1)} \cup \{y_1, y_2\} \) is graceful with the graceful labeling \( f^{(2)} \), where \( f^{(2)} \) is the same as \( f_{q+1}^{(1)} \) on \( T^{(1)} \) and gives the label \( q + 2 \) to \( y_2 \). We move all the components in \( f_{q+1}^{(1)}(A^{(2)}) \) from \( y_1 \) to \( y_2 \) and let the resulting tree be \( T^{(2)} \). Observe that the sums of consecutive terms in \( A^{(1)} \) are alternately \( q + 1 \) and \( q \) beginning and ending with \( q + 1 \) so by Lemma 3.1 the sums of consecutive terms in \( f_{q+1}^{(1)}(A^{(2)}) \) are alternately \( q + 2 \) and \( q + 1 \) beginning and ending with the sum \( q + 2 \), i.e. \( f_{q+1}^{(1)}(A^{(2)}) \) can be partitioned into pairs of labels whose sum is \( q + 2 \). Therefore, by Lemma 2.3(i), \( T^{(2)} \) is graceful.
Repeating the above procedure for \( l + 1 \) times we find that the tree \( T^{(l+1)} \) with the vertex set \( V(T) \cup \{y_0, y_1, \ldots, y_l, y_{l+1}\} \), is graceful with the graceful labeling \( f^{(l+1)} \) in which the vertex \( y_{l+1} \) gets the label \( q + l + 1 \) and the components of \( f_q^{(l)}f_{q+1}^{(l)}(A^{(l+1)}) \) are incident on \( x_{l+1} \). By Lemma 3.1, we find that the sums of consecutive terms in \( f_q^{(l)}f_{q+1}^{(l)}(A^{(l+1)}) \) are \( q + l + 1 \) and \( q + l \) beginning and ending with the sum \( q + l + 1 \). Since \( B_j \) contains the first 2\( r_j \) terms of \( S \) and for \( 2 \leq j \leq n, B_j \) contains the first 2\( r_j \) terms of \( S^{(l+1)} \setminus B_1 \cup B_2 \cup \ldots \cup B_{j-1} \), the sums of consecutive terms in \( f_q^{(l)}f_{q+1}^{(l)}(B_j^{(l+1)}) \), \( 1 \leq j \leq n \), are \( q + l + 1 \) and \( q + l \) beginning and ending with the sum \( q + l + 1 \).

Next, we take the inverse transformation \( f_{q+1}^{(l+1)} \) of \( f^{(l+1)} \) of \( T^{(l+1)} \). By Lemma 2.1, \( f_{q+1}^{(l+1)} \) is a graceful labeling of \( T^{(l+1)} \) and the label of \( y_{l+1} \) in \( f_{q+1}^{(l+1)}(T^{(l+1)}) \) is 0. Next, we make \( y_{l+2} \) adjacent to \( y_{l+1} \) and give the label \( q + l + 2 \) to \( y_{l+2} \). Obviously, \( T^{(l+1)} \cup \{y_{l+1}, y_{l+2}\} \) is graceful with the graceful labeling \( f^{(l+2)} \), where \( f^{(l+2)} \) is the same as \( f_{q+1}^{(l+1)} \) on \( T^{(l+1)} \) and gives the label \( q + l + 2 \) to \( y_{l+2} \).

For those \( j \) with \( s_j \geq l + 2, 1 \leq j \leq n \), we move all the components in \( f_{q+1}^{(l+1)}f_{q+2}^{(l+1)}f_{q+3}^{(l+1)}(B_j^{(l+2)}) \) from \( y_{l+1} \) to \( y_{l+2} \) and let the resulting tree be \( T^{(l+2)} \). By Lemma 3.1, the sums of consecutive terms in \( f_{q+1}^{(l+1)}f_{q+2}^{(l+1)}f_{q+3}^{(l+1)}(B_j^{(l+2)}) \) are alternately \( q + l + 2 \) and \( q + l + 1 \) beginning and ending with \( q + l + 2 \). One sees that each \( f_{q+1}^{(l+1)}f_{q+2}^{(l+1)}f_{q+3}^{(l+1)}(B_j^{(l+2)}) \) can be partitioned into pairs of labels whose sum is \( q + l + 2 \). By Lemma 2.3(i), \( T^{(l+2)} \) is graceful.

Let \( s^* = \max\{s_1, s_2, \ldots, s_n\} \). Repeating the above procedure \( s^* - l - 1 \) times we get the graceful tree \( T^{(s^*)} \) with vertex set \( V(T) \cup \{y_1, \ldots, y_s^*\} \) in which the vertex \( y_s^* \) gets the label \( q + s^* \). If \( s^* = m \) then we stop, otherwise we proceed as follows.

We apply inverse transformation to the graceful tree \( T^{(s^*)} \) so that the vertex \( y_s^* \) gets the label 0. Then make the vertex \( y_{s^*+1} \) adjacent to \( y_s^* \) and give the label \( q + s^* + 1 \) to \( y_{s^*+1} \). If \( s^* + 1 = m \) then we stop, otherwise we repeat this procedure until the vertex \( y_m \) gets a label. The graceful tree that is obtained on the vertex set \( V(T) \cup V(H') \) is easily seen to be the tree \( T_1 \).

