139 (2014) MATHEMATICA BOHEMICA No. 4, 597–605

ENTROPY OF SCALAR REACTION-DIFFUSION EQUATIONS

Siniša Slijepčević, Zagreb

(Received September 27, 2013)

Abstract. We consider scalar reaction-diffusion equations on bounded and extended domains, both with the autonomous and time-periodic nonlinear term. We discuss the meaning and implications of the ergodic Poincaré-Bendixson theorem to dynamics. In particular, we show that in the extended autonomous case, the space-time topological entropy is zero. Furthermore, we characterize in the extended nonautonomous case the space-time topological and metric entropies as entropies of a pair of commuting planar homeomorphisms.

Keywords: reaction-diffusion equation; attractor; invariant measure; entropy; Poincaré-Bendixson theorem

MSC 2010: 37L30, 37A35, 37B40, 35B40

1. Introduction

We consider the autonomous scalar reaction-diffusion equation

$$u_t = u_{xx} + f(x, u, u_x), \tag{DC}$$

where f is C^2, 1-periodic in x, and more generally a nonautonomous equation

$$u_t = u_{xx} + f(t, x, u, u_x), \tag{AC}$$

where f is also 1-periodic in t (we will refer to them as (DC) and (AC) case, respectively). For both Cauchy problems, we consider them first in the standard setting, on a bounded interval with periodic boundary conditions, with solutions in the space $H^2(S^1)$ (the *bounded* case). We are, however, also interested in them as extended systems, analyzing time evolution of functions $u\colon \mathbb{R} \to \mathbb{R}$ not necessarily spatially periodic or decaying at infinity. More precisely, the phase space is $H^2_{\rm ul}(\mathbb{R})$, which stands for a uniformly local space with the appropriate weighting function ϱ (the *extended* case). We will be interested in the asymptotic behavior of uniformly bounded solutions in all four cases, and in particular whether in the systems considered chaotic behavior can occur.

The asymptotic behavior and the structure of the attractor in the bounded, (DC) case is relatively well understood. The attractor is generically Morse-Smale, i.e., it consists of finitely many hyperbolic equilibria and their transversal connections ([6], [8]). Both the nonautonomous (AC) and the extended cases are more complex. It is, for example, known that the attractor in the extended case is typically infinitely dimensional ([8], Section 5), and interesting dynamics can occur even in the case when the dynamics is "formally gradient" ([4], [5], [10], [11]). The structure of the attractor is still open even in some simple examples, such as the real Ginzburg-Landau equation ([2]).

Let us first be more specific on the setting. We fix a constant $M > 0$ and let X be the (forward-time) invariant set of initial conditions u_0 for which the entire solution of (DC) is bounded so that $\|u(t)\|_{H^2(S^1)} \leqslant M$ for all $t \geqslant 0$, or the entire solution of (AC) is bounded so that $\|u(t)\|_{H^2_{\mathrm{ul}}(\mathbb{R})} \leqslant M$. We equip the space X with the C^1 topology in the bounded, and the weighted C^1_ϱ topology in the extended case. By the compact embedding theorems, in both cases X is compact, and the equations (DC), (AC) generate a semiflow on X. We list in the Appendix the definitions of the spaces $H^2_{\mathrm{ul}}(\mathbb{R}), C^1_\varrho$ and the required results on existence, uniqueness of solutions as well as on continuous dependence on initial conditions.

(Note that the solutions of the bounded case can be viewed as a small invariant subset of spatially periodic solutions $u(x+1) = u(x)$ in the extended case. The topologies $H^2(\mathbb{R})$ and $H^2_{\mathrm{ul}}(\mathbb{R})$; C^1 and C^1_ϱ coincide on that set.)

