ON THE INTERSECTION OF TWO DISTINCT
k-GENERALIZED FIBONACCI SEQUENCES

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Abstract. Let \( k \geq 2 \) and define \( F^{(k)} := (F_n^{(k)})_{n \geq 0} \), the \( k \)-generalized Fibonacci sequence whose terms satisfy the recurrence relation

\[
F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \ldots + F_{n-k}^{(k)},
\]

with initial conditions \( 0, 0, \ldots, 0, 1 \) (\( k \) terms) and such that the first nonzero term is \( F_1^{(k)} = 1 \). The sequences \( F := F^{(2)} \) and \( T := F^{(3)} \) are the known Fibonacci and Tribonacci sequences, respectively. In 2005, Noe and Post made a conjecture related to the possible solutions of the Diophantine equation \( F^{(k)} = F^{(l)} \). In this note, we use transcendental tools to provide a general method for finding the intersections \( F^{(k)} \cap F^{(m)} \) which gives evidence supporting the Noe-Post conjecture. In particular, we prove that \( F \cap T = \{0, 1, 2, 13\} \).

Keywords: \( k \)-generalized Fibonacci numbers, linear forms in logarithms, reduction method

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1. Introduction

Several problems in number theory are actually questions about the intersection of two known sequences (or sets). Before giving examples, let us recall some terminology: let \( F := (F_n)_{n \geq 0} \) be the Fibonacci sequence, \( \mathbb{P} := \{p: p \text{ prime}\} \), \( \mathbb{P} := \{y^t: y, t \in \mathbb{Z}, t > 1\} \) (the perfect powers), \( \mathcal{F} := \{n!: n \in \mathbb{Z}, n \geq 0\} \), \( \mathcal{R} := \{a(10^n - 1)/9: 1 \leq a \leq 9, n \in \mathbb{Z}, n > 0\} \) (the repdigits or unidigital numbers). Below, we cite some results about the intersection of these sets:

\( \triangleright \) Erdős and Selfridge [8] proved that \( \mathcal{F} \cap \mathbb{P} = \{1\} \).
\( \triangleright \) In 2000, Luca [25] proved that \( F \cap \mathcal{R} = \{0, 1, 2, 3, 5, 8, 55\} \).
\( \triangleright \) Luca [26] also proved that \( F \cap \mathcal{F} = \{1, 2\} \).
\( \triangleright \) In 2003, Bugeaud et al [4] showed that \( F \cap \mathbb{P} = \{0, 1, 8, 144\} \) (see [28] for a generalization).
Let \((a_n)_{n \geq 1}\) be the tower given by \(a_1 = 1\) and \(a_n = n^{a_{n-1}}\), for \(n \geq 2\). Luca and the author [27] proved that \(\{a_1 + \ldots + a_n: n \geq 1\} \cap \mathcal{P} = \{1\}\).

However, some related questions are still open problems, as for instance the sets \(\mathbb{P} \cap F\) and \(\mathbb{P} \cap R\) are unknown.

Let \(k \geq 2\) and denote \(F^{(k)} := (F^{(k)}_n)_{n \geq 0}\), the \(k\)-generalized Fibonacci sequence whose terms satisfy the recurrence relation

\[
F^{(k)}_n = F^{(k)}_{n-1} + F^{(k)}_{n-2} + \ldots + F^{(k)}_{n-k},
\]

with initial conditions \(0, 0, \ldots, 0, 1\) (\(k\) terms) and such that the first nonzero term is \(F^{(k)}_1 = 1\).

The above sequence is one among the several generalizations of Fibonacci numbers. Such a sequence is also called \(k\)-step Fibonacci numbers, the Fibonacci \(k\)-sequence, or \(k\)-bonacci numbers. Clearly, for \(k = 2\) we obtain the well-known Fibonacci numbers and for \(k = 3\), Tribonacci numbers.

Recall that Tribonacci numbers have a long history. For the first time, they were studied in 1914 by Agronomoff [1] and subsequently by many others. The name Tribonacci was coined in 1963 by Feinberg [9]. The basic properties of Tribonacci numbers can be found in [18], [24], [36], [38]. For recent papers, we refer the reader to [3], [19], [20], [33] and to the collection [21], [22], [23].

Recently, Alekseyev [2] described how to compute the intersection of two Lucas sequences including the sequences of Fibonacci, Pell, Luca’s and Lucas-Pell numbers. In general, we refer the reader to [34], [35], [37] for results on the intersection of two recurrence sequences.