In the following theorem we demonstrate how we give graceful labeling to certain classes of lobsters by applying Theorem 3.3 to a graceful diameter four tree.

**Theorem 3.4.** The lobsters in Tables 3.1 and 3.2 below are graceful.

**Description of Tables.** In the column headings, the triple \((x, y, z)\) represents the number of odd, even and pendant branches, respectively, where \( e \) means any even number of branches (nonzero, unless otherwise stated), \( o \) means any odd number of branches and 0 means no branch. For example, \((e, o, o)\) means an even number of odd branches, no even branch and an odd number of pendant branches. If
in a triple $e$ or $o$ appear more than once then it does not mean that the corresponding branches are equal in number. For example, $(e, e, o)$ does not mean that the number of odd branches is equal to the number of even branches.

<table>
<thead>
<tr>
<th>Lobsters</th>
<th>$(e, 0, o)$</th>
<th>$(o, 0, o)$</th>
<th>$(o, e, o)$</th>
<th>$(e, o, o)$</th>
<th>$(o, 0, o)$</th>
<th>$(e, e, o)$</th>
<th>$(e, 0, 0)^\dagger$ or $(0, e, 0)^\dagger$</th>
<th>$(0, 0, e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3$, $t_3 &lt; m$</td>
<td>$- t_3 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* &lt; m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* &lt; m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* &lt; m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* &lt; m$</td>
</tr>
<tr>
<td>b</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3$, $t_3 &lt; m$</td>
<td>$- t_3 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>c</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3$, $t_3 \leq m$</td>
<td>$- t_3 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>d</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 1$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m$</td>
<td>$t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>e</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 1$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 \leq m$</td>
<td>$t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>f</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 1$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 \leq m$</td>
<td>$t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>g</td>
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<td>$t_1 + 1 \rightarrow t_2$, $t_2 \leq m$</td>
<td>$t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>h</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 1$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 \leq m$</td>
<td>$t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(1)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>i</td>
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<td>$t_1 + 1 \rightarrow t_2$, $t_2 \leq m$</td>
<td>$t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
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<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
</tr>
<tr>
<td>j</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 1$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 \leq m$</td>
<td>$t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t_2 + 1 \rightarrow t^<em>$, $t^</em> \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
<td>$- t^* + 1 \rightarrow m(2)$, if $t^* \leq m$</td>
</tr>
</tbody>
</table>

Table 3.1

1st column: 0 means that $x_0$ is attached to any one of the mentioned combinations of branches. The notation $0(r)$, $r = 1, 2$ (or $r = 1, 2, 3, 4, 5, 6, 7$), means that $x_0$ is attached to the combination of branches mentioned in the column heading in which $r$ is the superscript.

Other columns: $i \rightarrow j$ (or $i \rightarrow j(r)$, $r = 1, 2$) means that each $x_i$, $i \leq l \leq j$, is attached to the mentioned combination or any one of the combinations of branches (respectively, the branches mentioned in the triple with superscript $r$).

Further, when some vertex $x_i$ on the central path is attached to two combinations $(x, y, 0)$ and $(0, 0, e)$, we mean that $x_i$ is attached to the combination $(x, y, e)$. For
example, in Table 3.1(c), \(x_{t_3+1}\) is attached to the combinations \((e, 0, 0)\) and \((0, 0, e)\), which means that \(x_{t_3+1}\) is attached to the combination \((e, 0, e)\).

<table>
<thead>
<tr>
<th>Lobsters</th>
<th>((e, o, 0)) (^1) or ((e, o, e)) (^2) or ((0, o, 0)) (^3) or ((0, o, e)) (^4) or ((e, e, o)) (^5) or ((0, e, o)) (^6) or ((o, o, o)) (^7) or ((e, 0, o)) (^8)</th>
<th>((o, o, 0)) (^1) or ((0, o, o)) (^2)</th>
<th>((e, e, 0))</th>
<th>((e, 0, 0)) (^1) or ((0, e, e)) (^2)</th>
<th>((0, 0, e))</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0 (any one of the combinations from 1 to 6)</td>
<td>(t \rightarrow t(1), t &lt; m)</td>
<td>(t+1 \rightarrow t', t' \leq m)</td>
<td>(t' + 1 \rightarrow m) if (t' &lt; m)</td>
<td>—</td>
</tr>
<tr>
<td>b</td>
<td>0 (any one of the combinations from 5 to 8)</td>
<td>(t \rightarrow t(2), t &lt; m)</td>
<td>—</td>
<td>(t+1 \rightarrow m(2))</td>
<td>(t+1 \rightarrow s, s \leq m)</td>
</tr>
</tbody>
</table>

Table 3.2

**Proof.** For every lobster \(L\) we first construct a diameter four tree, say \(T(L)\), by successively merging the vertices \(x_i, i = 1, 2, \ldots, m\) of \(H\) with \(x_0\). It is clear that \(x_0\) is the center of \(T(L)\) and its degree is odd. Let \(|E(T(L))| = q\) and \(\deg(x_0) = 2k + 1\). We give the label 0 to \(x_0\). We consider the sequence \(A\) in Construction 3.2. We use the notation \(l, n, k_1, k_2, A^{(i)}\), \(i \geq 1\), and \(B_j, j = 1, 2, \ldots, n\), of Construction 3.2 and determine them for each lobster of this theorem.