We consider solutions of (DC), (AC) as a dynamical system. Denote the semiflow generated by it with φ, where φ^t is the t-time map on X, $t \geqslant 0$, so that $\varphi^t(u_0) = \varphi^t(u(0)) = u(t)$. In the extended case both the (DC) and (AC) systems have a spatial symmetry. We denote by $Su(x) = u(x-1)$ the translation operator on X, then S is a homeomorphism of X which commutes with φ. Now we consider different dynamical systems in the (AC)/(DC) and bounded/extended cases, specifically we consider the following semiflows or maps on X:

(1.1)

Semiflow/map:	(DC)	(AC)
Bounded	φ^t	φ^1
Extended	φ^t, S	$\varphi^1, S.$

Invariant sets and measures with respect to (DC), (AC) will always be invariant with respect to the actions in (1.1).

In the following, we summarize recent results of the author on asymptotics of scalar reaction-diffusion equations which partially generalize the well-known

Poincaré-Bendixson theorem proved by Fiedler and Mallet-Paret in the bounded, (DC) case. We then focus on implications of this to the topological entropy of (DC), (AC); i.e., existence of chaotic behavior with respect to φ and S.

2. Ergodic Poincaré-Bendixson theorem

We first recall the Poincaré-Bendixson theorem by Fiedler and Mallet-Paret [3]. The classical Poincaré-Bendixson theorem ([7], Theorem 14.1.1) describes the asymptotics of bounded smooth planar flows. Fiedler and Mallet-Paret essentially showed that the asymptotics of $u_0 \in X$ in the bounded (DC) case is 2-dimensional, so the classical theorem applies. The projection is defined by

$$\pi(u) = (u(0), u_x(0)).$$

The key result from [3], Theorem 2, says that for any $u_0 \in X$, $\pi\colon \omega(u_0) \to \mathbb{R}^2$ is a homeomorphism onto $\pi(\omega(u_0)) \subset \mathbb{R}^2$. Here $\omega(u_0)$ denotes the ω-limit set of u_0, as usual. The set $\omega(u_0)$ consists then either of a single periodic orbit or a connected family of equilibria and their heteroclinic or homoclinic connections.

This result does not generalize to other cases from (1.1) (for a counter example see [12]). Our approach is to consider invariant measures rather than asymptotics. We denote by $\mathcal{A}$ the union of supports of all invariant (Borel, probability) measures on X. Here the measures are assumed to be invariant with respect to all the actions in (1.1).

As $\mathcal{A}$ is union of closed sets, it may seem that it is not necessarily closed. The following lemma clarifies it.

Lemma 2.1. *The set $\mathcal{A}$ is closed, thus compact.*

Proof. Assume u_n is a sequence in $\mathcal{A}$ converging to v, such that u_n is in the support of some invariant measure μ_n. Let

$$\nu = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n.$$

Then ν is also a (Borel, probability) invariant measure, containing u_n in its support. As the support of ν is closed, it contains also v, thus $v \in \mathcal{A}$. By the choice of the topology, the entire state space X is compact, and so is $\mathcal{A}$. $\square$

The set $\mathcal{A}$ is closely related to the notion of the non-wandering set, and it contains all uniformly recurrent orbits (with respect to all the actions in (1.1)). It is also related to the notion of the global attractor of (DC), (AC) (see e.g. [8]), and is strictly contained in the global attractor (and typically much smaller). Recall that the attractor is characterized as the union of all bounded, complete trajectories ([8], Theorem 2.13).

Lemma 2.2. *The set $\mathcal{A}$ consists of complete orbits. In particular, φ generates a continuos flow on $\mathcal{A}$ in the* (DC) *case; and φ^1 is a homeomorphism in the* (AC) *case.*

Proof. If μ is any φ^t or φ^1 invariant measure (in the (DC) or (AC) case, respectively), then by invariance the set of all complete trajectories has full measure. By continuity, $\operatorname{supp}(\mu)$ consists of complete trajectories, and so does $\mathcal{A}$. As φ^1 is a continuous bijection on a compact set, it is a homeomorphism. □

