BOUNDEDNESS OF SOLUTIONS TO PARABOLIC-ELLIPTIC CHEMOTAXIS-GROWTH SYSTEMS WITH SIGNAL-DEPENDENT SENSITIVITY

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(Received September 30, 2013)

Abstract. This paper deals with parabolic-elliptic chemotaxis systems with the sensitivity function $\chi(v)$ and the growth term $f(u)$ under homogeneous Neumann boundary conditions in a smooth bounded domain. Here it is assumed that $0 < \chi(v) \leq \frac{\chi_0}{v^k}$ ($k \geq 1$, $\chi_0 > 0$) and $\lambda_1 - \mu_1 u \leq f(u) \leq \lambda_2 - \mu_2 u$ ($\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$). It is shown that if $\chi_0$ is sufficiently small, then the system has a unique global-in-time classical solution that is uniformly bounded. This boundedness result is a generalization of a recent result by K. Fujie, M. Winkler, T. Yokota.

Keywords: chemotaxis; global existence; boundedness

MSC 2010: 35B40, 35K60

1. INTRODUCTION AND MAIN RESULT

In this paper we consider the global existence and boundedness in the parabolic-elliptic chemotaxis-growth system

$$\begin{cases}
t_u = \Delta u - \nabla \cdot (u\chi(v)\nabla v) + f(u), & x \in \Omega, \ t > 0, \\
0 = \Delta v - v + u, & x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary $\partial \Omega$. We assume

Partially supported by Grant-in-Aid for Scientific Research (C), No. 25400119.
that the initial data $u_0$ satisfies

(1.2) \[ u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \quad \text{and} \quad \int_{\Omega} u_0 > 0. \]

As for the chemotactic sensitivity function, we assume that

(1.3) \[ \chi \in C^1((0, \infty)) \quad \text{with} \quad \chi > 0. \]

Also we assume that $f \in C^1([0, \infty))$ and there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ such that

(1.4) \[ \lambda_1 - \mu_1 s \leq f(s) \leq \lambda_2 - \mu_2 s \quad \text{for all} \quad s \in [0, \infty). \]

This system was introduced by Keller and Segel [6], [7] (see also [4], [14], [15]), and the mathematical study of this system has developed extensively. In this paper we especially focus on the signal-sensitivity function and the growth term. There are some known results related to this system in [1], [2], [8]–[13], [16]–[19]. The present work is devoted to the global existence and boundedness. We remark that the existence of classical solutions to (1.1) is shown by a similar way as in [3]. Since $f(0) \geq \lambda_1 > 0$ by (1.4), the solution to (1.1) is nonnegative.

In order to formulate our main result, given a nonnegative $0 \neq u_0 \in C^0(\overline{\Omega})$, let us define a constant $\gamma > 0$ as

(1.5) \[ \gamma := \min \left\{ \| u_0 \|_{L^1(\Omega)}, \frac{\lambda_1}{\mu_1} |\Omega| \right\} \int_0^{\infty} \frac{1}{(4\pi t)^{n/2}} e^{-\left( t + \frac{\text{diam} \Omega}{4t} \right)^2} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\text{diam} \Omega}{4t}} \, dt < \infty, \]

where $\text{diam} \Omega := \max_{x, y \in \overline{\Omega}} |x - y|$. We remark that the integrand in (1.5) decays exponentially not only as $t \to \infty$ but also as $t \to 0$, and so $\gamma < \infty$ for all $n \in \mathbb{N}$. The constant $\gamma$ marks an a priori pointwise lower bound on the solution component $v$, as we shall see below. In what follows, when $k = 1$ we regard the value of $k^k/(k - 1)^{k-1}$ as 1.

**Theorem 1.1.** Let $n \in \mathbb{N}$, and suppose that $u_0, \chi$ and $f$ satisfy (1.2), (1.3) and (1.4), respectively. Moreover, assume that $\chi$ satisfies

\[ \chi(s) \leq \frac{\chi_0}{s^k} \quad \text{for all} \quad s \in [\gamma, \infty), \]

with some $k \geq 1$ and some $\chi_0 > 0$ fulfilling

\[ \chi_0 < \frac{2}{n} \frac{k^k}{(k - 1)^{k-1}} \gamma^{k-1}. \]
Then (1.1) possesses a unique global classical solution \((u, v)\) which satisfies

\[
\|u(\cdot, t)\|_{L^\infty} \leq M_\infty \quad \text{for all } t \in [0, \infty)
\]

with some constant \(M_\infty > 0\).