Let \(L\) be a lobster of type (a) in Table 3.1. We follow the steps given below.

1. For \(i = 0, 1, \ldots, t_1\), the centers of the odd (pendant) branches incident on \(x_i\) in \(L\) get labels consecutively from the beginning (end) of the sequence \(A^{(i)}\), where \(A^{(0)} = A\). We take \(k_1, k_2\) as the sum total of the number of odd (respectively, pendant) branches incident on \(x_i, 0 \leq i \leq t_1\).

2. (i) Take \(l = t_1\), \(n = 2\) and determine \(B_1\) and \(B_2\). For \(i = t_1 + 1, \ldots, t_2\), let the number of odd branches incident on \(x_i\) be \(2\lambda_i + 1\), where \(\lambda_i \geq 0\). The centers of these branches will get labels from \(B_1\). For \(i = t_1 + 1, \ldots, t_2\), let the number of even branches incident on \(x_i\) be \(2\alpha_i\), \(\alpha_i > 0\), among which the centers of \(2\beta_i + 1\) branches for arbitrary integers \(\beta_i, 0 \leq \beta_i < \alpha_i\), will get labels from \(B_1\), and the centers of the rest of these branches will get labels from \(B_2\). Let \(\sum_{i=t_1+1}^{t_2} (2\lambda_i + 1) + \sum_{i=t_1+1}^{t_2} (2\beta_i + 1) = 2p_1\).

(ii) Let the number of odd branches incident on \(x_i, i = t_2 + 1, \ldots, t_3\), be \(2\lambda_i, \lambda_i \geq 1\). The centers of these branches get labels from the sequence \(B_1\). Let \(\sum_{i=t_2+1}^{t_3} 2\lambda_i = 2p_2\).

3. We give labelings to the centers of the branches incident on \(x_i, t_1 + 1 \leq i \leq m\), in the following manner.

   (i) For \(i = t_1 + 1, \ldots, t_2\), the centers of \(2\lambda_i + 1\) odd branches incident on \(x_i\) get labels consecutively from the beginning of \(B_1^{(i)}\), the centers of \(2\alpha_i\) even branches incident on \(x_i\) get \(2\beta_i + 1\) labels consecutively from the end of \(B_1^{(i)}\) and \(2\alpha_i - 2\beta_i - 1\) labels
consecutively from the beginning of $B_2^{(i)}$, and the centers of the pendant branches incident on $x_i$ get labels consecutively from the end of $B_2^{(i)}$.

(ii) For $i = t_2 + 1, \ldots, t_3$, among the odd branches incident on $x_i$, the centers of any odd number of branches get labels consecutively from the beginning of $B_1^{(i)}$ and the centers of the rest of these branches get labels consecutively from the end of $B_1^{(i)}$.

(iii) For $i = t_2 + 1, \ldots, t^*$, the centers of the even (pendant) branches incident on $x_i$ get labels consecutively from the beginning (respectively, end) of $B_2^{(i)}$.

If $t^* < m$ we do the following additional step.

(iv) For $i = t^* + 1, \ldots, m$, among the even branches incident on $x_i$, the centers of any odd number of branches get labels consecutively from the beginning of $B_2^{(i)}$ and the centers of the rest of these branches get labels consecutively from the end of $B_2^{(i)}$.

Here we notice that the above labeling of the centers of the branches incident on the center $x_0$ of $T(L)$ follows Lemma 2.4. Therefore, by Lemma 2.4 there exists a graceful labeling of $T(L)$ with the above labels of the center $x_0$ and the centers of the branches incident on $x_0$ (i.e. we give labeling to the remaining vertices of $T(L)$ using the techniques of [4]). Finally, we apply Theorem 3.3 for $n = 2$ to $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of $L$ (see example below). This approach will be the same for all the remaining cases of this theorem and hence we will just indicate the modifications we do in steps 1 to 3.

Example 1. Consider the lobster $L$ presented in Figure 3 which is of the type (a) in Table 3.1. We construct the diameter four tree $T(L)$ shown in Figure 4. $|E(T(L))| = q = 73$ and $k = 13$. Therefore, $A = (73, 1, 72, 2, \ldots, 61, 13, 60)$. Here

![Figure 3](image.png)

Figure 3. A lobster $L$ of type (a) in Table 3.1. Here $m = 4, t_1 = 1, t_2 = 2$ and $t^* = 3$

$m = 5, t_1 = 1, t_2 = 2, t_3 = 3, t^* = 4, k_1 = 3, k_2 = 4$. Therefore, $A^{(t_1+1)} = A^{(2)} = (2, 71, 3, \ldots, 61, 11, 62)$. Here $\lambda_2 = 0, \lambda_3 = 1, \alpha_2 = 1, \beta_2 = 0$, so $|B_1| = 4$, i.e. $B_1 = (2, 71, 3, 70)$ and $B_2 = (4, 69, 5, \ldots, 61, 11, 62)$. We give the label 0 to the vertex $x_0$ and give labelings to the centers of the branches incident on $x_0$ as per the steps 1 and 3. Using the techniques of [4] (Theorem 1 of [4]) we obtain a graceful labeling of $T(L)$ given in Figure 4. Then in Figure 5 we make $x_1$ adjacent to $x_0$, give the label 74 to $x_1$ and move all the components in $A^{(1)}$ to $x_1$. The tree in Figure 6 is obtained by applying inverse transformation to the lobster found in Figure 5,
making $x_2$ adjacent to $x_1$, giving the label 75 to $x_2$ and moving all the components in $f_{74}^{(1)}(A^{(2)})$ to $x_2$. Then we proceed as per the technique described in Theorem 3.3 and get a graceful labeling of $L$. Figure 8 represents $L$ with a graceful labeling.

