BASE-BASE PARACOMPACTNESS AND SUBSETS OF THE SORGENFREY LINE

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Abstract. A topological space $X$ is called base-base paracompact (John E. Porter) if it has an open base $B$ such that every base $B' \subseteq B$ has a locally finite subcover $C \subseteq B'$. It is not known if every paracompact space is base-base paracompact. We study subspaces of the Sorgenfrey line (e.g. the irrationals, a Bernstein set) as a possible counterexample.

Keywords: base-base paracompact space, coarse base, Sorgenfrey irrationals, totally imperfect set

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1. Introduction

The irrationals as a topological subspace of the reals have a coarse base \cite{4}, i.e. an open base that has no locally finite subcover. Also, the base of all bounded, open, convex sets in a reflexive, infinite-dimensional Banach space is coarse \cite{4}, \cite{5}. A space is totally paracompact \cite{8} if every open base has a locally finite subcover. Equivalently, if no base is coarse. In totally paracompact metric spaces small and large inductive dimensions coincide \cite{8}. The irrationals with the usual metric topology are not totally paracompact \cite{1}, \cite{4}, \cite{11}, \cite{12}. Non-metrizable, paracompact spaces that are not totally paracompact are the Sorgenfrey line and the Michael line \cite{2}, \cite{14}, \cite{26}. A similar property that holds for all metrizable spaces was defined by John E. Porter:

Definition 1.1 \cite{21}. A space $X$ is base-base paracompact if it has an open base $B$ such that every base $B'$ contained in $B$ has a locally finite subcover $C$. Equivalently, if there exists an open base $B$ for $X$ such that $B$ contains no coarse base.
Base-base paracompact spaces are paracompact since every subcover is a refinement. Although base properties are stronger than covering properties, no example is known of a paracompact space that is not base-base paracompact [21], also [19], [20]. John E. Porter proved that base-base paracompact spaces are D-spaces [21] (i.e., for every open neighborhood assignment \( \{U_x: x \in X\} \) there is a closed discrete \( D \subseteq X \) such that \( \bigcup \{U_x: x \in D\} = X \)). Thus, it would be enough to find a paracompact space that is not a D-space, but this is an old problem of Eric van Douwen [7], [10].

Base-base paracompact spaces include base-cover paracompact and base-family paracompact spaces studied by the author in [16], [17], [18]. The latter two classes are distinct from each other and from paracompact spaces. Only the \( F_\sigma \) subspaces of the Sorgenfrey line are base-cover paracompact. Only countable subspaces of the Sorgenfrey line are base-family paracompact. Consistently, there are subspaces of the Sorgenfrey line which are not base-cover paracompact (i.e. not \( F_\sigma \)), yet that are base-base paracompact. Such is any Lusin set or, under MA, any uncountable set of cardinality less than continuum. These sets are Hurewicz [9], [13], and hence totally paracompact [6], see also [2], [12], [14], [15], [25]. G. Gruenhage gave a direct proof for Lusin subspaces that the base of all half-open intervals contains no coarse base.

A. Lelek [11] gave a necessary condition for a subset of a complete metric space to be totally paracompact. He constructed a coarse base \( B' \) such that if \( C \subseteq B' \) is a point-finite family, then the complement of \( \bigcup C \) contains a Cantor set (that is, a homeomorphic copy of the Cantor middle-third set). In the context of base-base paracompactness, a similar construction would naturally be subject to the requirement that the elements of \( B' \) come from a base \( B \) that is given in advance. For some subspaces of the Sorgenfrey line, we show that such a construction works for common bases \( B \) defined below. It remains open if all bases for such subspaces are common.

2. Common bases for the Sorgenfrey line

Recall that the Sorgenfrey line \( S \) is the set of all reals having all half-open intervals \([a, b)\) as a base for its topology. If \( X \) is a subspace of \( S \), then a base \( B \) for \( X \) in \( S \) is a family \( B \) of open subsets of \( S \) such that if \( U \subseteq S \) is open and \( x \in X \cap U \) then \( x \in B \subseteq U \) for some \( B \in B \). We denote the set of integers \( \{0, 1, \ldots \} \) by \( \omega \).

We first discuss the intuition behind the definition that follows. Suppose that \( B' \) is a base, and we are to pick open sets from it, one at a time, to form a cover \( C \subseteq B' \). If we pick sets that are too big too often, then these sets may overlap too much, and as a result we may end up with a cover \( C \) that is not locally finite, or even not point-finite. If we pick sets that are too small then there will be too many gaps that are not covered, and at the end \( C \) may not be a cover. Think of the usual construction of the Cantor set, as being an attempt to use the middle-third open
intervals to form a cover of the unit interval, but at the end the Cantor set is exactly
the part that was not covered.

