Abstract. The problem of uniqueness of an entire or a meromorphic function when it shares a value or a small function with its derivative became popular among the researchers after the work of Rubel and Yang (1977). Several authors extended the problem to higher order derivatives. Since a linear differential polynomial is a natural extension of a derivative, in the paper we study the uniqueness of a meromorphic function that shares one small function CM with a linear differential polynomial, and prove the following result: Let \( f \) be a nonconstant meromorphic function and \( L \) a nonconstant linear differential polynomial generated by \( f \). Suppose that \( a = a(z) (\neq 0, \infty) \) is a small function of \( f \). If \( f - a \) and \( L - a \) share \( 0 \) CM and

\[
(k + 1)N(r, \infty; f) + N(r, 0; f') + N_k(r, 0; f') < \lambda T(r, f') + S(r, f')
\]

for some real constant \( \lambda \in (0, 1) \), then \( f - a = (1 + c/a)(L - a) \), where \( c \) is a constant and \( 1 + c/a \neq 0 \).

Keywords: meromorphic function; differential polynomial; small function; sharing

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1. Introduction, Definitions and Results

Let \( f, g \) be nonconstant meromorphic functions defined in the open complex plane \( \mathbb{C} \). For \( a \in \mathbb{C} \cup \{ \infty \} \) we say that \( f, g \) share the value \( a \) CM (counting multiplicities) if \( f, g \) have the same \( a \)-points with the same multiplicities, and we say that \( f, g \) share the value \( a \) IM (ignoring multiplicities) if \( f, g \) have the same \( a \)-points but the multiplicities are not taken into account.

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We refer the reader to [6] for the standard notation and definitions of the value distribution theory. However, in the following we explain some notation used in the paper.

**Definition 1.1.** For a meromorphic function $f$ and for $a \in \mathbb{C} \cup \{\infty\}$ and for a positive integer $k$

(i) $N_k(r,a;f) (N_k(a;f))$ denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not less than $k$;

(ii) $N_k(r,a;f) (N_k(a;f))$ denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not greater than $k$;

(iii) $N_k(r,a;f)$ denotes the sum $\overline{N}(r,a;f) + \sum_{j=2}^{k} \overline{N}(j(r,a;f))$.

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$ and $N_k(r,a;f) \leq k\overline{N}(r,a;f)$.

Rubel-Yang [10], Mues-Steinmetz [9], Gundersen [5], Yang [12] and others considered the uniqueness problem of entire functions when their first and $k$th derivatives share two values CM or IM.

Brück [4] considered the uniqueness problem of an entire function when it shares a single value CM with its first derivative and proved the following theorem.

**Theorem A ([4]).** Let $f$ be a nonconstant entire function. If $f$ and $f'$ share the value 1 CM and $N(r,0;f') = S(r,f)$, then $f - 1 = c(f' - 1)$, where $c$ is a nonzero constant.

Yang [11] considered an entire function of finite order and proved the following result.

**Theorem B ([11]).** Let $f$ be a nonconstant entire function of finite order and let $a \neq 0$ be a finite constant. If $f$, $f^{(k)}$ share the value $a$ CM, then $f - a = c(f^{(k)} - a)$, where $c$ is a nonzero constant and $k \geq 1$ is an integer.

Zhang [14] extended Theorem A to meromorphic functions and proved the following results.

**Theorem C ([14]).** Let $f$ be a nonconstant meromorphic function. If $f$ and $f'$ share 1 CM and if

$$2\overline{N}(r,\infty;f) + 2N(r,0;f') < \lambda T(r,f') + S(r,f')$$

for some constant $\lambda \in (0,1)$, then $f - 1 = c(f' - 1)$, where $c$ is a nonzero constant.
Theorem D ([14]). Let $f$ be a nonconstant meromorphic function. If $f$ and $f^{(k)}$ share 1 CM and if

$$2N(r, \infty; f) + N(r, 0; f') + N(r, 0; f^{(k)}) < \lambda T(r, f^{(k)}) + S(r, f^{(k)})$$

for some constant $\lambda \in (0, 1)$, then $f - 1 = c(f^{(k)} - 1)$, where $c$ is a nonzero constant.

Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$. A meromorphic function $a = a(z)$, defined in $\mathbb{C}$, is called a small function of $f$ if $T(r, a) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f)/T(r, f) \to 0$ as $r \to \infty$, possibly outside a set of finite linear measure.

Yu [13] considered the uniqueness problem of an entire function or a meromorphic function when it shares one small function with its derivative. The next two theorems are the results of Yu [13].

Theorem E ([13]). Let $f$ be a nonconstant entire function and let $a = a(z)$ ($\not\equiv 0, \infty$) be a small function of $f$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and

$$\delta(0; f) > 3/4,$$

then $f \equiv f^{(k)}$, where $k$ is a positive integer.

Theorem F ([13]). Let $f$ be a nonentire meromorphic function and $a = a(z)$ ($\not\equiv 0, \infty$) a small function of $f$. If

(i) $f$ and $a$ have no common pole,
(ii) $f - a$ and $f^{(k)} - a$ share the value 0 CM,
(iii) $4\delta(0; f) + 2(8 + k)\Theta(\infty; f) > 19 + 2k$,

then $f \equiv f^{(k)}$, where $k$ is a positive integer.


Theorem G ([8]). Let $f$ be a nonconstant meromorphic function and $a = a(z)$ ($\not\equiv 0, \infty$) a small function of $f$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM, $f^{(k)}$ and $a = a(z)$ do not have any common pole of the same multiplicity and $2\delta(0; f) + 4\Theta(\infty; f) > 5$, then $f \equiv f^{(k)}$, where $k$ is a positive integer.

Al-Khaladi [3] observed by considering $f(z) = 1 + \exp(e^z)$ and $a(z) = e^z/(e^z - 1)$ that in Theorem A it is not possible to replace the value 1 by a small function. Instead, he proved the following result.
Theorem H ([3]). Let \( f \) be a nonconstant entire function satisfying \( N(r, 0; f') = S(r, f) \) and let \( a = a(z) (\neq 0, \infty) \) be a small function of \( f \). If \( f - a \) and \( f' - a \) share 0 CM, then \( f - a = (1 + c/a)(f' - a) \), where \( 1 + c/a = e^\beta \), \( c \) is a constant and \( \beta \) is an entire function.

In 2005 Al-Khaladi [2] considered the general order derivative of an entire function and proved the following result.

Theorem I ([2]). Let \( f \) be a nonconstant entire function satisfying \( N(r, 0; f^{(k)}) = S(r, f) \) and let \( a = a(z) (\neq 0, \infty) \) be a small function of \( f \). If \( f - a \) and \( f^{(k)} - a \) share 0 CM, then \( f - a = (1 + P_{k-1}/a)(f^{(k)} - a) \), where \( 1 + P_{k-1}/a = e^\beta \), \( P_{k-1} \) is a polynomial of degree at most \( k - 1 \) and \( \beta \) is an entire function.

Recently Al-Khaladi [1] extended Theorem I to meromorphic functions and proved the following theorem.

Theorem J ([1]). Let \( f \) be a nonconstant meromorphic function and let \( a = a(z) (\neq 0, \infty) \) be a small function of \( f \). If \( f - a \) and \( f^{(k)} - a \) share 0 CM and
\[
(k + 1)N(r, \infty; f) + (k + 1)N(r, 0; f^{(k)}) < \lambda T(r, f^{(k)}) + S(r, f^{(k)})
\]
for some constant \( \lambda \in (0, 1) \), then \( f - a = (1 + P_{k-1}/a)(f^{(k)} - a) \), where \( P_{k-1} \) is a polynomial of degree at most \( k - 1 \) and \( 1 + P_{k-1}/a \neq 0 \).

For a nonconstant meromorphic function \( f \) we denote by \( L = L(f) \) a linear differential polynomial of the form
\[
L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)},
\]
where \( a_1, a_2, \ldots, a_k (\neq 0) \) are constants.

In the paper we prove the following theorem, which involves the sharing of a small function by \( f \) and \( L \).