Figure 4. The diameter four tree $T(L)$ corresponding to $L$ with a graceful labeling.

Figure 5. The graceful lobster with the graceful labeling $f^{(1)}$ obtained by making $x_1$ adjacent to $x_0$, giving the label 74 to $x_1$, and moving all the branches in $A^{(1)}$ to $x_1$.

268
Figure 6. The graceful lobster with the graceful labeling \( f^{(2)} \) obtained by applying inverse transformation to the lobster in Figure 5, making \( x_2 \) adjacent to \( x_1 \), giving the label 75 to \( x_2 \), and moving all the components in \( f^{(1)}(A^{(2)}) \) to \( x_2 \).

Figure 7. The graceful lobster with the graceful labeling \( f^{(3)} \) obtained by applying inverse transformation to the lobster in Figure 6, making \( x_3 \) adjacent to \( x_2 \), giving the label 76 to \( x_3 \), and moving all the components in \( f^{(2)}(A^{(2)}) \) to \( x_3 \).

Figure 8. The graceful lobster with the graceful labeling \( f^{(4)} \) obtained by applying inverse transformation to the lobster in Figure 7, making \( x_4 \) adjacent to \( x_3 \), giving the label 77 to \( x_4 \), and moving all the components in \( f^{(2)}(A^{(3)}) \) to \( x_4 \).
For lobsters of type \((x)\), \(x = b, \ldots, j\), in Table 3.1, the proof follows if we proceed as in the proof involving the lobsters of type (a) in Table 3.1 by modifying steps 1, 2 and 3. For lobsters of type (b) we first define an integer \(p\) as \(p = m\) if either \(t' = m\) or \(t' < m\) with each \(x_i, i = t' + 1, \ldots, m\), being attached to an even number of odd branches and \(p = t'\) if \(t' < m\) with each \(x_i, i = t' + 1, \ldots, m\), being attached to an even number of even branches; and this definition of \(p\) will hold henceforth in the text. Next, we set \(t_3 = p\), \(t^* = t_3\) and \(m = m + t' - p\) in steps 1, 2, and 3. For lobsters of type (c), we set \(t_3 = m\), \(t^* = t_3\) and \(m = s\) in steps 1, 2 and 3, and replace even branches by pendant branches in step 3(iv). For lobsters of type (d), we set \(t_3 = p\) and \(t^* = t'\) in steps 1, 2, 3(i), 3(ii), and 3(iii), and furthermore, if \(p = t'\) then we set \(t^* = t'\) in step 3(iv).

For lobsters \(L\) of type (e), we repeat steps 1 and 2(i), set \(t_3 = m\), and replace the number of odd branches by the sum total of the number of odd and even branches in step 2(ii), repeat step 3(i), and modify steps 3(ii) to 3(iv) in the following manner.

3(ii) For \(i = t_2 + 1, \ldots, t'\), the centers of the odd (even) branches incident on \(x_i\) get labels consecutively from the beginning (respectively, end) of \(B_1(i)\).

3(iii) Set \(t^* = t_2\) and \(m = s\), and replace \(B_1\) by \(B_2\) and even branches by pendant branches in step 3(iv).

If \(t' < m\) then we do the following additional step.

3(iv) For \(i = t' + 1, \ldots, m\), among the odd (or even) branches incident on \(x_i\), the centers of any odd number of these branches get labels consecutively from the beginning of \(B_1(i)\) and the centers of the rest of these branches get labels consecutively from the end of \(B_1(i)\).

For lobsters \(L\) of type (f), we repeat steps 1 and 2(i), set \(t_3 = p\) in step 2(ii), repeat step 3(i), set \(t_3 = p\) in step 3(ii), and set \(t^* = t_2\) and \(m = m + t' - p\) in step 3(iv). For lobsters \(L\) of type (g), we repeat steps 1 and 2(i), set \(t_3 = m\) and replace odd branches by odd (or even) branches in step 2(ii), repeat step 3(i), set \(t_3 = m\) and replace odd branches by odd (or even) branches in step 3(ii), and set \(t^* = t_2\) and \(m = s\) and replace even branches with pendant branches in step 3(iv). For lobsters \(L\) of type (h), we repeat steps 1, 2, and 3 excluding step 3(iii) in the proof involving the lobsters of type (e). For lobsters \(L\) of type (i), we repeat steps 1 and 2(i) and let
\[ |B_1| = 2r_1 = 2p_1, \] repeat step 3(i), set \( t_3 = t_2 \) and \( t^* = t' \) in step 3(iii). Furthermore, if \( t' < m \) then we set \( t^* = t' \) in step 3(iv). For lobsters \( L \) of type \((j)\), we set \( t_1 = t \) and \( t_2 = t' \) in steps 1 and 2(i). If \( t' < m \) then we set \( t_2 = t' \) and \( t_3 = m \) and replace odd branches with odd (or even) branches in step 2(ii). Here \( |B_1| = 2r_1 = 2p_1 \) if \( t' = m \) and \( 2p_1 + 2p_2 \) if \( t' < m \). Set \( t_1 = t \) and \( t_2 = t' \) in step 3(i). Furthermore, if \( t' < m \) then we set \( t_2 = t' \) and \( t_3 = m \) and replace odd branches with odd (or even) branches in step 3(ii).