Suppose that we want to make it difficult for \( C \) to be a point-finite cover, then
what we want is for \( B' \) to have only sets that are either too big or too small. Suppose
we are to construct \( B' \) first, with \( B' \subseteq B \), where \( B \) is given. To do this, we need \( B \) to
have enough sets of suitable sizes. The following definition works.

**Definition 2.1.** Assume that \( X \) is dense in \( S \). Call a base \( B \) for \( X \) in \( S \) a common
base if there are an interval \( T \) and sets \( A_n, n \in \omega \), such that \( T \cap X = \bigcup_{n \in \omega} A_n \) and
for each interval \( I \subseteq T \) and each \( n \) there are \( \varepsilon > 0 \) and an interval \( J \subseteq I \) such
that for each \( x \in J \cap A_n \), there is \( B \in B \) (depending on \( I, n, \varepsilon, J \) and \( x \)) with
\([x, x + \varepsilon] \subseteq B \subseteq [x, \infty) \cap I \).

Recall that a non-empty set of reals is perfect if it is closed and has no isolated
points. A set of reals is totally imperfect if it does not contain any perfect set.

**Theorem 2.2.** Let \( X \) be a dense subspace of the Sorgenfrey line \( S \) such that
\( S \setminus X \) is dense and totally imperfect. Then every common base \( B \) for \( X \) contains a
crude base \( B' \). Thus, if \( X \) is base-base paracompact, then only a base that is not
common could possibly witness this.

**Proof.** Note that in the above definition we may replace “for each \( x \in J \cap A_n \)” by
“for each \( x \in J \cap \left( \bigcup_{m \leq n} A_m \right) \).” We may also assume that the length \( \lambda(J) < \varepsilon \)
and therefore the right endpoint of \( J \) belongs to \([x, x + \varepsilon] \), and to \( B \).

Let \( \Sigma = \{ s : s \text{ is a finite sequence of non-negative integers} \} \). If \( s = \langle k, l, \ldots, i \rangle \) and \( j \in \omega \) let \( s^{-\langle j \rangle} \) denote the sequence \( \langle k, l, \ldots, i, j \rangle \). \( I_s \) and \( J_s \) will always denote
left-closed, right-open intervals, with the left endpoint of \( J_s \) in \( S \setminus X \), and its right
endpoint in \( X \). Start with any \( I_0 \subseteq T \) (where \( 0 \) is the empty sequence). This defines
\( I_s \) for all \( s \) with \( |s| = 0 \), where \( |s| \) is the length of \( s \). Recursively, assume \( n \geq 0 \) and
\( I_s \) were defined whenever \( |s| \leq n \). We will define \( I_s \) for \( |s| = n + 1 \).

For each \( s \) with \(|s| = n \) fix \( \varepsilon_s > 0 \) and \( J_s \subseteq I_s \) with \( \lambda(J_s) < \varepsilon_s \) such that for every
\( x \in J_s \cap \left( \bigcup_{m \leq n} A_m \right) \) we can fix \( B_s(x) \in B \) with \([x, x + \varepsilon_s] \subseteq B_s(x) \subseteq [x, \infty) \cap I_s \).

Using a sequence of points decreasing to the left endpoint of \( J_s \), represent \( J_s \) minus its
left endpoint as the disjoint union of countably many left-closed, right-open intervals
\( I_{s^{-\langle l \rangle}} \), \( l \in \omega \), where the left endpoint of \( I_{s^{-\langle l \rangle}} \) is the right endpoint of \( I_{s^{-\langle l+1 \rangle}} \), i.e.
\( I_{s^{-\langle l+1 \rangle}} \) is “the next and to the left of” \( I_{s^{-\langle l \rangle}} \). We also require that \( \lambda(I_s) < |s|^{-1} \)
for each \( s \in \Sigma \), and that the right endpoint of \( J_s \), and therefore the right endpoints
of all \( I_{s^{-\langle l \rangle}} \), \( l \in \omega \), are bounded away a distance at least \( \varepsilon_s \) from the right endpoint
of \( I_s \). It is easily seen by induction on \(|s| \) that \( I_s \cap I_{s'} = \emptyset \) if \(|s| = |s'| \) and \( s \neq s' \).
If $\sigma$ is an infinite sequence of non-negative integers let $\sigma|n$ denote the sequence of the first $n$ many members of $\sigma$. There is a unique point $p_\sigma \in \bigcap_{n \in \omega} I_{\sigma|n}$. Such a $p_\sigma$ may or may not be in $X$. If a point $x \in X$ happens to be $p_\sigma$ for some $\sigma$ we say that $x$ is of type one. Then $\{x\} = \bigcap_{n \in \omega} J_{\sigma|n}$. Pick $n(x)$ with $x \in A_{n(x)}$. Then the family $B_1(x) = \{B_{\sigma|n}(x) : n \geq n(x)\}$ is a local base at $x$.