In a very recent paper, Togbé and the author [29] proved that only finitely many terms of a linear recurrence sequence whose characteristic polynomial has a dominant root can be repdigits. As an application, since the characteristic polynomial of the recurrence in (1.1), namely \(x^k - x^{k-1} - \ldots - x - 1\), has just one root \(\alpha\) such that \(|\alpha| > 1\) (see for instance [39]), hence \(F^{(k)} \cap \mathcal{R}\) is a finite set, for all \(k \geq 2\). See also the article [32] for some results on the set \(F^{(k)} \cap \mathbb{P}\) and a conjecture on the intersection \(F^{(k)} \cap F^{(m)}\). We point out that this last intersection is, to the best of our knowledge, not known even in the easiest case \((k, m) = (2, 3)\), that is, for numbers that are both Fibonacci and Tribonacci. A possible way to find this intersection is to look at the Fibonacci and Tribonacci sequences modulo \(p^t\), where \(p\) is a prime number. We refer the reader to [5], [13], [16], [17] for results of this nature. However, this approach seems to be hard to work in practice. This observation prompted the author to look for a more interesting and constructive approach which could be useful in the general case.

It is important to notice that Mignotte (see [31]) showed that if \((u_n)_{n \geq 0}\) and \((v_n)_{n \geq 0}\) are two linearly recurrence sequences then, under some weak technical as-
assumptions, the equation

\[ u_n = v_m \]

has only finitely many solutions in positive integers \( m, n \). Moreover, all such solutions are effectively computable. Therefore, it seems reasonable to think that \( F^{(k)} \cap F^{(m)} \) is a finite set for all \( k \neq m \).

The goal of this paper is to apply transcendental tools to provide a method for studying the intersection \( F^{(k)} \cap F^{(m)} \), for integers \( 2 \leq k < m \) and determine completely this set for \( (k, m) = (2, 3) \) (confirming the expectation). More precisely, our result is the following.

**Theorem 1.** The only solution of the Diophantine equation

\[ F_n = T_m \]

in positive integer numbers \( m \) and \( n \) with \( n > 3 \), is \((n, m) = (7, 6)\). Hence, \( F \cap T = \{0, 1, 2, 13\} \).

We organize this paper as follows. In Section 2, we will recall some useful properties such as a result of Matveev on linear forms in three logarithms and the reduction method of Baker-Davenport that we will use in the proof of Theorem 1. In Section 3, we first use Baker’s method to obtain a bound for \( n \), then we completely prove Theorem 1 by means of the Baker-Davenport reduction method.

2. Auxiliary results

We recall the well-known Binet’s formula:

\[ F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \quad \text{for all } n \geq 0, \]

where \( \varphi = (1 + \sqrt{5})/2 \). It is almost unnecessary to stress that this is a very helpful formula which moreover allows to deduce that

\[ \varphi^{n-2} < F_n < \varphi^{n-1} \quad \text{for all } n \geq 1. \]

In 1982, Spickerman [36] found the following “Binet-style” formula for the Tribonacci sequence:

\[ T_n = \frac{\alpha^n}{-\alpha^2 + 4\alpha - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1} \quad \text{for all } n \geq 0, \]
where \( \alpha, \beta, \gamma \) are the roots of \( x^3 - x^2 - x - 1 = 0 \). Explicitly, we have
\[
\begin{align*}
\alpha &= \frac{1}{3} + \frac{1}{3} (19 - 3\sqrt{33})^{1/3} + \frac{1}{3} (19 + 3\sqrt{33})^{1/3}, \\
\beta &= \frac{1}{3} - \frac{1}{6} (1 + i\sqrt{3}) (19 - 3\sqrt{33})^{1/3} - \frac{1}{6} (1 - i\sqrt{3}) (19 + 3\sqrt{33})^{1/3}, \\
\gamma &= \frac{1}{3} - \frac{1}{6} (1 - i\sqrt{3}) (19 - 3\sqrt{33})^{1/3} - \frac{1}{6} (1 + i\sqrt{3}) (19 + 3\sqrt{33})^{1/3}.
\end{align*}
\]

Another interesting formula due to Spickermann is
\[
T_n = \text{Round}\left[\frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n\right],
\]
where, as usual, \( \text{Round}[x] \) is the nearest integer to \( x \).

