RECENT PROGRESS IN ATTRACTORS FOR QUINTIC WAVE EQUATIONS

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Abstract. We report on new results concerning the global well-posedness, dissipativity and attractors for the quintic wave equations in bounded domains of $\mathbb{R}^3$ with damping terms of the form $(-\Delta_x)^\theta\partial_t u$, where $\theta=0$ or $\theta=1/2$. The main ingredient of the work is the hidden extra regularity of solutions that does not follow from energy estimates. Due to the extra regularity of solutions existence of a smooth attractor then follows from the smoothing property when $\theta=1/2$. For $\theta=0$ existence of smooth attractors is more complicated and follows from Strichartz type estimates.

Keywords: damped wave equation; fractional damping; critical nonlinearity; global attractor; smoothness

MSC 2010: 35B40, 35B45

1. Introduction

We consider the nonlinear damped wave equation

$$\begin{cases} \partial_t^2 u+\gamma(-\Delta_x)^\theta\partial_t u-\Delta_x u+f(u)=g(x),\\ u|_{t=0}=u_0,\quad \partial_t u|_{t=0}=u_0' \end{cases} \tag{1.1}$$

in a bounded smooth domain $\Omega\subset\mathbb{R}^3$ endowed with Dirichlet boundary conditions. Here Δ_x is the Laplacian with respect to the variable $x=(x^1,x^2,x^3)$, $\theta\in[0,1]$ and $\gamma>0$ are given exponents, $g\in L^2(\Omega)$ is a given external force and f is a nonlinearity which satisfies some natural dissipativity and growth assumptions, say,

$$-C+\kappa|u|^q\leqslant f'(u)\leqslant C(1+|u|^q) \tag{1.2}$$

for some positive C and q.

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Wave equations of the form (1.1) are of big interest from both the theoretical and applied points of view and have been studied by many authors, see [1], [5]–[11], [18], [23], [24] and references therein. It is remarkable that even on the linear level ($f = g = 0$), these equations demonstrate rather nontrivial analytic properties in a strong dependence on the value of the exponent θ. For the convenience of the reader, we briefly summarize them in the following table:

θ	Semigroup	Smoothing	Maximal regularity
0	C_0	asymptotic	no
$(0, \frac{1}{2})$	C^∞	instantaneous	no
$[\frac{1}{2}, 1)$	analytic	instantaneous	yes
1	analytic	instantaneous for $\partial_t u$, asymptotic for u	yes

See [7]–[9], [24] for more details. As we can see from this table, there are three important borderline cases: the first ($\theta = 0$) corresponds to the classical (weakly) damped wave equation, the second ($\theta = 1$) gives the so-called *strongly damped* wave equations and the third ($\theta = 1/2$) is often referred to as the wave equation with *structural* damping although the intermediate choices of θ are also interesting, see e.g. [7], [18], [24] and references therein.

The situation becomes much more delicate in the presence of the nonlinearity f since the analytic properties of solutions start to depend also on the growth rate of $f(u)$ as $u \to \infty$ (on the exponent q in (1.2)). Recall that the solutions $u(t)$ of problem (1.1) satisfy (at least formally) the energy identity

$$\frac{\mathrm{d}}{\mathrm{dt}} E(u(t), \partial_t u(t)) = -\gamma \|(-\Delta_x)^{\theta/2} \partial_t u(t)\|_{L^2}^2, \tag{1.3}$$

where $E(u, v) = \frac{1}{2}\|\partial_t v\|_{L^2}^2 + \frac{1}{2}\|\nabla_x u\|_{L^2}^2 + (F(u), 1) - (g, u)$. Here and below (u, v) stands for the usual inner product in $L^2(\Omega)$ and $F(u) := \int_0^u f(v)\,\mathrm{d}v$ is a potential of the nonlinearity f. Thus, a weak *energy* solution of problem (1.1) on the interval $t \in [0, T]$ is naturally defined as a function $u(t)$ which has the regularity

$$\begin{gathered} u \in L^\infty(0, T; H_0^1(\Omega) \cap L^{q+2}(\Omega)), \quad \partial_t u \in L^\infty(0, T; L^2(\Omega)), \\ (-\Delta_x)^{\theta/2} \partial_t u \in L^2(0, T; L^2(\Omega)) \end{gathered} \tag{1.4}$$

(here $H^s(\Omega)$ stands for the usual Sobolev space of distributions whose derivatives up to order s belong to L^2, and $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$) and satisfies equation (1.1) in the sense of distributions. The corresponding energy phase space is

$$\mathcal{E} := [H_0^1(\Omega) \cap L^{q+2}(\Omega)] \times L^2(\Omega), \quad \xi_u := (u, \partial_t u) \in \mathcal{E}. \tag{1.5}$$

Recall also that, due to the Sobolev embedding $H^1_0 \subset L^6$, we can take $\mathcal{E} = H^1_0(\Omega) \times L^2(\Omega)$ if the growth exponent is $q \leqslant 4$ (in particular, for the case of quintic nonlinearities), but the term L^{q+2} in the definition (1.5) of the energy phase space looks unavoidable if the growth exponent is $q > 4$. The next standard result shows that weak energy solutions exist and are dissipative for all admissible values of θ and q.