Remark 2.1. Dynamical interpretation of $\mathcal{A}$: The set $\mathcal{A}$ contains an important part of the dynamics. In the bounded case, we can interpret $\mathcal{A}$ as the set such that for any initial condition u_0, $u(t)$ is in any neighborhood of $\mathcal{A}$ "almost surely with respect to time". More precisely, for any neighborhood U of $\mathcal{A}$, in the (DC) case we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_U(u(t))\,\mathrm{d}t = 1$$

(in the (AC) case, the integral is replaced by a sum over integer times). This follows from the Birkhoff ergodic theorem and the properties of the weak* topology on the space of measures; see [7], Section 4.1. In the extended case, we can show that $u(t)$ is in any neighborhood of $\mathcal{A}$ "with space-time probability 1". This means that, if μ is any S-invariant measure, then (in the (DC) case),

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_X 1_U(u(t))\,\mathrm{d}\mu(u_0)\,\mathrm{d}t = 1.$$

The (AC) case is analogous (see [12], [13]). We will also see in the next section that $\mathcal{A}$ contains "chaotic behavior", if it exists.

The main result from [12] now says that the set $\mathcal{A}$ is 2-dimensional:

Theorem 2.1 (Ergodic Poincaré-Bendixson)**.** *The map $\pi\colon\, \mathcal{A} \to \mathbb{R}^2$ is a homeomorphism onto a compact subset $\pi(\mathcal{A}) \subset \mathbb{R}^2$.*

Let us summarize for reader's convenience the key argument. Fiedler and Mallet-Paret used in [3] the intersection-counting (or "lap-number") function which in the bounded, (DC) case associates with u, v the number of intersections of u, v, or equivalently the number of zeroes $z(u - v)$ of $u - v$. It is then enough to apply the classical result of Angenent [1] that for $u, v \in H^2(S^1)$, $z(u(t) - v(t))$ is always finite for $t > 0$, and the ideas going back to Sturm that the function $t \mapsto z(u(t) - v(t))$ is non-increasing, and that it is strictly decreasing if $\pi(u(t)) = \pi(v(t))$.

In the extended case, $u(t)$ and $v(t)$ are defined on the entire $\mathbb{R}$, thus the number of intersections $z(u(t)-v(t))$ is typically infinite, and the Fiedler and Mallet-Paret argument does not apply. In [12] we consider instead the *average number of intersections* with respect to two (Borel, probability) measures

$$Z(\mu, \nu) = \int_{X \times X} z(u - v)\, \mathrm{d}\mu(u)\, \mathrm{d}\nu(v),$$

where $z(u - v)$ is the number of zeroes of $u - v$ in $[0, 1)$, and in the extended case we also assume that μ, ν are S-invariant. We then in [12] show that:

(i) $t \mapsto Z(\mu(t), \nu(t))$ is non-decreasing along the evolution $\mu(t)$, $\nu(t)$ of measures along the solutions of (AC), (DC).

(ii) If for some u, v in the supports of $\mu(t)$, $\nu(t)$ respectively, $\pi(u) = \pi(v)$, then $Z(\mu(t), \nu(t))$ is strictly decreasing at t.

For invariant measures μ, ν, $t \mapsto Z(\mu(t), \nu(t))$ is by definition constant, thus π must be injective on $\mathcal{A}$.

In the (DC) case, we can now describe $\mathcal{A}$.