2. Preliminaries

We begin with the following lemma shown in [3]. This lemma is key to deriving a uniform-in-time estimate for \(v\).

**Lemma 2.1.** Let \(w \in C^0(\Omega)\) be a nonnegative function such that \(\int_\Omega w > 0\). If \(z\) is a weak solution to

\[
\begin{aligned}
-\Delta z + z &= w, \quad x \in \Omega, \\
\frac{\partial z}{\partial \nu} &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

then

\[
z \geq \left( \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-t + (\text{diam } \Omega)^2/(4t)} \, dt \right) \int_\Omega w > 0 \quad \text{in } \Omega.
\]

Here we give an a priori pointwise lower bound on the solution component \(v\). The first equation in (1.1) and the condition (1.4) imply

\[
\frac{d}{dt} \int_\Omega u = \int_\Omega f(u) \geq \lambda_1 |\Omega| - \mu_1 \int_\Omega u.
\]

Integrating this inequality, we have

\[
\int_\Omega u \geq \frac{\lambda_1}{\mu_1} |\Omega| + e^{-\mu_1 t} \left( \|u_0\|_{L^1(\Omega)} - \frac{\lambda_1}{\mu_1} |\Omega| \right) \quad \text{for all } t \in (0, \infty),
\]

and then

\[
\int_\Omega u \geq \min \left\{ \|u_0\|_{L^1(\Omega)}, \frac{\lambda_1}{\mu_1} |\Omega| \right\}.
\]

By virtue of Lemma 2.1 we can thereby estimate \(v\) from below as follows:

\[
(2.1) \quad v(x, t) \geq \gamma
\]

for all \(x \in \Omega\) and \(t \in (0, T)\), whenever \((u, v)\) solves (1.1) in \(\Omega \times (0, T)\) for some \(T > 0\). Here \(\gamma > 0\) is a constant defined as (1.5).

**Remark 2.1.** The maximum principle yields the lower pointwise estimate for \(v(\cdot, t)\) for fixed \(t > 0\). On the other hand, Lemma 2.1 and the uniform-in-time estimate for mass imply the uniform estimate (2.1).

We next collect some known facts concerning the Neumann Laplacian in \(\Omega\). For the proof of (iii) see [5], Lemma 2.1.
Lemma 2.2. For \( r \in (1, \infty) \), let \( \Delta \) denote the realization of the Laplacian in \( L^r(\Omega) \) with domain \( \{ w \in W^{2,r}(\Omega); \partial w/\partial n = 0 \text{ on } \partial \Omega \} \). Then the operator \(-\Delta + 1\) is sectorial and possesses closed fractional powers \((-\Delta + 1)^\theta\), \( \theta \in (0, 1) \), with dense domain \( D((-\Delta + 1)^\theta) \). Moreover, the following statements hold:

(i) If \( m \in \{0, 1\}, p \in [1, \infty] \) and \( q \in (1, \infty) \), then there exists a constant \( c_{m,p} > 0 \) such that for all \( w \in D((-\Delta + 1)^\theta) \),

\[
\|w\|_{W^{m,p}(\Omega)} \leq c_{m,p}\|(-\Delta + 1)^\theta w\|_{L^q(\Omega)},
\]

provided that \( m < 2\theta \) and \( m - n/p < 2\theta - n/q \).

(ii) Let \( p \in (1, \infty) \). Then there exist \( c > 0 \) and \( \nu_1 > 0 \) such that for all \( u \in L^p(\Omega) \) and any \( t > 0 \),

\[
\|(-\Delta + 1)^\theta e^{t(\Delta - 1)}u\|_{L^p(\Omega)} \leq ct^{-\theta}e^{-\nu_1 t}\|u\|_{L^p(\Omega)}.
\]

(iii) Let \( p \in (1, \infty) \). Then there exists \( \nu_2 > 1 \) such that for \( \varepsilon > 0 \) there exists \( c_\varepsilon > 0 \) such that for all \( \mathbb{R}^n\)-valued \( z \in C_0^\infty(\Omega) \),

\[
\|(-\Delta + 1)^\theta e^{t(\Delta - 1)}\nabla \cdot z\|_{L^p(\Omega)} \leq c_\varepsilon t^{-\theta - 1/2 - \varepsilon}e^{-\nu_2 t}\|z\|_{L^p(\Omega)}, \quad t > 0.
\]