Theorem 1.1. *Let $\theta \in [0,1]$, $q \geqslant 0$, $g \in L^2(\Omega)$, let the nonlinearity f satisfy* (1.2) *and the initial data $\xi_u(0) := (u_0, u_0') \in \mathcal{E}$. Then there exists at least one weak energy solution $u(t)$ defined for all $t \geqslant 0$ which satisfies the estimate:*

$$\|\xi_u(t)\|_{\mathcal{E}} + \int_t^{t+1} \|(-\Delta_x)^{\theta/2}\partial_t u(s)\|^2_{L^2}\,\mathrm{d}s \leqslant Q(\|\xi_u(0)\|_{\mathcal{E}})\mathrm{e}^{-\alpha t} + Q(\|g\|_{L^2}), \tag{1.6}$$

where the positive constant α and the monotone function Q are independent of u and t.

The proof of this result is straightforward. Indeed, the dissipative estimate (1.6) formally follows by multiplication of equation (1.1) by $\partial_t u + \beta u$ for a properly chosen positive constant β followed by Gronwall's inequality and the existence of a solution can be verified, say, by Galerkin approximations, see [1], [10].

In contrast to this, the uniqueness and further regularity of energy solutions is more difficult and requires essential restrictions on the growth exponent q. To the best of our knowledge this problem has been solved before only for the values of θ and q collected in the following table:

θ	0	$\left(0,\frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2},\frac{3}{4}\right)$	$\left[\frac{3}{4},1\right]$
q	$[0,2]$	?	$[0,4)$	$0 \leqslant q(\theta) < 8\theta/(3-4\theta)$	$(0,\infty)$

See [1], [7], [18], [24] for more details. E.g., for the case of structural damping $\theta = 1/2$, the open problem was to treat the "critical" case of quintic nonlinearities $q = 4$, and for the classical case $\theta = 0$ in bounded domains with Dirichlet boundary conditions the theory has been developed only for cubic and sub-cubic nonlinearities ($q \leqslant 2$) although $q = 4$ has been conjectured as the critical exponent here. For domains without boundary: $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$ (periodic boundary conditions), a reasonable theory exists for $q < 4$ in [13], [19]. In the quintic case ($\theta = 0$, $q = 4$) and $\Omega = \mathbb{R}^3$, the global existence of regular solutions has not been known for a long time (see [16], [20], [26]), but the attractor theory has been not developed for this case.

The aim of these notes is to present our recent results concerning global well-posedness, dissipativity and existence of smooth global attractors for quintic ($q = 4$) wave equations in bounded domains, see [17] and [25] for a detailed exposition. We restrict ourselves to discussing only two borderline cases $\theta = 1/2$ and $\theta = 0$ although our conjecture is that analogous results hold for all $\theta \in (0, 1/2)$. Note that in both cases, the regularity of solutions provided by the energy estimate is not sufficient and the results are obtained by verifying some extra space-time regularity of the solutions although how this regularity is obtained is very different: in the case of structural damping, we derive an extra Lyapunov type estimate based on the multiplication of equation (1.1) by $(-\Delta_x)^{1/2}u$ while in the weakly damped case it is achieved by utilizing the so-called Strichartz type estimates which have been recently extended to the case of bounded domains, see [3], [4]. We discuss each of these two cases in more details below.

2. Quintic wave equation: the case of structural damping $\theta = 1/2$

The key novelty here is the following theorem proved in [25].

Theorem 2.1. *Let $\theta = 1/2$, $g \in L^2(\Omega)$ and let the nonlinearity $f(u)$ be odd and satisfy* (1.2) *with $q = 4$. Then any weak energy solution $u(t)$ of problem* (1.1) *belongs to the space $L^2(0, T; H^{3/2}(\Omega))$ and satisfies the estimate*

$$\|u\|_{L^2(t,t+1;H^{3/2}(\Omega))} \leqslant Q(\|\xi_u\|_{L^\infty(t,t+1;\mathcal{E})}) + Q(\|g\|_{L^2}), \tag{2.1}$$

where the monotone function Q is independent of t and u.

