ON HENSTOCK-KURZWEIL METHOD TO STRATONOVICH INTEGRAL

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Abstract. We use the general Riemann approach to define the Stratonovich integral with respect to Brownian motion. Our new definition of Stratonovich integral encompass the classical Stratonovich integral and more importantly, satisfies the ideal Itô formula without the "tail" term, that is,

$$f(W_t) = f(W_0) + \int_0^t f'(W_s) \circ \mathrm{d}W_s.$$

Further, the condition on the integrands in this paper is weaker than the classical one.

Keywords: Itô formula; Henstock-Kurzweil approach; Stratonovich integral

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1. Introduction

Since the 1950s, Henstock and Kurzweil independently introduced a Riemann-type definition of integrals using non-uniform meshes (where meshes vary from point to point) [5], [6]. This approach makes it possible to give an alternative definition of the Itô integral with respect to Brownian motion using Riemann sums [7], [9], [13], [14], [16], [17]. Protter [13] and Toh [1], [15] used this Riemann approach to define the stochastic integral with respect to a semimartingale. This new approach turned out to encompass the classical stochastic integral.

Based on the success of using the Henstock-Kurzweil method in defining the Itô integral, we believe that we also can define the Stratonovich integral using this method as well.

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Gradinaru and Nourdin [3], [10] showed that the classical Stratonovich formula holds with respect to fractional Brownian motion and converges in law in the Skorohod space of the right continuous functions with left limit. One approach of defining the Stratonovich integral is to define it as the sum of an Itô integral and an additional term (see [4], [12]). We tap on this definition to consider a new way to define the Stratonovich integral using the Henstock-Kurzweil approach. We will also show that the two integrals, that is, the classical Stratonovich integral and the Henstock-Kurzweil Stratonovich integral, agree.

To align with the classical definition of the Stratonovich integral with Brownian motion, we consider the case where the integrands are continuous.

2. Classical Stratonovich integral

In this section, we present the definition of the Brownian motion and of its properties, and the definition of the classical Stratonovich integral.

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A one-dimensional *standard Brownian motion* is a real-valued process $W_t = \{W(\omega, t),\ t \geqslant 0,\ \omega \in \Omega\}$ that has the following properties:

(a) If $t_0 < t_1 < \ldots < t_n$, then $W_{t_0},\ W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.
(b) If $s, t \geqslant 0$ then $P(W_{s+t} - W_s \in A) = \int_A (2\pi t)^{-1/2} \exp(-x^2/2t)\,\mathrm{d}x$.
(c) With probability one, $t \to W_t$ is continuous and $W_0 = 0$.

Due to (a) and (b), W_t has independent increment and the increment $W_{s+t} - W_s$ has a normal distribution with mean 0 and variance t. It is not difficult to verify that $E(W_t) = 0$ and $E[W_t W_s] = \min\{s, t\} = s \wedge t$ and $E(W_t - W_s)^4 = 3(t-s)^2$ (see, for example [2], page 302).

Let $C^2(\mathbb{R})$ denote the class of functions which have continuous second derivatives.

Definition 2.2 (see [12], page 75, and [4], page 156)**.** If $f \in C^2(\mathbb{R})$, then the *classical Stratonovich integral* (henceforth, Stratonovich integral) of $f(W_t)$ with respect to the standard Brownian motion W_t is

$$\int_0^t f(W_s) \circ \mathrm{d}W_s \triangleq \int_0^t f(W_s)\,\mathrm{d}W_s + \frac{1}{2}\int_0^t f'(W_s)\,\mathrm{d}s \tag{2.1}$$

where the first integral on the right-hand side of (2.1) is the classical Itô integral and the other one is the Lebesgue integral.

We know that if $f \in C^2(\mathbb{R})$ and W is a Brownian motion, then $f(W_t)$ is a semimartingale (see [4], page 149). This is the setting for which the classical Stratonovich integral is defined.

R e m a r k. For any fixed $\omega \in \Omega$ and $t > 0$, W_t is bounded for $0 \leqslant s \leqslant t$, so $f(W_s)$ is bounded on this interval. By Itô integral definition (see [4], page 146), $\int_0^t f(W_s)\,\mathrm{d}W_s$ is a continuous local martingale. Also, $\int_0^t f'(W_t)\,\mathrm{d}s$ is well-defined since $f'(W_t)$ is continuous. Therefore, the condition on $f \in C^2$ guarantees that $\int_0^t f(W_t) \circ \mathrm{d}W_s$ is well-defined. In other words, for $f \in C^2$ the Stratonovich integral of $f(W_t)$ with respect to the standard Brownian motion W_t exists.

3. Henstock-Kurzweil approach to define Stratonovich integral with respect to Brownian motion

In this section, we still keep the δ-fine division introduced in [9], page 52, and [15], to define the Stratonovich integral. To be consistent with the result of the Stratonovich integral, we shall modify the definition of the Itô-McShane integral in [9], page 52.

Definition 3.1 (see [14], [15]). Let $0 = \xi_1 < \ldots < \xi_n < \xi_{n+1} = 1$ and let I_k be a compact subinterval of $[0,1]$ for $k = 1, \ldots, n$. Let $\delta\colon [0,1] \to \mathbb{R}^+$. A finite collection of interval-point pairs $D = (I_k, \xi_k)_{k=1}^n$ is called a (δ, η)-*fine belated partial division* of $[0,1]$ if I_k, $k = 1, \ldots, n$, are non-overlapping subintervals of T; and each $I_k \subset [\xi_k, \xi_k + \delta(\xi_k))$; and $\left|[0,1] \setminus \bigcup\limits_{k=1}^n I_k\right| < \eta$.