**Theorem 1.1.** Let \( f \) be a nonconstant meromorphic function such that \( L \) is nonconstant. Suppose that \( a = a(z) (\neq 0, \infty) \) is a small function of \( f \). If \( f - a \) and \( L - a \) share 0 CM and
\[
(k + 1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N_k(r, 0; f') < \lambda T(r, f') + S(r, f')
\]
for some real constant \( \lambda \in (0, 1) \), then \( f - a = (1 + c/a)(L - a) \), where \( c \) is a constant and \( 1 + c/a \neq 0 \).
2. Lemmas

In this section we present some necessary lemmas.

**Lemma 2.1** ([6], page 55, Theorem 3.1). Let \( f \) be a nonconstant meromorphic function. Then

\[
T(r, L) \leq (k + 1)T(r, f') + S(r, f).
\]

**Lemma 2.2.** Let \( f \) be a nonconstant meromorphic function such that \( L \) is nonconstant. Suppose that \( a = a(z) (\neq 0, \infty) \) is a small function of \( f \). If \( f - a \) and \( L - a \) share \( 0 \) IM, then

\[
T(r, f) \leq \left( \frac{1}{k + 1} + k + 2 \right) T(r, L) + S(r, f) \leq \{ (k + 1)(k + 2) + 1 \} T(r, f') + S(r, f).
\]

**Proof.** By Milloux’s basic result [6], page 57, Theorem 3.2, we get

\[
T(r, f) \leq \overline{N}(r, \infty; f) + N(r, 0; f) + \overline{N}(r, 1; L) - N_0(r, 0; L') + S(r, f),
\]

where \( N_0(r, 0; L') \) is the counting function of those zeros of \( L' \) which are not the 1-points of \( L \).

Now \( N(r, 0; f) - N_0(r, 0; L') \leq (k + 1)\overline{N}(r, 0; f) \) and \( (k + 1)\overline{N}(r, \infty; f) \leq N(r, \infty; L) \leq T(r, L) \). Therefore

\[
(2.1) \quad T(r, f) \leq T(r, L) + \overline{N}(r, 1; L) + (k + 1)\overline{N}(r, 0; f) + S(r, f)
\]

\[
\leq \left( \frac{1}{k + 1} + 1 \right) T(r, L) + (k + 1)\overline{N}(r, 0; f) + S(r, f).
\]

Since \( L(f - a) = L(f) - \sum_{j=1}^{k} a_j a^{(j)} \), we have \( T(r, L(f - a)) = T(r, L) + S(r, f) \).

Now replacing \( f \) by \( f - a \) in (2.1) and noting that \( f - a \) and \( L - a \) share 0 IM we get

\[
T(r, f - a) \leq \left( \frac{1}{k + 1} + 1 \right) T(r, L) + (k + 1)\overline{N}(r, 0; f - a) + S(r, f)
\]

and so

\[
(2.2) \quad T(r, f) \leq \left( \frac{1}{k + 1} + k + 2 \right) T(r, L) + S(r, f).
\]

By Lemma 2.1 we get

\[
(2.3) \quad T(r, L) \leq (k + 1)T(r, f') + S(r, f).
\]

Now the lemma follows from (2.2) and (2.3). \( \square \)
Lemma 2.3 ([6], page 47, Theorem 2.5). Let \( f \) be a nonconstant meromorphic function and \( a_1, a_2, a_3 \) three distinct small functions of \( f \). Then

\[
T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).
\]

Lemma 2.4 ([7]). Let \( f \) be a nonconstant meromorphic function and \( k \) a positive integer. If \( f \) and \( f^{(k)} \) share 1 IM and \( f^{(k)} = (Af + B)/(Cf + D) \), where \( A, B, C, D \) are constants, then \( (f^{(k)} - 1)/(f - 1) \) is a nonzero constant.

3. Proof of Theorem 1.1

Proof. Let \( h = (f - a)/(L - a) \). Then \( f - a = h(L - a) \) and differentiating we get

\[
f' - a' = (hL)' - (ha)'.
\]

We now consider the following cases.