P r o o f. We sketch the proof of this theorem for the case of periodic boundary conditions $\Omega = \mathbb{T}^3$. The case of Dirichlet boundary conditions can be reduced to that one by using the odd extension of the solution u through the boundary (to this end, we need the assumption that the nonlinearity f is odd) together with a proper cut-off procedure, see [25] for the details. To this end, we use the following identity obtained by multiplying (1.1) by $(-\Delta_x)^{1/2}u$:

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big((\partial_t u, (-\Delta_x)^{1/2}u) + \frac{\gamma}{2}\|(-\Delta_x)^{1/2}u\|^2_{L^2}\Big) + \|(-\Delta_x)^{3/4}u\|^2_{L^2} + (f(u), (-\Delta_x)^{1/2}u) \\ = (g, (-\Delta_x)^{1/2}u) + \|(-\Delta_x)^{1/4}\partial_t u\|^2_{L^2}.$$

The integration of this identity over the interval $(t, t+1)$ gives the desired estimate (2.1) in a straightforward way if we are able to estimate the term containing the nonlinearity f. To this end, we utilize the following lemma which can be verified using Fourier series and the Parseval equality.

Lemma 2.1. *Let $s \in (0,1)$ and $u, v \in H^s(\mathbb{T}^3)$. Then*

$$(v, (-\Delta_x)^s u) = c \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \frac{(v(x+h) - v(x))(u(x+h) - u(x))}{|h|^{3+2s}} \, \mathrm{d}x \, \mathrm{d}h, \tag{2.2}$$

where the constant c depends only on s.

This lemma, together with the assumption $f'(u) \geqslant -K$ and the mean value theorem gives the desired estimate

$$\big(f(u), (-\Delta_x)^{1/2} u\big) \geqslant -K \|(-\Delta_x)^{1/4} u\|_{L^2}^2,$$

which completes the proof of the theorem in the case of periodic boundary conditions. □

With the extra regularity of energy solutions proved, their uniqueness and further regularity can be obtained in a standard way, so we omit the technicalities and only recall below (see [1], [12], [21]) the definition of the global attractor followed by the statement of the main result, proved in [25].

Definition 2.1. *A compact subset $\mathcal{A}$ of a Banach space $\mathcal{E}$ is called a global attractor for a semigroup $S(t)\colon \mathcal{E} \to \mathcal{E}$, if*

(1) $\mathcal{A}$ is strictly invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$;
(2) for any bounded set $B \subset \mathcal{E}$ and any neighbourhood $\mathcal{O}(\mathcal{A})$ of the set $\mathcal{A}$, there exists a time $T = T(B, \mathcal{O})$ such that $S(t)B \subset \mathcal{O}(\mathcal{A})$ for all $t \geqslant T$.

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *hold. Then the energy solution of problem* (1.1) *is unique and the solution semigroup $S(t)$ associated with problem* (1.1) *in the phase space $\mathcal{E}$ possesses a global attractor $\mathcal{A}$ which is bounded in the more regular space $\mathcal{E}_1 := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$.*

Remark 2.1. The existence of an exponential attractor bounded in $\mathcal{E}_1$ that, in particular, implies finite fractal dimension of the global attractor $\mathcal{A}$ is also verified in [25].

3. Quintic wave equation: the case of weak damping $\theta = 0$

The extra space-time regularity of energy solutions is based here on the following nontrivial result concerning Strichartz type estimates for the linear wave equation in a bounded domain, see [3], [4].

Lemma 3.1. *Let $\xi_0 \in \mathcal{E}$, $G(t) \in L^1([0,T];L^2(\Omega))$ and let v solve the linear equation*

$$\partial_t^2 v - \Delta v = G(t), \quad v|_{\partial\Omega} = 0,\ \xi_v|_{t=0} = \xi_0. \tag{3.1}$$

Then $v \in L^4([0,T];L^{12}(\Omega))$ and the following estimate holds:

$$\|v\|_{L^4([0,T];L^{12}(\Omega))} \leqslant C_T\big(\|\xi_0\|_{\mathcal{E}} + \|G\|_{L^1([0,T];L^2(\Omega))}\big), \tag{3.2}$$

where C may depend on T but it is independent of ξ_0 and G.

However, in contrast to the previous case, we do not know whether or not *all* energy solutions possess this extra regularity. For this reason, we restrict ourselves to considering only such energy solutions $u(t)$ of (1.1) which belong to the space $L^4(0,T;L^{12}(\Omega))$. We will refer in the sequel to such solutions as *Shatah-Struwe* solutions of problem (1.1).

The next theorem which gives the global well-posedness of problem (1.1) in the class of Shatah-Struwe solutions can be proved with help of the so-called Pohozhaev-Morawetz identity and the above mentioned Strichartz estimate, see [4].