For lobsters \( L \) of type \((a)\) in Table 3.2, the proof follows if we proceed as in the proof involving the lobsters of type \((a)\) in Table 3.1 with the changes in steps 1 to 3 as per the following.

1. The centers of odd (pendant branches followed by even branches) incident on \( x_0 \) get labels from the beginning (respectively, end) of \( A \). For \( i = 1, 2, \ldots, t \), the centers of the odd (even) branches incident on \( x_i \) get labels consecutively from the beginning (respectively, end) of the sequence \( A^{(i)} \). We take \( k_1 \) (\( k_2 \)) as the sum of the total number of odd (respectively, sum total of even and pendant) branches incident on \( x_i \), \( 0 \leq i \leq t \).

2. Take \( n = 2 \) and \( l = t \) and determine \( B_1 \) and \( B_2 \). Take \( |B_1| = 2r_1 \) as the total number of odd branches incident on the vertices \( x_i \), \( i = t + 1, t + 2, \ldots, p \).

3. Omit step 3(i). Set \( t_1 = t \) and \( t_2 = p \) in step 3(ii). Omit step 3(iii). Set \( t^* = t \) and \( m = m + t' - p \) in step 3(iv).

For lobsters \( L \) of type \((b)\) in Table 3.2, the proof follows if we proceed as in the proof involving the lobsters of type \((a)\) in Table 3.1 with the changes in steps 1 to 3 as per the following.

1. The centers of odd branches followed by even branches (pendant branches) incident on \( x_0 \), get labels from the beginning (end) of \( A \). For \( i = 1, 2, \ldots, t \), the centers of the even (pendant) branches incident on \( x_i \) get labels consecutively from the beginning (end) of the sequence \( A^{(i)} \). We take \( k_1 \) (\( k_2 \)) as the sum of the total number of odd and even branches (respectively, number of pendant branches) incident on \( x_i \), \( 0 \leq i \leq t \).

2. Take \( n = 2 \) and \( l = t \) and determine \( B_1 \) and \( B_2 \). Take \( |B_1| = 2r_1 \) as the sum total of number of even branches incident on the vertices \( x_i \), \( i = t + 1, t + 2, \ldots, m \).

3. Omit step 3(i). Set \( t_1 = t \) and \( t_2 = m \) and replace odd branches with even branches in step 3(ii). Omit step 3(iii). Set \( t^* = t \) and \( m = s \) and replace even branches by pendant branches in step 3(iv).

Next, we show that for the case \( n = 2 \) in Construction 3.2 by distributing the branches in \( B_j \), \( j = 1, 2 \), to the vertices \( y_i \), \( 0 \leq i \leq m \) of \( H' \) in a slightly different manner we get a graceful tree \( T_2 \) (may be different from \( T_1 \)). By applying this result to diameter four trees we obtain some more graceful lobsters.

271
Construction 3.5. Let the tree $T$, the path $H'$, the graceful labeling $f$ (of $T$) and the sequence $A$ be the same as in Construction 3.2 (see Figure 1). We construct a tree $T_2$ (see Figure 10) from $T$ by identifying the vertex $y_0$ of $H'$ with $a_0$ and distributing the components (incident on the vertex $a_0$) in $A$ to $y_i$, $i = 0, 1, 2, \ldots, m$, in the following manner.

(1) The components in $A$ are distributed to the vertices $y_0, y_1, \ldots, y_l$ in the same manner as described in Construction 3.2. The integers $k_1$ and $k_2$ are defined as in Construction 3.2.

(2) We take $n = 2$ in Construction 3.2, i.e. we partition the sequence $A^{(l+1)}$ into two parts: $A^{(l+1)} = B_1 \cup B_2$. Let $\{l_1, l_2\} = \{1, 2\}$. For $l + 1 \leq i \leq s_{l_j}$, we move $2\alpha_i^{(l_j)}$ components from $B_i^{(l_j)}$ to $y_i$, where $\alpha_i^{(l_j)} \geq 1$.

The components in $B_{l_1}$ are distributed to the vertices $y_{l+1}, y_{l+2}, \ldots, y_{s_{l_1}}$, in the same manner as described in Construction 3.2. The components in $B_{l_2}$ are distributed to the vertices $y_{l+1}, y_{l+2}, \ldots, y_{s_{l_2}}$, in the following way:

(i) For some integer $s'$, $l + 1 \leq s' < s_{l_2}$, the components of $B_{l_2}$ are distributed to $y_i$, $l + 1 \leq i \leq s'$, in the same manner as described in Construction 3.2.