Case I: Let \( a' \neq 0 \). We put

\[
W = \frac{(hL)'}{hf'} - \frac{(ha)'}{ha'}.
\]

If \( z_0 \) is a zero of \( f' - a' \) with \( a'(z_0) \neq 0, \infty \), then we get from (3.1) that \( W(z_0) = 0 \). Let \( W \neq 0 \). Then

\[
\overline{N}(r, 0; f' - a') \leq \overline{N}(r, 0; W) + S(r, f) \leq T(r, W) + S(r, f) = \overline{N}(r, W) + m(r, W) + S(r, f) = \overline{N}(r, W) + S(r, f).
\]

From (3.2) we get

\[
W = \frac{(hL)'}{hL} \cdot \frac{L}{f'} + \frac{(ha)'}{ha} \cdot \frac{a}{a'}.
\]

Let \( z_1 \) be a pole of \( f \) with multiplicity \( p \) such that \( a(z_1) \neq 0, \infty \) and \( a'(z_1) \neq 0 \). Then \( z_1 \) is a pole of \( hL \) with multiplicity \( p \) and a pole of \( L/f' \) with multiplicity \( k - 1 \). Hence \( z_1 \) is a pole of \( W \) with multiplicity at most \( k \).

Let \( z_2 \) be a zero of \( f' \) with multiplicity \( q \) such that \( a(z_2) \neq 0, \infty \) and \( a'(z_2) \neq 0 \). If \( q \leq k - 1 \) and \( L(z_2) \neq 0 \), then \( z_2 \) is a pole of \( (hL)'/(hL) \cdot L/f' \) with multiplicity \( q \leq k - 1 \). Also, if \( q \leq k - 1 \) and \( z_2 \) is a zero of \( L \) with multiplicity \( t \ (\geq 1) \), then \( z_2 \) is a pole of \( (hL)'/(hL) \cdot L/f' \) with multiplicity \( q - (t - 1) \leq q \leq k - 1 \).
If \( q \geq k \), then \( z_2 \) is a pole of \( L/f' \) with multiplicity \( k - 1 \) and a pole of \((hL)'/(hL)\) with multiplicity 1. Hence \( z_2 \) is a pole of \((hL)'/(hL) \cdot L/f' \) with multiplicity \( k \).

Therefore from (3.4) we get

\[
N(r, W) \leq k\overline{N}(r, \infty; f) + N_k(r, 0; f') + S(r, f).
\]

From (3.3) and (3.5) we obtain

\[
N(r, 0; f' - a') \leq k\overline{N}(r, \infty; f) + N_k(r, 0; f') + S(r, f).
\]

Since by Lemma 2.1 and Lemma 2.2, \( a' = a'(z) \) is a small function of \( f' \) and \( S(r, f) \) is interchangeable with \( S(r, f') \), we get by Lemma 2.3 and (3.6)

\[
T(r, f') \leq \overline{N}(r, 0; f' - a') + \overline{N}(r, 0; f') + \overline{N}(r, \infty; f') + S(r, f') \leq (k + 1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N_k(r, 0; f') + S(r, f'),
\]

which contradicts the hypothesis.

Therefore \( W \equiv 0 \) and so by (3.1) and (3.2) we get \( (f' - a')(ha)' = (f' - a')a' \)

Since \( f' \not\equiv a' \), we have \( (ha)' = a' \) and so \( ha = a + c \), where \( c \) is a constant. Hence

\[
f - a = h(L - a) = \left(1 + \frac{c}{a}\right)(L - a),
\]

where \( 1 + c/a \not\equiv 0 \).

Case II: Let \( a' \equiv 0 \) so that \( a \) is a constant. We now consider the following subcases.