Accordingly, for all \( t > 0 \) the operator \((-\Delta + 1)^\theta e^{t\Delta} \cdot \) admits a unique extension to all of \( L^p(\Omega) \) which, again denoted by \((-\Delta + 1)^\theta e^{t\Delta} \cdot \), satisfies the above estimate for all \( \mathbb{R}^n\)-valued \( z \in L^p(\Omega) \).

3. Proof of main result

We first deduce \( L^p \)-boundedness of solutions to (1.1). Next let us show that \( L^p \)-boundedness with sufficiently large \( p \) implies \( L^\infty \)-boundedness. Combining these results will prove our main theorem.

Lemma 3.1. Let \( p > 1 \), and suppose that \((u, v)\) is a classical solution to (1.1) in \( \Omega \times (0, T) \) for some \( T > 0 \). Then there exist \( C_1, C_2 > 0 \) such that

\[
\frac{d}{dt} \int_\Omega u^p \leq -\frac{p(p-1)}{2} \int_\Omega u^{p-2} |\nabla u|^2 + \frac{p(p-1)}{2} \int_\Omega u^p \chi^2(v) |\nabla v|^2 + C_1 \int_\Omega u^p + C_2 \quad \text{for all } t \in (0, T).
\]
The condition (1.4) yields
\[ \phi \in \Omega \text{ in } (s_\phi, T) \]
for some constants \( C \) and \( T > 0 \). Using integration by parts, we see that
\[ p \in \exists k \geq 1 \text{ such that } \int_\Omega u^{p-1}f(u) \leq C_1 \int_\Omega u^p + C_2 \text{ for some constants } C_1, C_2 > 0, \text{ and hence we obtain the desired inequality}. \]

The next lemma is obtained in [3]. For convenience we give the sketch of the proof.

**Lemma 3.2.** Let \( p > 1 \), and suppose that \((u, v)\) is a classical solution to (1.1) in \( \Omega \times (0, T) \) for some \( T > 0 \). Moreover, for \( \gamma > 0 \) given by (1.5) (see also (2.1)), let \( \varphi \in C^1([\gamma, \infty)) \) such that \( \varphi \geq 0 \) and there exists a constant \( M > 0 \) satisfying \( s\varphi(s) \leq M \) for all \( s \geq \gamma \). Let \( A \) and \( B \) be positive constants such that \( AB = p \).

Then
\[ \int_\Omega u^p (-\varphi'(v) - \frac{B^2}{2} \varphi^2(v)) |\nabla v|^2 \leq \frac{A^2}{2} \int_\Omega u^{p-2} |\nabla u|^2 + M \int_\Omega u^p \text{ for all } t \in (0, T). \]

**Sketch of the proof.** Multiplying the second equation in (1.1) by \( u^p \varphi(v) \) and using integration by parts, we see that
\[ -\int_\Omega u^p \varphi'(v) |\nabla v|^2 = p \int_\Omega u^{p-1} \varphi(v) \nabla u \cdot \nabla v + \int_\Omega u^p \varphi(v) v - \int_\Omega u^{p+1} \varphi(v). \]

Applying Young's inequality completes the proof. \( \Box \)

Now we give \( L^p \)-boundedness of solutions to (1.1).

**Proposition 3.3.** Suppose that \( n \in \mathbb{N} \), and that \( u_0, \chi \) and \( f \) satisfy (1.2), (1.3) and (1.4), respectively. Let \((u, v)\) be a classical solution to (1.1) in \( \Omega \times (0, T) \) for some \( T > 0 \). Moreover, let \( \gamma > 0 \) be as in (1.5) and (2.1). Suppose that there exist \( k \geq 1 \) and \( \chi_0 > 0 \) such that \( \chi(s) \leq \chi_0/s^k \) for all \( s \geq \gamma \). Then for any \( p \in [1, \chi_0^{-1}[k^k/(k - 1)^{k-1}]^{k-1} \) there exists a constant \( M_p > 0 \) fulfilling
\[ \|u(\cdot, t)\|_{L^p} \leq M_p \text{ for all } t \in [0, T). \]