Corollary 2.1. *In the* (DC) *case, $\mathcal{A}$ consists of equilibria and periodic orbits.*

Proof. Consider the semiflow φ^* on $\mathcal{A}^* = \pi(\mathcal{A}) \subset \mathbb{R}^2$ defined by $\varphi^* = \pi \circ \varphi \circ \pi^{-1}$. It is well-defined and continuous because of Theorem 2.1. Analogously as in [3], Lemma 2.3, one can show that φ is a semiflow (actually a flow by Lemma 2.2) generated by a continuous vector field on $\mathcal{A}^*$. Analogously as in the proof of [3], Proposition 1 (by proving a particular version of planar Poincaré-Bendixson), we deduce that the only recurrent orbits of φ^* on $\mathcal{A}^*$ are equilibria and periodic orbits. A support of any probability invariant measure is contained in the closure of the set of recurrent orbits ([7], Proposition 4.1.1), thus all the invariant measures of φ^* and consequently of φ are supported on the union of equilibria and periodic orbits. $\square$

Note that in the bounded (DC) case, Corollary 2.1 follows directly also from [3]. In the extended (DC) case, it is new.

Corollary 2.1 has important dynamical implications to the extended (AC) dynamics in the sense of Remark 2.1, discussed in more detail in [12].

We note that in [13] we prove an analogue of the Ergodic Poincaré-Bendixson theorem to a discrete-space, continuous-time case (the generalized Frenkel-Kontorova model), and deduce several dynamical implications.

Here we consider the notions of topological h_{top} and metric h_μ entropy of (DC), (AC), and prove new results in the extended case. The definition of entropy will depend on the actions in the table (1.1). We will denote them by $h_{\mathrm{top}}(\mathcal{A})$, $h_\mu(\mathcal{A})$, $h_{\mathrm{top}}(X)$, $h_\mu(X)$ (when considered on $\mathcal{A}$ or X), where μ is an invariant measure, implicitly always assuming different definitions in all four cases studied.

In the bounded case, h_{top} is the usual topological entropy which measures complexity of a semiflow φ or a map φ^1, and h_μ is the metric entropy associated with an invariant measure μ (see e.g. [7], Section 4.3). In the extended case, h_{top} is usually called the space-time topological entropy, as it measures mutual complexity of the space- and time-evolutions (the spatial translation S and time evolution φ). Similarly, h_μ is a space-time metric entropy with respect to a space-time invariant measure μ (see [15] and references therein).

Entropy is always non-negative; positivity of entropy is interpreted as "existence of chaos". We first recall the well-known variational principle:

Proposition 3.1. *Let $\mathcal{Y} = \mathcal{A}$ or X. Then*

$$h_{\mathrm{top}}(\mathcal{Y}) = \sup_\mu h_\mu(\mathcal{Y}), \tag{3.1}$$

where supremum is taken over all invariant measures μ.

P r o o f. In the bounded case, this is a special case of [7], Theorem 4.5.3. In the extended case, this follows from [9] (proved in general for group actions). □

Note that in particular $h_{\mathrm{top}}(\mathcal{A})$ $=h_{\mathrm{top}}(X)$, as the right-hand sides of (3.1) coincide; so we drop the argument $\mathcal{A}$, X.

First note that in the bounded (DC) case, h_{top}=h_μ $= 0$. Here is why: by the Poincaré-Bendixson theorem [3], $\mathcal{A}$ consists of periodic orbits and equilibria (Corollary 2.1 gives an alternative proof of this in the bounded case), so by definition and the variational principle entropy must be zero. The same logic applies now to the extended (DC) case.

Corollary 3.1. *In the extended* (DC) *case, the space-time topological and metric entropies are zero.*

P r o o f. By Corollary 2.1, $\mathcal{A}$ consists of periodic orbits and equilibria. By definition, for any invariant measure μ, $h_\mu = 0$. The claim follows from the variational principle. □

This result complements what is currently known. Zelik [15] has shown that for scalar extended (DC) systems with a formal gradient structure, the space-time entropy is zero, while Turaev and Zelik [14] constructed examples in one complex variable with positive entropy.

It remains an open question whether there exists an extended (AC) system with positive entropy. Mielke and Zelik [16] constructed a closely related example, a scalar perturbed Swift-Hohenberg equation (not monotone, so our results do not apply).