Subcase (i): Let \( k \geq 2 \). From (3.1) we get

\[
f' = (hL)' - ah' = h\left\{\frac{(hL)'}{h} - a\frac{h'}{h}\right\}
\]

and so

\[
\frac{1}{h} = \frac{(hL)'}{hf'} - a\frac{h'}{h} \cdot \frac{1}{f'}.
\]

We put \( F = f' \), \( G = (hL)'/(hf') \) and \( b = ah'/h \). Then

\[
\frac{1}{h} = G - \frac{b}{F}.
\]

Differentiating (3.7) we obtain

\[
-\frac{1}{h} \frac{h'}{h} = G' - \frac{b'}{F} + \frac{b}{F} \frac{F'}{F'}.
\]
Eliminating $1/h$ from (3.7) and (3.8) we get

\begin{equation}
\frac{A}{F} = G' + \frac{G h'}{h},
\end{equation}

where $A = b \cdot h' + b' - b \cdot F'/F$.

First we suppose that $G \equiv 0$. Then $hL = d$, a nonzero constant. Putting $h = (f - a)/(L - a)$ we have $L(f - a) = d(L - a)$. This implies that $f$ is an entire function. Therefore, $h$ is an entire function having no zero. We now put $h = e^\alpha$, where $\alpha$ is an entire function.

Now $f = a + h(L - a) = a + d - ae^\alpha$ and $L = de^{-\alpha}$. Also we see that $L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)} = P(\alpha')e^\alpha$, where $P(\alpha')$ is a differential polynomial in $\alpha'$. Therefore $P(\alpha')e^\alpha = de^{-\alpha}$ and so $P(\alpha')e^{2\alpha} = d$. This implies $2T(r, e^\alpha) = T(r, P(\alpha')) = S(r, e^\alpha)$, a contradiction. Hence $G \not\equiv 0$.

Next we suppose that $A \equiv 0$. Then from (3.9) we get $G'/G + h'/h = 0$. Integrating we obtain $Gh = K$, where $K$ is a nonzero constant. Hence $(hL)' = kf'$ and again integration yields $hL = Kf + M$, where $M$ is a constant. Since $f - a = hL - ah$, we get

\begin{equation}
(1 - K)f = a(1 - h) + M.
\end{equation}

If $K = 1$, from (3.10) we see that $h$ is a constant. Hence $f - a = (1 + c/a)(L - a)$, where we put $h = 1 + c/a$ for some constant $c$ such that $1 + c/a \neq 0$.

Let $K \neq 1$. Then from (3.10) we see that $h$ is nonconstant. Since $h$ is entire, (3.10) implies that $f$ is also entire. Therefore $h = (f - a)/(L - a)$ has no zero. So we can put $h = e^\beta$, where $\beta$ is an entire function. Hence from (3.10) we get

$$f = \frac{a + M}{1 - K} - \frac{ae^\beta}{1 - K}$$

and so

\begin{equation}
L = K \frac{f}{h} + \frac{M}{h} = \frac{Ka + M}{1 - K}e^{-\beta} - \frac{a}{1 - K}.
\end{equation}

Also

\begin{equation}
L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)} = Q(\beta')e^\beta,
\end{equation}

where $Q(\beta')$ is a differential polynomial in $\beta'$.

Since $L$ is nonconstant, we see that $Ka + M \neq 0$. Hence from (3.11) and (3.12) we get

$$Q(\beta')e^{2\beta} = \frac{Ka + M}{1 - K} - \frac{a}{1 - K}e^\beta.$$
This implies by the first fundamental theorem

\[ 2T(r, e^\beta) \leq T(r, e^\beta) + T(r, Q(\beta')) + O(1) = T(r, e^\beta) + S(r, e^\beta), \]
a contradiction.

Finally we suppose that \( A \not\equiv 0 \). Now \( m(r, A) \leq 2m(r, b) + m(r, b') + m(r, h'/h) + m(r, F'/F) = S(r, f) \). Since \( A = a(h'/h)^2 + a(h'/h)' - h'/h \cdot F'/F \), we see that \( N(r, \infty; A) \leq 2N(r, \infty; f) + \mathcal{N}(r, 0; f') \). Hence

\[ (3.13) \quad T(r, A) \leq 2\mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; f') + S(r, f). \]

Now from (3.9) and (3.13) we get

\[ (3.14) \quad m\left( r, \frac{1}{F} \right) \leq m\left( r, \frac{1}{A} \right) + m\left( r, G' + G \frac{h'}{h} \right) \leq T(r, A) + S(r, f) \leq 2\mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; f') + S(r, f). \]

