A SIMPLE PROOF OF WHITNEY’S THEOREM ON CONNECTIVITY IN GRAPHS

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Abstract. In 1932 Whitney showed that a graph $G$ with order $n \geq 3$ is 2-connected if and only if any two vertices of $G$ are connected by at least two internally-disjoint paths. The above result and its proof have been used in some Graph Theory books, such as in Bondy and Murty’s well-known Graph Theory with Applications. In this note we give a much simple proof of Whitney’s Theorem.

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We consider a finite undirected simple graph $G$ with the vertex set $V(G)$. If $x, y \in V(G)$ then $d(x, y)$ denotes the distance between $x$ and $y$, a path in $G$ with end-vertices $x$ and $y$ will be denoted by $(x, y)$.

In 1932 Whitney [2], [3] showed the following well-known result.

**Theorem.** A graph $G$ with order $n \geq 3$ is 2-connected if and only if any two vertices of $G$ are connected by at least two internally-disjoint paths.

Whitney’s Theorem is Theorem 3.2 in [1]. However, the proof in [1] (pp.44–45) used Theorem 2.3 [1] (pp.27–28), so the proof is more complex than the one given here.

**Simple Proof of Theorem.** If any two vertices of $G$ are connected by at least two internally-disjoint paths, then, clearly, $G$ is connected and has no 1-vertex cut. Hence $G$ is 2-connected.

Conversely, let $G$ be a 2-connected graph and assume there exist two vertices $u$ and $v$ without two internally-disjoint $(u,v)$-paths. Let $P$ and $Q$ be two $(u,v)$-paths with the common vertex set $S$ as small as possible. Let $w \in S \setminus \{u,v\}$ and $P_1, P_2$
denote the sections of $P$ from $u$ to $w$ and $w$ to $v$ and $Q_1$, $Q_2$ denote the sections of $Q$ from $u$ to $w$ and $w$ to $v$, respectively. Since $G$ is 2-connected, let $R$ denote a shortest path from some vertex $x$ of $(V(P_1) \cup V(Q_1)) \setminus \{w\}$ to some vertex $y$ of $(V(P_2) \cup V(Q_2)) \setminus \{w\}$ without passing through $\{w\}$. We may assume, without loss of generality, that $x$ is in $P_1$ and $y$ in $Q_2$. Let $T$ denote the $(u, v)$-path composed of the section of $P_1$ from $u$ to $x$ and the section of $Q_2$ from $y$ to $v$ together with $R$. Clearly the common vertices of $T$ and the $(u, v)$-path composed of $Q_1$ and $P_2$ are all in $S \setminus \{w\}$. This contradicts the choice of both $P$ and $Q$ as having the smallest number of vertices. \hfill \Box

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References


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