(ii) For $i = s' + 1, \ldots, s_{l_2}$, the components of $B_{l_2}$ are distributed to $y_i$ in the following manner. Suppose $|B_{l_2}^{(s'+1)}| = 2k_3$. We first partition $B_{l_2}^{(s'+1)}$ as $B_{l_2}^{(s'+1)} = C_1 \cup C_2$, where for some integer $k_4$, $1 \leq k_4 < k_3$, $C_1$ consists of $2k_4$ terms from the beginning of $B_{l_2}^{(s'+1)}$ and $C_2 = B_{l_2}^{(s'+1)} \setminus C_1$. Let $s_{l_2}^{(1)}$ and $s_{l_2}^{(2)}$ be integers, where $s_{l_2}^{(1)}, s_{l_2}^{(2)} \geq s' + 1$ and $\max(s_{l_2}^{(1)}, s_{l_2}^{(2)}) = s_{l_2}$. For $l' = 1, 2$ and $i = s' + 1, \ldots, s_{l_2}^{(l')}$, we move $2\beta_i^{(l')}$, $\beta_i^{(l')} \geq 1$, components from $C_{l'}$ to $y_i$. In particular, we move $2\gamma_i^{(l')} + 1$ components for arbitrary integers $\gamma_i^{(l')}, 0 \leq \gamma_i^{(l')} < \beta_i$, whose labels appear consecutively from the beginning of $C_{l'}^{(i)}$ and $2\beta_i^{(l')} - 2\gamma_i^{(l')} - 1$ components whose labels appear consecutively from the end of $C_{l'}^{(i)}$, where $C_{l'}^{(s'+1)} = C_{l'}$ and for $i \geq s' + 2$, $C_{l'}^{(i)}$ is obtained from $C_{l'}^{(i-1)}$ by deleting the components which are retained at $y_{i-1}$. The numbers $\beta_i^{(l')}$, $i = s' + 1, \ldots, s_{l_2}^{(l')}$, $l' = 1, 2$, are chosen in such a way that

$$\sum_{i = s' + 1}^{s_{l_2}^{(1)}} \beta_i^{(l)} = k_4 \quad \text{and} \quad \sum_{i = s' + 1}^{s_{l_2}^{(2)}} \beta_i^{(2)} = k_3 - k_4.$$ 

The numbers $\alpha_i^{(l_1)}$, $i = l + 1, \ldots, s_{l_1}$, and $\alpha_i^{(l_2)}$, $i = l + 1, \ldots, s'$, $j = 1, 2$, are chosen in such a way that

$$\sum_{i = l + 1}^{s_{l_1}} \alpha_i^{(l_1)} = r_{l_1} \quad \text{and} \quad \sum_{i = l + 1}^{s'} \alpha_i^{(l_2)} = r_{l_2} - k_3.$$ 

272
Theorem 3.6. The tree $T_2$ in Construction 3.5 is graceful.

Proof. We first proceed as in the proof of Theorem 3.3 for $n = 2$ by setting $\max(s_1, s_2, \ldots, s_n) = s + 1$. Next, we continue the above process for $n = 3$, i.e. the max($s_1, s_2, \ldots, s_n) = s + 1$. Next, we continue the above process for $n = 3$, i.e. the

Figure 10. The tree $T_2$ obtained from $T$. Here we take $l_1 = 1$, $l_2 = 2$, $s_1 = s_2^{(1)} = s_3^{(2)} = m$. Theorem 3.6. The tree $T_2$ in Construction 3.5 is graceful. Theorem 3.6. The tree $T_2$ in Construction 3.5 is graceful.

Proof. We first proceed as in the proof of Theorem 3.3 for $n = 2$ by setting $\max(s_1, s_2) = s_1 + 1$. Next, we continue the above process for $n = 3$, i.e. the max($s_1, s_2, s_3) = s + 1$. Next, we continue the above process for $n = 3$, i.e. the

Figure 10. The tree $T_2$ obtained from $T$. Here we take $l_1 = 1$, $l_2 = 2$, $s_1 = s_2^{(1)} = s_3^{(2)} = m$.
parts are $B_{l_1}$, $C_1$, $C_2$ (or $C_1$, $C_2$, $B_{l_1}$) if $s_{l_1} > s'$ ($n = 2$, i.e. the parts are $C_1^{(i)}$ and $C_2^{(i)}$ if $s_{l_1} < s'$).