Since \( A \not\equiv 0 \), it is clear that \( b \not\equiv 0 \). Let \( z_3 \) be a zero of \( F \) with multiplicity \( q \) \((\geq k+1)\). Then \( z_3 \) is a zero of \( b = af'/\left(f-a\right) - aL'/\left(L-a\right) \) with multiplicity at least \( q-k \). Hence

\[ N_{k+1}(r, \frac{1}{F}) - k\mathcal{N}_{k+1}(r, \frac{1}{F}) \leq N(r, 0; b) \]

and so

\[ N_{k+1}(r, \frac{1}{F}) \leq k\mathcal{N}_{k+1}(r, \frac{1}{F}) + N(r, 0; b) \]
\[ \leq k\mathcal{N}_{k+1}(r, \frac{1}{F}) + T(r, b) + O(1) \]
\[ = k\mathcal{N}_{k+1}(r, \frac{1}{F}) + N(r, b) + S(r, f) \]
\[ \leq k\mathcal{N}_{k+1}(r, \frac{1}{F}) + \mathcal{N}(r, \infty; f) + S(r, f). \]

So

\[ (3.15) \quad N\left( r, \frac{1}{F} \right) = N_k\left( r, \frac{1}{F} \right) + N_{k+1}\left( r, \frac{1}{F} \right) \leq N_k(r, 0; f') + \mathcal{N}(r, \infty; f) + S(r, f). \]

Adding (3.14) and (3.15) and using the first fundamental theorem we get

\[ T(r, f') \leq 3\mathcal{N}(r, \infty; f) + N_k(r, 0; f') + \mathcal{N}(r, 0; f') + S(r, f), \]

which is a contradiction with the hypothesis for \( k \geq 2 \).
Subcase (ii): Let $k = 1$. We put $g = f/a$ and $R = L/a$. Then $g$ and $R$ share the value 1 CM. Let

$$H = \left( \frac{g''}{g'} - \frac{2g'}{g-1} \right) - \left( \frac{R''}{R'} - \frac{2R'}{R-1} \right).$$

We first suppose that $H \not\equiv 0$. Since $g$ and $R$ share 1 CM, we get

$$N(r, H) = \overline{N}(r, H) \leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f') - \overline{N}(r, a; f) + \overline{N}_*(r, 0; f^{(2)}),$$

where $\overline{N}_*(r, 0; f^{(2)})$ denotes the reduced counting function of those zeros of $f^{(2)}$ which are not the zeros of $(f - a)f'$.

Since $g$ and $R$ share the value 1 CM, it is easy to see that

$$N_1(r, a; f) = N_1(r, 1; g) \leq N(r, 0; H) \leq T(r, H) + O(1) = N(r, H) + S(r, f)$$

and so

$$N_1(r, a; f) = N_1(r, 1; g) \leq N(r, \infty; f) + \overline{N}(r, 0; f') - \overline{N}(r, a; f) + \overline{N}_*(r, 0; f^{(2)}) + S(r, f)$$

and so

$$N(r, H) = \overline{N}(r, H) \leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f') - \overline{N}(r, a; f) + \overline{N}_*(r, 0; f^{(2)}) + S(r, f).$$

Now by the second fundamental theorem and (3.16) we get in view of the fact that $L - a$ and $f - a$ share 0 CM:

$$T(r, f') = T(r, L) + O(1)$$

and

$$= \overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + \overline{N}(r, a; f) - \overline{N}_*(r, 0; f^{(2)}) + S(r, f')$$

a contradiction with the hypothesis.

Therefore $H \equiv 0$ and so integration yields $R = (Ag + B)/(Cg + D)$, where $A$, $B$, $C$, $D$ are constants. Hence by Lemma 2.4 we get $(g - 1)/(R - 1)$ is a nonzero constant. So we can put $f - a = (1+c/a)(L-a)$, where $c$ is a constant and $1+c/a \neq 0$. This proves the theorem. \(\square\)
References


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