Theorem 3.1. *Let $\theta = 0$, $g \in L^2(\Omega)$ and let the nonlinearity f satisfy* (1.2) *with $q = 4$ as well as the following extra assumptions*

1. $|f''(u)| \leqslant C(1 + |u|^3)$,
2. $f(u)u - 4F(u) \geqslant -C$.

Then, for any $(u_0, u_0') \in \mathcal{E}$, there exists a unique Shatah-Struwe solution $u(t)$ of problem (1.1) *defined for all $t \geqslant 0$.*

Thus, the (Shatah-Struwe) solution semigroup $S(t)\colon \mathcal{E} \to \mathcal{E}$ associated with problem (1.1) is well-defined. Moreover, these Shatah-Struwe solutions have a number of good properties: they satisfy the energy identity (1.3) and the dissipative estimate (1.6); they are more regular, say, $\xi_u(t) \in \mathcal{E}_1$ for all t if $\xi_u(0) \in \mathcal{E}_1$, etc.

However, in contrast to the previous case, we do not know whether or not the analogue of estimate (2.1) holds for the Strichartz norm $\|u\|_{L^4(t,t+1;L^{12}(\Omega))}$, and the previous theorem actually does not give any control of this norm as $t \to \infty$. Thus, the control of the Strichartz norm may be a priori lost when passing to the limit $t \to \infty$ and even if we initially consider the Shatah-Struwe solutions only, the other types of energy solutions (for which we have neither the energy equality nor the uniqueness theorem) may a priori appear on the attractor.

For this reason, despite the fact that the considered Shatah-Struwe solutions are unique, as an intermediate step, we need to exploit the existence of a weak attractor in the class of energy solutions where the uniqueness theorem is not known. Namely, as proved in [27], if we restrict ourselves to considering the energy solutions which can be obtained as limits of Galerkin approximations only, then the trajectory of the dynamical system associated with these solutions possesses a trajectory attractor in the weak-star topology of $L^\infty_{\rm loc}(\mathbb{R}_+,\mathcal{E})$ and (which is almost the same), the *multi-valued* semigroup $\overline{S}(t)$ associated with these solutions (the uniqueness is not known for that type of solutions) possesses a global attractor $\mathcal{A}_w$ in a weak topology of $\mathcal{E}$, see also [10] for more details. Moreover, as shown in [27] (see also [14], [15]), the solutions on the attractor $\mathcal{A}_w$ possess the following backward regularity.

Theorem 3.2. *Under the above assumptions, the global attractor $\mathcal{A}_w$ is generated by complete energy solutions $u(t)$, $t \in \mathbb{R}$ which are bounded in $\mathcal{E}$ for all $t \in \mathbb{R}$. Moreover, for any such solution u there exists $T = T(u)$ such that $u(t) \in \mathcal{E}_1$ for $t \leqslant -T$ and*

$$\|u\|_{L^\infty(-\infty,-T;\mathcal{E}_1)} \leqslant C,$$

where the constant C is independent of u.

Combining the result of Theorem 3.1 with the above backward regularity is enough to verify the analogue of Theorem 2.2 for that case as well.

Theorem 3.3. *Let the assumptions of Theorem* 3.1 *hold. Then the (Shatah-Struwe) solution semigroup $S(t)$ associated with problem* (1.1) *possesses a global attractor $\mathcal{A}$ in $\mathcal{E}$ which is a bounded set in $\mathcal{E}_1$.*

Indeed, combining the aforementioned backward regularity, the fact that for any initial data $\xi_0 \in \mathcal{E}_1$, the corresponding Shatah-Struwe solution remains in $\mathcal{E}_1$, and the proper version of weak-strong uniqueness, we establish that the weak attractor satisfies $\mathcal{A}_w \subset \mathcal{E}_1$, see [17] for more details. Thus, the energy equality holds for any solution belonging to this attractor. The asymptotic compactness of the solution semigroup in $\mathcal{E}$ can be then established using the so-called energy method, see [2], [22]. Finally, the additional regularity of the global attractor $\mathcal{A}$ is based on this asymptotic compactness by using more or less standard bootstrapping arguments, see [17].

R e m a r k 3.1. In the *subcritical* case $q < 4$, the following analogue of the dissipative estimate for the Strichartz norm holds:

$$\|u\|_{L^4([t,t+1];L^{12}(\Omega))} \leqslant Q(\|\xi_0\|_{\mathcal{E}})\mathrm{e}^{-\alpha t} + Q(\|g\|), \tag{3.3}$$

where the monotone function Q is independent of t and u and the attractor theory in this case is essentially simpler, see [17].

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