We, however, can give a characterization of space-time topological and metric entropies in the extended (AC) case in terms of entropies of a pair of planar homeomorphisms. We define $T, U\colon\ \mathcal{A}^* \to \mathcal{A}^*$, where $\mathcal{A}^* = \pi(\mathcal{A})$, with

$$\begin{aligned} T &= \pi \circ S \circ \pi^{-1}, \\ U &= \pi \circ \varphi^1 \circ \pi^{-1}. \end{aligned}$$

Lemma 2.2 and Theorem 2.1 imply that T, U are commuting homeomorphisms of $\mathcal{A}^*$, which is a compact subset of $\mathbb{R}^2$. The following observation now follows directly from the definitions of entropies:

Corollary 3.2. *The space-time topological and metric entropies of the extended* (AC) *system coincide with the topological and metric entropies of the pair* (T, U) *acting on* $\mathcal{A}^*$.

(Here metric entropies of an invariant measure μ on $\mathcal{A}$ and the pulled measure $\pi_*\mu$ on $\mathcal{A}^*$ coincide.) If extended (AC) systems can have positive entropy, Corollary 3.2 can be used to visualize space-time chaos and study bifurcations related to its occurrence in the planar setting.

4. Appendix: uniformly local Sobolev spaces and existence of solutions

We recall the definitions of the uniformly local Sobolev space $H^2_{\mathrm{ul}}(\mathbb{R})$, a coarser C^1_ϱ topology, as well as the results on existence of solutions and a continuous semiflow required in the extended case.

Let $\varrho\colon\ \mathbb{R} \to (0,\infty)$ be a smooth, rapidly decreasing function, for instance $\varrho(x) = 1/\cosh(|x|)$, and let $(T_y u)(x) = u(x-y)$ be the translation operator. Given a Borel-measurable function u, we define

$$\begin{aligned} \|u\|_{L^2_{\mathrm{ul}}} &= \sup_{y\in\mathbb{R}} \left(\int_{\mathbb{R}} \varrho(x+y) u(x)^2 \, \mathrm{d}x \right), \\ L^2_{\mathrm{ul}}(\mathbb{R}) &= \Big\{ \|u\|_{L^2_{\mathrm{ul}}(\mathbb{R})} < \infty,\ \lim_{y\to 0} \|T_y u - u\|_{L^2_{\mathrm{ul}}} = 0 \Big\}, \end{aligned}$$

$$H^2_{\mathrm{ul}}(\mathbb{R}) = \{u \in L^2_{\mathrm{ul}}(\mathbb{R}),\ \partial_x u \in L^2_{\mathrm{ul}}(\mathbb{R}),\ \partial_x^2 u \in L^2_{\mathrm{ul}}(\mathbb{R})\},$$
$$\|u\|_{H^2_{\mathrm{ul}}} = (\|u\|^2_{L^2_{\mathrm{ul}}(\mathbb{R})} + \|\partial_x u\|^2_{L^2_{\mathrm{ul}}(\mathbb{R})} + \|\partial_x^2\|^2_{L^2_{\mathrm{ul}}(\mathbb{R})})^{1/2},$$
$$\|u\|_{C^1_\varrho} = \max\Big\{\sup_{x\in\mathbb{R}}(\varrho(x)|u(x)|),\ \sup_{x\in\mathbb{R}}(\varrho(x)|\partial_x u(x)|)\Big\}.$$

In the extended case, it is well-known that (DC), (AC) generate a continuous semiflow on X; the semiflow is also continuous if X is equipped with the coarser topology C^1_ϱ (for details on that see sections 7.1 and 7.3 of [4]).

Recall that X is the set of $u \in H^2_{\mathrm{ul}}(\mathbb{R})$ whose semiorbit is bounded in the $H^2_{\mathrm{ul}}(\mathbb{R})$ norm by a constant. To be able to view φ as a dynamical system on X, we rely on the following compact-embedding observation.