Call an $x \in X$ of type two if $x$ is not of type one. If $x \in X \setminus I_0$ then clearly $x$ is of type two: Then let $B_2(x) = \{B \in \mathcal{B} : x \in B \subseteq S \setminus I_0\}$. If $x \in X \cap I_0$ and $x$ is of type two then there is an $s \in \Sigma$ such that $x \in I_s$ but $x \notin I_{s-(l)}$ for any $l \in \omega$, or equivalently $x \notin J_s$. Let $B_2(x) = \{B \in \mathcal{B} : x \in B \subseteq I_s \setminus J_s\}$. If $x$ is of type two then $B_2(x)$ is a local base at $x$. Note that if $p \in X$ is of type one and $x \in X$ is of type two then no member of $B_2(x)$ contains $p$.

Let $B_1 = \bigcup \{B_1(x) : x$ is of type one$\}$ and $B_2 = \bigcup \{B_2(x) : x$ is of type two$\}$. Then the family $B' = B_1 \cup B_2$ is a base for $X$ in $S$ contained in $\mathcal{B}$.

Suppose that $\mathcal{C} \subseteq \mathcal{B}'$ and $\mathcal{C}$ is point-finite at each $x \in X$. Then for each $s \in \Sigma$ there could be at most finitely many $x \in J_s \cap \bigcup_{n \leq |s|} A_n$ for which $B_s(x) \in \mathcal{C}$, since all such $B_s(x)$ contain the right endpoint of $J_s$ which is in $X$. Let $x_s$ be the minimal such $x$ (and if there are no such $x$ let $x_s = \infty$). Then $x_s$ is strictly larger than the left endpoint of $J_s$ since the latter is not in $X$. Hence $I_{s-(l)}$ is to the left of $x_s$ for infinitely many $l$. Recursively we may construct the smallest set $\Sigma' \subset \Sigma$ and simultaneously pick distinct $k_s$ and $l_s$ for each $s \in \Sigma'$ such that: (a) $\emptyset \in \Sigma'$, and (b) $I_{s-(k_s)}$ and $I_{s-(l_s)}$ both are to the left of $x_s$, and therefore they do not intersect $B_s(x)$, if $B_s(x) \in \mathcal{C}$ for some $x$. Note that they also do not intersect any $B_{s'}(x')$ with $|s'| = |s|$ and $s' \neq s$.

Hence the set $P = \bigcap_{n \in \omega} \left(\bigcup \{I_s : s \in \Sigma', |s| = n\}\right)$ is a Cantor set (in the usual topology of the reals) that does not intersect any element of $\mathcal{C} \cap B_1$. Since $S \setminus X$ is totally imperfect there is $p \in X \cap P$. Then $p$ is not covered by $\mathcal{C} \cap B_1$. Since $p$ is of type one, $p$ is not covered by $\mathcal{C} \cap B_2$ either. Therefore $\mathcal{C}$ does not cover $X$. \hfill \Box

3. Examples and problems

If $X$ is as in Theorem 2.2 we do not know if every base for $X$ in $S$ is common. If so, then $X$ would be paracompact but not base-base paracompact. The base of all half-open intervals is common (which by Theorem 2.2 implies that e.g. the irrationals as a subspace of $S$ are not totally paracompact, relating to Problem 3.1 of [2]). Given any common base $\mathcal{B}$, for simplicity consisting of half-open intervals, and any partition $\{E_m : m \in \omega\}$ of $X$, we obtain another common base $\mathcal{B}'$ by removing from $\mathcal{B}$ all
$[x, x+t)$ with $x \in E_m$ and $t > 1/m$. The sets $A_n$ from Definition 2.1 that work for $B$ need not work for $\tilde{B}$, but the sets $A_n \cap E_m$, $n, m \in \omega$, would.