**Proof.** Taking any \( p \in [1, \chi_0^{-1}[k^k/(k - 1)^{k-1}]^{k-1} \), we have \( \chi_0 < p^{-1}[k^k/(k - 1)^{k-1}]^{k-1} \). Now we take \( \varepsilon > 0 \) and \( L > 0 \) such that
\[ \varepsilon < p(p-1), \quad L < \gamma < \frac{k}{k-1}L \text{ and } \chi_0 \leq \frac{1}{p} \sqrt{\frac{p(p-1)-\varepsilon}{p(p-1)}} \frac{k^k}{(k - 1)^{k-1}}L^{k-1}. \]
Applying Lemma 3.2 to \( \varphi(s) := 1/(B^2(s - L)) \), \( A := \sqrt{p(p - 1) - \varepsilon} \) and \( B := p/\sqrt{p(p - 1) - \varepsilon} \), we infer that

\[
\int_\Omega u^p \left(- \varphi'(v) - \frac{B^2}{2} \varphi^2(v) \right)|\nabla v|^2 \leq \frac{p(p - 1) - \varepsilon}{2} \int_\Omega u^{p-2} |\nabla u|^2 + M \int_\Omega u^p
\]

and

\[
\frac{p(p - 1)}{2} \chi^2(s) \leq -\varphi'(s) - \frac{B^2}{2} \varphi^2(s) \quad \text{for all } s \geq \gamma.
\]

Now by (3.2), we can combine (3.1) with Lemma 3.1 to see that

\[
\frac{d}{dt} \int_\Omega u^p \leq - \frac{p(p - 1)}{2} \int_\Omega u^{p-2} |\nabla u|^2 + \frac{p(p - 1) - \varepsilon}{2} \int_\Omega u^{p-2} |\nabla u|^2 + (M + C_1) \int_\Omega u^p + C_2
\]

for all \( t \in (0, T) \). Since the first equation in (1.1) and the condition (1.4) yield

\[
\frac{d}{dt} \int_\Omega u = \int_\Omega f(u) \leq \lambda_2 |\Omega| - \mu_2 \int_\Omega u,
\]

we see that for all \( t \in (0, \infty) \),

\[
\int_\Omega u \leq \frac{\lambda_2}{\mu_2} |\Omega| + e^{-\mu_2 t} \left( \|u_0\|_{L^1(\Omega)} - \frac{\lambda_2}{\mu_2} |\Omega| \right) \leq \max \left\{ \|u_0\|_{L^1(\Omega)} , \frac{\lambda_2}{\mu_2} |\Omega| \right\}.
\]

By virtue of this estimate, proceeding similarly as in [3], Proposition 4.3, we can complete the proof from (3.3).

Next, assuming \( L^p \)-boundedness, we derive \( L^\infty \)-boundedness.

\textbf{Proposition 3.4.} Let \( n \in \mathbb{N} \), and assume that \( u_0, \chi \) and \( f \) satisfy (1.2), (1.3) and (1.4), respectively. Let \((u, v)\) be the classical solution to (1.1) in \( \Omega \times (0, T) \), and assume further that \( \chi \in L^\infty((\gamma, \infty)) \) with \( \gamma > 0 \) given by (1.5) (see also (2.1)). Then if there exist \( p > n/2 \) and a constant \( M_p > 0 \) such that \( \|u(\cdot, t)\|_{L^p} \leq M_p \) for all \( t \in (0, T) \), then there exists a constant \( M_\infty > 0 \) independent of \( T \) such that

\[
\|u(\cdot, t)\|_{L^\infty} \leq M_\infty \quad \text{for all } t \in (0, T).
\]
Proof. Let \( p > n/2 \). We may assume that \( p < n \). We see from (1.4) that \( f(s) + s \leq C(1 + s) \) for some \( C > 0 \). We can take \( q > n \) so that \( q > p \). Then we have

\[
\|f(u) + u\|_{L^q(\Omega)} \leq C\|1 + u\|^{p/q}_{L^p(\Omega)} \!
\leq C' \|1 + u\|^{1-p/q}_{L^\infty(\Omega)} \!
\leq C'' + C''' \|u\|^{1-p/q}_{L^\infty(\Omega)},
\]