**Theorem 3.7.** The lobsters in Table 3.3 below are graceful.

<table>
<thead>
<tr>
<th>Lobsters</th>
<th>(e, 0, 0)</th>
<th>(e, 0, a)</th>
<th>(e, e, a)</th>
<th>(e, 0, o)</th>
<th>(e, o, 0)</th>
<th>(e, e, 0)</th>
<th>(e, 0, 0)$^1$ or (0, e, 0)$^2$</th>
<th>(0, 0, e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 3$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 2$</td>
<td>$t_2 + 1 \rightarrow t_3, t_3 &lt; m - 1$</td>
<td>$t_3 + 1 \rightarrow t_4, t_4 &lt; m$</td>
<td>$t_4 + 1 \rightarrow s$</td>
<td>$s &lt; m$</td>
<td>$t_4 + 1 \rightarrow s$</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3, t_3 &lt; m$</td>
<td>$t_3 + 1 \rightarrow t'$, $t' \leq m$</td>
<td>$t' + 1 \rightarrow m$, if $t' &lt; m$</td>
<td>$t_3 + 1 \rightarrow s$</td>
<td>$s &lt; m$</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3, t_3 &lt; m$</td>
<td>$t_3 + 1 \rightarrow t_2$, $t_2 &lt; m$</td>
<td>$t_2 + 1 \rightarrow t'$, $t' \leq m$</td>
<td>$t_2 + 1 \rightarrow m$, if $t' &lt; m$</td>
<td>$t_3 + 1 \rightarrow s$</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3, t_3 &lt; m$</td>
<td>$t_3 + 1 \rightarrow t_2$, $t_2 &lt; m$</td>
<td>$t_2 + 1 \rightarrow t'$, $t' \leq m$</td>
<td>$t_2 + 1 \rightarrow m$, if $t' &lt; m$</td>
<td>$t_3 + 1 \rightarrow s$</td>
</tr>
<tr>
<td>e</td>
<td>0</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3, t_3 &lt; m$</td>
<td>$t_3 + 1 \rightarrow t'$, $t' \leq m$</td>
<td>$t' + 1 \rightarrow m$, if $t' &lt; m$</td>
<td>$t_2 + 1 \rightarrow s$</td>
<td>$s &lt; m$</td>
</tr>
<tr>
<td>f</td>
<td>0</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 2$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m - 1$</td>
<td>$t_2 + 1 \rightarrow t_3, t_3 &lt; m$</td>
<td>$t_3 + 1 \rightarrow t'$, $t' \leq m$</td>
<td>$t' + 1 \rightarrow m$, if $t' &lt; m$</td>
<td>$t_2 + 1 \rightarrow s$</td>
<td>$s &lt; m$</td>
</tr>
<tr>
<td>g</td>
<td>0</td>
<td>$1 \rightarrow t_1$, $t_1 &lt; m - 1$</td>
<td>$t_1 + 1 \rightarrow t_2$, $t_2 &lt; m$</td>
<td>$t_2 + 1 \rightarrow t_3, t_3 &lt; m$</td>
<td>$t_3 + 1 \rightarrow t'$, $t' \leq m$</td>
<td>$t' + 1 \rightarrow m$, if $t' &lt; m$</td>
<td>$t_2 + 1 \rightarrow s$</td>
<td>$s &lt; m$</td>
</tr>
</tbody>
</table>

**Table 3.3**

Description of Table 3.3. Same as Table 3.1.

Proof. As in the proof of Theorem 3.4, for every lobster $L$ of this theorem we first construct the diameter four tree $T(L)$. Let $|E(T(L))| = q$ and $\deg(x_0) = 2k + 1$. We give the label 0 to the center $x_0$. Here we use the notation $A_l$, $l$, $k_1$, $k_2$, $k_3$, $k_4$, $l$, $l_2$, $A^{(i)}$, $i \geq 1$, $B_j, j = 1, 2$, and $C_j, j = 1, 2$, $s_{l_1}$, $s_{l_2}$, and $s_{l_3}$ of Construction 3.5 and determine them for each lobster of this theorem.

Let $L$ be a lobster of type (a) in Table 3.3. We proceed as per the steps given below.

Set $t^* = t_4$ and repeat steps 1, 2, 3(i), 3(ii), and 3(iii) in the proof involving the lobsters of type (a) in Table 3.1. Set $l_1 = 1$, $l_2 = 2$, and $s_{l_1} = s_1 = t_3$.

4. Set $s_{l_2} = s_{l_2} = m$ and $s_{l_2} = s_{l_2} = s$. Take $s' = t_4$; hence $k_3$ is determined, i.e. $2k_3 = |B_2^{(s'+1)}| = |B_2^{(t_4+1)}|$. Next, we determine $k_4$ and hence $C_1$ and $C_2$. The terms of $C_1$ (C2) will be the labels given to the centers of the even (pendant) branches incident on each $x_i, i = t_4 + 1, t_4 + 2, \ldots, m$ (i.e. $t_4 + 1, t_4 + 2, \ldots, s$), $274$
then we proceed as per the technique described in Theorem 3.6 and get a graceful labeling of $L$ with the above labels of the center $x_0$ and the centers of the branches incident on $x_0$ (i.e. we can give labeling to the remaining vertices of $T(L)$ using the techniques of [4]). Finally, we apply Theorem 3.3 for $n = 2$ to $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of $L$ (see example below). This approach will be the same for all the remaining cases of this theorem and hence we will just indicate the modification we do in steps 1 to 5.

Example 2. Consider the lobster $L$ presented in Figure 11 which is of the type (a) in Table 3.3. We construct the diameter four tree $T(L)$ shown in Figure 12. $|E(T(L))| = q = 83$ and $k = 14$. Therefore, $A = (83, 1, 87, 2, 81, 3, \ldots, 13, 70, 14, 69)$. Here $m = 6, t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4, s = 5, l = t_1 = 1, k_1 = 3, k_2 = 2$. Therefore, $A^{(t_1+1)} = A^{(2)} = (2, 81, 3, 80, \ldots, 12, 71, 13, 70)$. Here $\lambda_2 = 0, \alpha_1 = 1, \beta_1 = 0, 2p_1 = (2\lambda_2 + 1) + (2\beta_1 + 1) = 2, \lambda_3 = 1, 2p_2 = 2\lambda_3 = 2$. So $|B_1| = 2(p_1 + p_2) = 4, B_1 = (2, 81, 3, 80)$ and $B_2 = (4, 79, 5, \ldots, 12, 71, 13, 70)$. Here $l_1 = 1, l_2 = 2, s' = t_4 = 4, s_{t_1} = s_1 = t_3 = 3, s_{t_2}^{(1)} = s_{t_2}^{(2)} = s_{t_2}^{(3)} = m = 6$, and $s_{t_2}^{(4)} = s_{t_2}^{(5)} = s = 5$. $|B_{l_2}^{(s'+1)}| = |B_2^{(5)}| = 2k_3 = 8$. $|C_1| = 2k_4 = 4$ is the sum of the number of even branches incident on $x_i, i = 5, 6$; and $|C_2| = 2(k_3 - k_4) = 4$ is the number of pendant branches incident on $x_5$. We give the label 0 to the vertex $x_0$ and give labelings to the centers of the branches incident on $x_0$ as per steps 1 to 5. Using the techniques of [4] (Theorem 1 of [4]) we obtain a graceful labeling of $T(L)$ given in Figure 12. Then we proceed as per the technique described in Theorem 3.6 and get a graceful labeling of $L$. Figure 13 represents the lobster $L$ with a graceful labeling.