Since \( \alpha^{-2} < \alpha/(\alpha - \beta)(\alpha - \gamma) = 0.33622 \ldots < \alpha \), the above identity yields the bounds
\[
\alpha^{n-3} < T_n < \alpha^{n+2} \quad \text{for all } n \geq 1.
\]

The Fibonacci and Tribonacci numbers can also be computed using the generating functions
\[
\begin{align*}
(2.3) \quad \frac{z}{1 - z - z^2} &= 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + 21z^7 + 34z^8 + \ldots, \\
(2.4) \quad \frac{z}{1 - z - z^2 - z^3} &= 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + 81z^8 + \ldots
\end{align*}
\]

In order to prove Theorem 1, we will use a lower bound for a linear form in three logarithms \( \text{à la Baker} \) and such a bound was given by the following result of Matveev [30].

**Lemma 1.** Let \( \alpha_1, \alpha_2, \alpha_3 \) be real algebraic numbers and let \( b_1, b_2, b_3 \) be nonzero rational numbers. Define
\[
\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.
\]

Let \( D \) be the degree of the number field \( \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) \) over \( \mathbb{Q} \) and let \( A_1, A_2, A_3 \) be positive real numbers which satisfy
\[
A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \text{for } j = 1, 2, 3.
\]

Assume that
\[
B \geq \max \left\{ 1, \max \{|b_j|A_j/A_1; 1 \leq j \leq 3\} \right\}.
\]
Define also
\[ C = 6750000 \cdot e^4(20.2 + \log(3^{5.5}D^2 \log(eD))). \]

If \( \Lambda \neq 0 \), then
\[ \log |\Lambda| \geq -CD^2A_1A_2A_3 \log(1.5eDB \log(eD))). \]

As usual, in the above statement, the \textit{logarithmic height} of an \( s \)-degree algebraic number \( \alpha \) is defined as
\[ h(\alpha) = \frac{1}{s} \left( \log |a| + \sum_{j=1}^{s} \log \max\{1, |\alpha(j)|\} \right), \]
where \( a \) is the leading coefficient of the minimal polynomial of \( \alpha \) (over \( \mathbb{Z} \)), \( \alpha(j) \) are the conjugates of \( \alpha \) and, as usual, the absolute value of the complex number \( z = a + bi \) is \( |z| = \sqrt{a^2 + b^2} \).

After finding an upper bound on \( n \) which is generally too large, the next step is to reduce it. For that, we need a variant of the famous Baker-Davenport lemma, which is due to Dujella and Pethö [6]. For a real number \( x \), we use \( \|x\| = \min\{|x-n|: n \in \mathbb{N}\} = |x - \text{Round}[x]| \) for the distance from \( x \) to the nearest integer.

**Lemma 2.** Suppose that \( M \) is a positive integer. Let \( p/q \) be a convergent of the continued fraction expansion of \( \gamma \) such that \( q > 6M \) and let \( \varepsilon = \|\mu q\| - M\|\gamma q\| \), where \( \mu \) is a real number. If \( \varepsilon > 0 \), then there is no solution to the inequality
\[ 0 < m\gamma - n + \mu < AB^{-m} \]
in positive integers \( m, n \) with
\[ \frac{\log(Aq/\varepsilon)}{\log B} \leq m < M. \]

See Lemma 5, a) in [6]. Now, we are ready to deal with the proofs of our results.
3. The proof of Theorem 1

3.1. Finding a bound on \( n \). By Binet’s formulae (2.1) and (2.2) we get

\[
\frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} = \frac{\alpha^m}{\alpha'} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'}.
\]

Let us denote by \( \alpha', \beta', \gamma' \) the values of \( Q(x) = -x^2 + 4x - 1 \) at \( x = \alpha, \beta, \gamma \), respectively. By (2.2) and equation (1.2), we have

\[
\frac{\varphi^n}{\sqrt{5}} = \frac{(-1)^n \varphi^{-n}}{\sqrt{5}} \frac{\alpha^m}{\alpha'} + \frac{\beta^m}{\beta'} + \frac{\gamma^m}{\gamma'},
\]

for any \( m, n \geq 1 \). More precisely,

\[
|\frac{\varphi^n}{\sqrt{5}} - \frac{\alpha^m}{\alpha'}| \leq |\frac{\varphi^{-1}}{\sqrt{5}}| + 2|\frac{\beta}{\beta'}| < 0.67 \quad \text{for any } m, n \geq 1
\]

where in the last inequality we have used \( |\beta| = |\gamma| = 0.73735 \ldots \) and \( |\beta'| = |\gamma'| = 3.84631 \ldots \).