where \( C', C'' \) are some positive constants. Recalling the choice of \( q \), we see that \( 1-p/q \in (0,1) \). Moreover, we choose \( q > n \) satisfying further that \( 1-(n-p)q/(np) > 0 \), which enables us to pick \( \lambda \in (1,\infty) \) fulfilling \( 1/\lambda < 1 - (n-p)q/(np) \). The elliptic regularity \( (\|\nabla v\|_{L^{np/(n-p)}(\Omega)} \leq k_p \|u\|_{L^p(\Omega)}) \) and Hölder’s inequality yield

\[
\|u\chi(v)\nabla v\|_{L^q(\Omega)} \leq \|\chi\|_{L^{\infty}(\Omega)} \|\nabla v\|_{L^{np/(n-p)}(\Omega)} \|u\|_{L^{\infty}(\Omega)}
\leq \|\chi\|_{L^{\infty}(\Omega)} \|\nabla v\|_{L^{np/(n-p)}(\Omega)} \|u\|_{L^{\infty}(\Omega)}
\leq \|\chi\|_{L^{\infty}(\Omega)} \|\nabla v\|_{L^{np/(n-p)}(\Omega)} \|u\|_{L^{\infty}(\Omega)}
\leq K_p \|u\|^{\beta}_{L^{\infty}(\Omega)},
\]

where \( \lambda' := \lambda/(\lambda - 1) \), for some \( \beta \in (0,1) \) and \( K_p > 0 \). Now let \( t \in (0,T) \). Then we have

\[
u(s) = e^{t(\Delta - 1)} u_0 - \int_0^t e^{(t-s)(\Delta - 1)}(\nabla \cdot (u(s)\chi(v(s))\nabla v(s)) + (f(u(s)) + u(s))) \, ds.
\]

Let \( \theta \in (n/(2q),1/2) \) and \( \varepsilon \in (0,1/2 - \theta) \). Using Lemma 2.2, we see that

\[|u(\cdot,t)|_{L^\infty(\Omega)} \leq |u_0|_{L^\infty(\Omega)} + c_0, \varepsilon \int_0^t (t-s)^{-\theta} e^{-\nu_1(t-s)} \|f(u(s)) + u(s)\|_{L^q(\Omega)} \, ds
\]

Combining (3.4) and (3.5) with the above inequality implies the uniform estimate:

\[
u(\cdot,t)|_{L^\infty(\Omega)} \leq K_0 + K_1 \left( \sup_{t \in [0,T]} |u(\cdot,t)|_{L^\infty(\Omega)} \right)^\beta + K_2 \left( \sup_{t \in [0,T]} |u(\cdot,t)|_{L^\infty(\Omega)} \right)^{1-p/q}
\]

for some \( K_0, K_1, K_2 > 0 \). Since \( \beta, 1-p/q \in (0,1) \), we obtain the desired inequality.

\( \square \)

We are now in a position to prove the main result.

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Proof of Theorem 1.1. As stated in Section 1, by a similar way as in [3] we can show that there exist $T_{\text{max}} \leq \infty$ (depending only on $\|u_0\|_{L^\infty(\Omega)}$) and exactly one pair $(u, v)$ of nonnegative functions $u \in C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})) \cap C^0([0, T_{\text{max}}); C^0(\Omega))$, and $v \in C^{2,0}(\overline{\Omega} \times (0, T_{\text{max}})) \cap C^0([0, T_{\text{max}}); C^0(\Omega))$ that solves (1.1) in the classical sense. According to the condition for $k$ and $\chi_0$, by Proposition 3.3 we can find some $p > n/2$ and $M_p > 0$ such that $\|u(\cdot, t)\|_{L^p} \leq M_p$ for all $t \in (0, T_{\text{max}})$. Therefore Proposition 3.4 completes the proof. □

Remark 3.1. The local-in-time existence of classical solutions to (1.1) can be provided under the only lower condition: $\lambda_1 - \mu_1 s \leq f(s)$. Moreover, if the growth term $f$ satisfies the relaxed condition: $\lambda_1 - \mu_1 s \leq f(s) \leq \lambda_2 + \mu_2 s$, then we have the upper mass estimate depending on time $t$ similarly, and so the global existence of solutions without uniform boundedness is proved.

References


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