Lemma 4.1. *The set X equipped with the C^1_ϱ topology is compact.*

Proof. The topology $H^2_{\mathrm{ul}}(\mathbb{R})$ projected to functions restricted to any bounded interval $[a,b]$ coincides with $H^2([a,b])$, thus $\{u|_{[a,b]}, \|u\|_{H^2_{\mathrm{ul}}(\mathbb{R})} \leqslant M\}$ is compact if equipped with the C^1 topology. The C^1_ϱ topology is generated by the C^1 topology on all bounded intervals. By the diagonalization argument we now see that X equipped with the C^1_ϱ norm is sequentially compact, thus compact. □

References

[1] *S. Angenent*: The zero set of a solution of a parabolic equation. J. Reine Angew. Math. *390* (1988), 79–96. zbl MR

[2] *J.-P. Eckmann, J. Rougemont*: Coarsening by Ginzburg-Landau dynamics. Commun. Math. Phys. *199* (1998), 441–470. zbl MR

[3] *B. Fiedler, J. Mallet-Paret*: A Poincaré-Bendixson theorem for scalar reaction diffusion equations. Arch. Ration. Mech. Anal. *107* (1989), 325–345. zbl MR

[4] *T. Gallay, S. Slijepčević*: Energy flow in formally gradient partial differential equations on unbounded domains. J. Dyn. Differ. Equations *13* (2001), 757–789. zbl MR

[5] *T. Gallay, S. Slijepčević*: Distribution of energy and convergence to equilibria in extended dissipative systems. To appear in J. Dyn. Differ. Equations.

[6] *R. Joly, G. Raugel*: Generic Morse-Smale property for the parabolic equation on the circle. Ann. Inst. Henri Poincaré, Anal. Non Linéaire *27* (2010), 1397–1440. zbl MR

[7] *A. Katok, B. Hasselblatt*: Introduction to the Modern Theory of Dynamical Systems. Encyclopedia of Mathematics and Its Applications 54, Cambridge University Press, Cambridge, 1995. zbl MR

[8] *A. Miranville, S. Zelik*: Attractors for dissipative partial differential equations in bounded and unbounded domains. Handbook of differential equations: Evolutionary Equations. Vol. IV (C. M. Dafermos, M. Pokorný, eds.). Elsevier/North-Holland, Amsterdam, 2008, pp. 103–200. zbl MR

[9] *J. Moulin Ollagnier, D. Pinchon*: The variational principle. Studia Math. *72* (1982), 151–159. zbl MR

[10] *S. Slijepčević*: Extended gradient systems: Dimension one. Discrete Contin. Dyn. Syst. *6* (2000), 503–518. zbl MR

[11] *S. Slijepčević*: The energy flow of discrete extended gradient systems. Nonlinearity *26* (2013), 2051–2079. zbl MR

[12] *S. Slijepčević*: Ergodic Poincaré-Bendixson theorem for scalar reaction-diffusion equations. Preprint.

[13] *S. Slijepčević*: The Aubry-Mather theorem for driven generalized elastic chains. Discrete Contin. Dyn. Syst. *34* (2014), 2983–3011. zbl MR

[14] *D. Turaev, S. Zelik*: Analytical proof of space-time chaos in Ginzburg-Landau equations. Discrete Contin. Dyn. Syst. *28* (2010), 1713–1751. zbl MR

[15] *S. Zelik*: Formally gradient reaction-diffusion systems in $\mathbb{R}^n$ have zero spatio-temporal topological entropy. Discrete Contin. Dyn. Syst. suppl. vol. (2003), 960–966. zbl MR

[16] *S. Zelik, A. Mielke*: Multi-pulse evolution and space-time chaos in dissipative systems. Mem. Am. Math. Soc. *198* (2009), 1–97. zbl MR

Author's address: *Siniša Slijepčević*, Department of Mathematics, University of Zagreb, Bijenička 30, Croatia, e-mail: `slijepce@math.hr`.