Define

\[
\Lambda = \Lambda(m, n) = m \log \alpha - n \log \varphi + \log \left(\frac{\sqrt{5}}{\alpha'}\right).
\]

Then

\[
\Lambda = \log \left(\frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'}\right),
\]

which yields

\[
|e^\Lambda - 1| = \left|\frac{\alpha^m \varphi^{-n} \sqrt{5}}{\alpha'} - 1\right|.
\]

On the other hand, from (3.1) we get

\[
\left|\varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'}\right| < 0.67 \cdot \sqrt{5} < 1.5.
\]

Hence

\[
|e^\Lambda - 1| = \frac{1}{\varphi^n} \left|\varphi^n - \frac{\alpha^m \sqrt{5}}{\alpha'}\right| < \frac{1.5}{\varphi^n}.
\]

Since \( \varphi = 1.61803 \ldots \), we have \( \frac{1.5}{\varphi^n} < \varphi^{-n+1} \) and then

\[
|e^\Lambda - 1| < \varphi^{-n+1}.
\]
We claim that $\Lambda \neq 0$. In fact, towards a contradiction, suppose that $\Lambda = 0$ and thus $\alpha^m \sqrt{5}/\alpha' = \varphi^n$. Therefore $\alpha^{2m}/\alpha'^2$ is a quadratic algebraic number. However $\alpha^{2m}/\alpha'^2 \in \mathbb{Q}(\alpha)$ which is absurd, because $\alpha$ is a $3$-degree algebraic number.

If $\Lambda > 0$, then $\Lambda < e^\Lambda - 1 < \varphi^{-n+2}$ (see (3.2)). If $\Lambda < 0$, then $1 - e^{-|\Lambda|} = |e^\Lambda - 1| < \varphi^{-n+2}$. Thus, for $\Lambda < 0$, we get

$$|\Lambda| < e^{|\Lambda|} - 1 < \frac{\varphi^{-n+1}}{1 - \varphi^{-n+1}} < \varphi^{-n+2},$$

where we have used the fact that $1 - \varphi^{-n+1} > 1/\varphi$ for all $n > 3$.

Hence, we have $|\Lambda| < \varphi^{-n+2}$ for any $\Lambda \neq 0$, which yields

$$\log |\Lambda| < -(n - 2) \log \varphi. \quad (3.3)$$

Now, we will apply Lemma 1. Take

$$\alpha_1 = \alpha, \quad \alpha_2 = \varphi, \quad \alpha_3 = \sqrt{5}/\alpha', \quad b_1 = m, \quad b_2 = -n, \quad b_3 = 1.$$  

Then $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\alpha, \varphi)$, $D = 6$ and $C < 1.2 \cdot 10^{10}$.

It is easy to verify that $1/\alpha'$ is a root of $44x^3 - 2x - 1$ and that $\sqrt{5}/\alpha'$ is a root of $1936x^6 - 880x^4 + 100x^2 - 125$. Since $\sqrt{5}/\alpha'$ is a 6-degree algebraic number, its minimal polynomial (over $\mathbb{Z}$) is $1936x^6 - 880x^4 + 100x^2 - 125$. Using direct calculation, we verify that the absolute value of every root of the minimal polynomial is less than 1. Hence $h(\alpha_3) < (\log 1936)/6 < 1.262$. Next, we have $h(\alpha_1) = (\log \alpha)/3 = 0.204$ and $h(\alpha_2) = (\log \varphi)/2 < 0.241$. We then take $A_1 = 1.22$, $A_2 = 1.45$ and $A_3 = 7.58$. Since (1.2) implies $n > m$, we have

$$\max \{1, \max\{|b_j|A_j/A_1; \; 1 \leq j \leq 3\}\} = \max\{m, 1.2n\} = 1.2n =: B.$$  

Hence, Lemma 1 yields

$$\log |\Lambda| > -6.8 \cdot 10^{12} \log(82n). \quad (3.4)$$

Combining the estimates (3.3) and (3.4), we get

$$6.8 \cdot 10^{12} \log(82n) > (n - 2) \log \varphi,$$

and this inequality implies $n < 6 \cdot 10^{14}$ and, by the trivial estimate $m < n$, we have $m < 6 \cdot 10^{14}$. In order to improve the estimates, we use the bounds on $F_n$ and $T_m$ together with Equation (1.2) to obtain $\alpha^{m-3} < T_m = F_n < \varphi^{n-1}$, which yields
\[ m < 0.8n + 2.2. \] Hence, \( m < 4.8 \cdot 10^{14} \). Similarly, \( \varphi^{n-2} < F_n = T_m < \alpha^{m+2} \) yields \( n < 1.3m + 4.6 \).