**Example 3.1.** Let $S \setminus X$ be dense and totally imperfect. Since $|X| = 2^\omega = \mathfrak{c}$ we may list $X = \{x_\alpha : \alpha < \mathfrak{c}\}$. As in the proof that under MA there is a scale [24], we may find a family of monotone increasing functions $\{f_\alpha : \alpha < \mathfrak{c}\}$ such that if $\beta < \alpha$ then $f_\alpha(n)$ goes to $\infty$ faster, as $n \to \infty$, than $f_\beta(n)$ does (e.g. $\lim_{n \to \infty} f_\alpha(n)/f_\beta(n + k) = \infty$ for all $k$). Then $1/f_\alpha(n)$ goes to 0 faster than $1/f_\beta(n)$ does. If $B_{x_\alpha} = \{[x_\alpha, x_\alpha + 1/f_\alpha(n)) : n \in \omega\}$ then $B = \bigcup_{\alpha < \mathfrak{c}} B_{x_\alpha}$ is a base for $X$ in $S$.

We do not know if the base $B$ in the above example is common or not. But the mere variety of “essentially different” local bases at different points (as in $B$ above) is not enough to produce a base that is not common, as the next example shows.

**Example 3.2.** Consider $S \cap [0, 1]$ instead of $S$. Let $X$ be the set of all dyadic irrationals in $[0, 1]$, i.e. all sums $x = \sum_{k=1}^{\infty} a_k/2^k$ where $a_k \in \{0, 1\}$ and both $a_k = 0$ and $a_k = 1$ occur infinitely often. Define a local base $B_x$ at $x$ as $B_x = \{[x, x + 2^{-k}) : a_k = 1\}$. Clearly $B_x = \{[x, x + 1/f_x(n)) : n \in \omega\}$ for a unique monotone increasing $f_x$. Then given any $g$: $\omega \to \omega$ there is $x \in X$ such that $f_x(n) \geq g(n)$ for all $n$. Nevertheless, we now show that the base for $X$ obtained in this way is common. Let $T = [0, 1]$ and $A_n = X$ for all $n$. Given any interval $I \subseteq T$, fix a finite sequence $u$ of 0’s and 1’s such that $[u] \subseteq I$ where $[u]$ is the set of all $x \in [0, 1]$ whose dyadic representation $\langle a_1, a_2, \ldots \rangle$ starts with $u$. Let $J = [u^\omega \setminus (0)^{\omega} \setminus (1)]$. Then for each $x$ in $J \cap X$ there is an element $B(x)$ in $B_x$ corresponding to the 1 at the end of $u^\omega \setminus (0)^{\omega} \setminus (1)$. All these $B(x)$ have the same length $\varepsilon = 2^{-|u^\omega \setminus (0)^{\omega} \setminus (1)|}$ and are contained in $I$.

Peter de Caux [3] proved that every finite power of $S$ is a hereditarily D-space. He used a special base, described below, easily seen to be common, too.

**Example 3.3** [3]. The base $B$ consists of all $[x, t_i)$ where $i \in \omega$ and $t_i$ is the smallest integer multiple of $2^{-i}$ larger than $x$.

**Question 3.4.** Let $X$ be as in Theorem 2.2. (a) Is there a base for $X$ that is not common? (b) Is $X$ an example of a space that is not base-base paracompact?

Since Lusin subsets of $S$ are base-base paracompact, one may inquire about other “small” subsets of $S$, including some that exist in ZFC, which leads to the following question:

**Question 3.5.** Is every Marczewski null subspace of the Sorgenfrey line base-base paracompact? (A set $M$ of real numbers is Marczewski null if for each perfect set $P$ there is a perfect set $Q$ contained in $P \setminus M$.)
Recall that $w(X)$ is the weight of $X$, i.e. the minimal possible cardinality of a base. $X$ is base-paracompact [21], [22] if it has an open base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$, such that every open cover has a locally finite refinement with elements of $\mathcal{B}$. (A base with only the latter property is called fine in [4], [12].) Base-base paracompact spaces are base-paracompact [21], but these two properties seem quite different for the following reason. Suppose $\mathcal{B}_1 \subseteq \mathcal{B} \subseteq \mathcal{B}_2$ are bases for a space $X$. If $\mathcal{B}$ witnesses base-base paracompactness, then so does $\mathcal{B}_1$, but $\mathcal{B}_2$ need not. The opposite holds for base-paracompactness: If $\mathcal{B}$ witnesses base-paracompactness, then so does $\mathcal{B}_2$ as long as $|\mathcal{B}_2| = w(X)$, but $\mathcal{B}_1$ need not. It is not known if paracompact spaces are base-paracompact [21], [22], see also [19], [20]. But, it is known that Lindelöf spaces, in particular every subspace of $S$, are base-paracompact [21], [22], see also [23].

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References


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