![Figure 11](image-url)  

Figure 11. A lobster $L$ of type (a) in Table 3.1. Here $m = 6, t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4, s = 5$.  

i.e. $|C_1| (|C_2|)$ is the sum of the number of even (pendant) branches incident on $x_i, i = t_4 + 1, t_4 + 2, \ldots, m$ (respectively, $i = t_4 + 1, t_4 + 2, \ldots, s$).

5. For $i = t_4 + 1, t_4 + 2, \ldots, m$ $(i = t_4 + 1, t_4 + 2, \ldots, s)$, among the even (pendant) branches incident on $x_i$, the centers of any odd number of branches get labels consecutively from the beginning of $C_1^{(i)} (C_2^{(i)})$ and the centers of rest of these branches get labels consecutively from the end of $C_1^{(i)}$ (respectively, $C_2^{(i)}$).

Here we notice that the above labeling of the centers of the branches incident on the center $x_0$ of $T(L)$ follows Lemma 2.4. Therefore, by Lemma 2.4 there exists a graceful labeling of $T(L)$ with the above labels of the center $x_0$ and the centers of the branches incident on $x_0$ (i.e. we can give labeling to the remaining vertices of $T(L)$ using the techniques of [4]). Finally, we apply Theorem 3.3 for $n = 2$ to $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of $L$ (see example below). This approach will be the same for all the remaining cases of this theorem and hence we will just indicate the modification we do in steps 1 to 5. □
Figure 12. The diameter four tree $T(L)$ corresponding to $L$ with a graceful labeling.

Figure 13. The lobster $L$ with a graceful labeling.

For lobsters of type $(x)$, $x = b, c, d, e$, in Table 3.3, the proof follows if we proceed as in the proof involving the lobsters of type (a) in Table 3.36 by modifying steps 1 to 5.

For lobsters of type (b) we do the following.

1. Repeat steps 1, 2, 3(i), 3(ii), and 3(iii) in the proof involving the lobsters of type (b) in Table 3.1.

2. Set $t_4 = t_3$ and $m = m + t' - p$ in steps 4 and 5, where $p$ is an integer defined as in the proof for the lobsters of type (b) in Table 3.1.

For lobsters of type (c), we repeat steps 1, 2, and 3(i), (ii) and set $t_4 = t_3$ in steps 3(iii), 4, and 5. For lobsters of type (d), we repeat steps 1, 2, 3(i), omit step 3(ii), set $t_4 = t_3$ in step 3(iii), and set $t_4 = t_3$ in steps 4 and 5.

For lobsters $L$ of type (e) and (f) we do the following.

Steps 1–3: If $L$ is of type (e) (respectively, (f)), then set $t' = t_3$ and repeat steps 1, 2, 3(i), 3(ii), and 3(iii) (steps 1, 2, 3(i), and 3(ii)) in the proof involving the lobsters of type (e) in Table 3.1. Set $l_1 = 2$, $l_2 = 1$, and $s_1 = s_2 = s$. 

276
Steps 4–5: Set $s_{l_2}^{(1)} = s_{l_1}^{(1)} = p$ and $s_{l_2}^{(2)} = s_{l_1}^{(2)} = m + t' - p$. Set $t_4 = t_3$, $m = p$, and $s = m + t' - p$, and replace even branches by odd branches and pendant branches by even branches in steps 4 and 5 in the proof for the lobsters of type (a).

For lobsters $L$ of type (g), the proof follows if we set $t' = t_2$ in steps 1, 2, 3(i), and 3(iii); and set $t_3 = t_2$ in steps 4 and 5 in the proof for the lobsters of type (e) in Table 3.3.

Remark 3.8. With some changes in steps 1 to 5, one can show that the lobsters obtained from the lobsters in Theorems 3.4 and 3.7 by eliminating one or more combinations of branches incident on the central path, are also graceful.

Remark 3.9. In all the lobsters to which we give graceful labelings in this paper, the vertex $x_m$ gets the largest label and $x_{m-1}$ gets the label 0. Therefore we get some more graceful lobsters by attaching a caterpillar to the vertex $x_m$ or by attaching a suitable caterpillar (any number of pendant branches or an odd (or even) branch or the combination of both) to the vertex $x_{m-1}$ in any of the lobsters discussed in Theorem 3.4 and 3.7.

References


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