### 3.2. Reducing the bound.

The next goal is to reduce the bound on \( m \). For that, let us suppose, without loss of generality, that \( \Lambda > 0 \) (the other case can be handled in a similar way by considering \( 0 < \Lambda' = -\Lambda \)).

We know that \( 0 < \Lambda < \varphi^{-n+2} \) and therefore
\[
0 < m \log \alpha - n \log \varphi + \log \left( \frac{\sqrt{5}}{\alpha'} \right) < \varphi^{-m+2}.
\]

Dividing by \( \log \varphi \), we get
\[
(3.5) \quad 0 < m \hat{\gamma} - n + \mu < 5.45 \cdot \varphi^{-m},
\]
with \( \hat{\gamma} = \log \alpha / \log \varphi \) and \( \mu = \log (\sqrt{5} / \alpha') / \log \varphi \).

Surely \( \hat{\gamma} \) is an irrational number (actually, this number is transcendental by the Gelfond-Schneider theorem: if \( \alpha \) and \( \beta \) are algebraic numbers with \( \alpha \neq 0 \) or \( 1 \), and \( \beta \) is irrational, then \( \alpha^{\beta} \) is transcendental). So, let us denote by \( p_n / q_n \) the \( n \)th convergent of its continued fraction.

In order to reduce our bound on \( m \), we will use Lemma 2. For that, taking \( M = 4.8 \cdot 10^{14} \), we have that
\[
\frac{p_{33}}{q_{33}} = \frac{53739149317980067}{42436582738078750},
\]
and then \( q_{33} > 6M \). Moreover, we get
\[
\| \mu q_{33} \| - M \| \hat{\gamma} q_{33} \| > 0.028 =: \varepsilon.
\]
Thus all the hypotheses of Lemma 2 are satisfied and we take \( A = 5.45 \) and \( B = \varphi \). It follows from Lemma 2 that there is no solution of the inequality in (3.5) (and then for the Diophantine equation (1.2)) in the range
\[
\left\lfloor \left\lfloor \frac{\log(Aq_{33}/\varepsilon)}{\log B} \right\rfloor + 1, M \right\rfloor = [91, 4.8 \cdot 10^{14}].
\]
Therefore \( m \leq 90 \) and then \( n \leq 120 \). To conclude, we use the formulas in (2.3) and (2.4) together with the Mathematica command
\[
\text{Intersection}[\text{CoefficientList}[\text{Series}[x/(1-x-x^2),x,0,120],x], \\
\text{CoefficientList}[\text{Series}[x/(1-x-x^2-x^3),x,0,90],x]]
\]
to find the possible solutions. Fastly, Mathematica returns us the set \{0, 1, 2, 13\} as its answer. This completes the proof.
We point out that the method in proof of Theorem 1 is quite general and that it can be used to work on the intersection of two arbitrary $k$-generalized Fibonacci sequences. In fact, in a similar fashion, we found the set $F^{(k)} \cap F^{(m)}$ for $4 \leq k < m \leq 10$. These cases suggest that the following statement (which is Conjecture 1 in [32]) should be true.

**Conjecture 1.** Let $k < m$ be positive integer numbers. Then

$$F^{(k)} \cap F^{(m)} = \begin{cases} 
\{0, 1, 2, 13\}, & \text{if } (k, m) = (2, 3), \\
\{0, 1, 2, 4, 504\}, & \text{if } (k, m) = (3, 7), \\
\{0, 1, 2, 8\}, & \text{if } k = 2 \text{ and } m > 3, \\
\{0, 1, 2, \ldots, 2^{k-1}\}, & \text{otherwise.} 
\end{cases}$$

When working on these cases it may be helpful that the polynomials $\psi_k(x) := x^k - x^{k-1} - \ldots - x - 1$ are irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1 - 2^{-k})$ and $2$ (as seen in [39]). Also, in a recent paper, G. Dresden [7, Theorem 1] gave a simplified “Binet-like” formula for $F^{(k)}_n$:

$$F^{(k)}_n = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1}$$

for $\alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x) = 0$. There are many other ways of representing these $k$-generalized Fibonacci numbers, as can be seen in [10], [11], [12], [14], [15]. Also, it was proved in [7, Theorem 2] that

$$F^{(k)}_n = \text{Round} \left[ \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \right],$$

where $\alpha$ is the dominant root of $\psi_k(x)$.

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References


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