Given $\delta(\xi) > 0$, unlike in the case of a δ-fine full division of $[0,1]$, a δ-fine belated full division of T may not exist. Given an example, $\delta(\xi) = (1-\xi)/2$ on $[0,1]$, the right hand side may not be covered. By the Vitali covering theorem for all $\eta > 0$, there exists a δ-fine belated partial division $D = \{(I_k, \xi_k)\}_{k=1}^n$ (see [9], page 52, and [15]) such that $\left|\bigcup\limits_{k=1}^n I_k - [0,1]\right| < \eta$ for sufficiently large k, see [9], page 52. In other words, we may not have a δ-fine full division of $[0,1]$, but we can have a (δ, η)-fine belated partial division which covers $[0,1]$ except for a set of arbitrarily small measure η.

Definition 3.2. An adapted process Y_t in $(\Omega, \mathcal{F}_t, \mathbb{P})$ is said to be *Stratonovich-Henstock belated* (denoted by SHB) integrable with respect to the Brownian motion W_t on $[0,1]$ to a random variable $A \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, if for all $\varepsilon > 0$ there exists a (δ, η)-fine belated partial division $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \ldots, n\}$ of $[0,1]$ such that

$$E\left|\sum_{i=1}^n \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A\right|^2 < \varepsilon$$

where and henceforth $\xi_{n+1} = 1$.

Proposition 3.3. *If an adapted process Y_t in $(\Omega, \mathcal{F}_t, \mathbb{P})$ is* SHB *integrable with respect to the Brownian motion W_t, then the Stratonovich-Henstock belated integral of Y_t is unique almost surely.*

Proof. Suppose both $A_1, A_2 \in L^2(\Omega)$ are the Stratonovich-Henstock belated integral of Y_t. By the definition of the Stratonovich-Henstock belated integral, for every $\varepsilon > 0$ there exists a (δ, η)-fine belated partial division $D_1 = \{([\xi_i, t_i], \xi_i) \colon i = 1, 2, \ldots, n\}$ of $[0, 1]$ such that

$$E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A_1\right|^2 < \frac{1}{4}\varepsilon.$$

And for the same $\varepsilon > 0$ there exists another (δ, η)-fine belated partial division $D_2 = \{([\xi_i', t_i'], \xi_i') \colon i = 1, 2, \ldots, n\}$ of $[0, 1]$ such that

$$E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'}) - A_2\right|^2 < \frac{1}{4}\varepsilon.$$

Hence, for every $\varepsilon > 0$,

$$\begin{aligned} E(|A_1 - A_2|^2) &= E\left|A_1 - \sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i})\right. \\ &\quad \left. + \sum_{i=1}^{n} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'}) - A_2\right|^2 \\ &\leqslant 2E\left|A_1 - \sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i})\right|^2 \\ &\quad + 2E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'}) - A_2\right|^2 < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have $E(|A_1 - A_2|^2) = 0$. Hence, $A_1 - A_2 = 0$, a.s. □

In view of Proposition 3.3, we will denote the integral of the process Y_t with respect to a Brownian motion W_t by (SH) $\int_0^1 Y_t \circ \mathrm{d}W_t \triangleq A$.

Proposition 3.4. *An adapted process Y_t in $(\Omega, \mathcal{F}_t, \mathbb{P})$ is* SHB *integrable with respect to a Brownian motion W_t if and only if for every $\varepsilon > 0$ there exists a positive function δ and a constant $\eta > 0$ such that whenever both $D = \{([\xi_i, t_i], \xi_i) \colon i = 1, 2, \ldots, n\}$ and $D' = \{([\xi_i', t_i'], \xi_i') \colon i = 1, 2, \ldots, n\}$ are (δ, η)-fine belated partial divisions of $[0, 1]$, then*

$$E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - \sum_{i=1}^{n} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'})\right|^2 < \varepsilon.$$

P r o o f. Suppose that the adapted process Y_t in $(\Omega, \mathcal{F}_t, \mathbb{P})$ is SHB integrable with respect to a Brownian motion W_t to A. Then for every $\varepsilon > 0$ there exists a positive function δ and a constant $\eta > 0$ such that for any (δ, η)-fine belated partial division $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \ldots, n\}$ of $[0, 1]$ we have

$$E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A\right|^2 < \frac{\varepsilon}{4}. \tag{3.1}$$

Let $D' = \{([\xi_i', t_i'], \xi_i')\colon\ i = 1, 2, \ldots, n\}$ be another (δ, η)-fine belated partial division of $[0, 1]$. Then we have

$$E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'}) - A\right|^2 < \frac{\varepsilon}{4}. \tag{3.2}$$

From (3.1) and (3.2) we have

$$\begin{aligned} E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - \sum_{i=1}^{n} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'})\right|^2 \\ < 2E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A\right|^2 \\ + 2E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'}) - A\right|^2 < \varepsilon. \end{aligned}$$

Conversely, for every $\varepsilon > 0$ there is a (δ, η)-fine belated partial division $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \ldots, n\}$ of $[0, 1]$ such that for any (δ, η)-fine belated partial division $D^j = \{([\xi_i^j, t_i^j], \xi_i^j)\colon\ i = 1, 2, \ldots, n\};\ j = 1, 2, \ldots$, of $[0, 1]$ we have

$$E\left|\sum_{i=1}^{n} \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - \sum_{i=1}^{n} \frac{Y_{\xi_i^j} + Y_{\xi_{i+1}^j}}{2}(W_{t_i^j} - W_{\xi_i^j})\right|^2 < \varepsilon.$$

For brevity, we may write $\|D(Y) - D^j(Y)\|^2 < \varepsilon$. Hence, for all j we have

$$\begin{aligned} \|D(Y) - D^j(Y)\|^2 < \varepsilon \Rightarrow 2\|D(Y) - D^j(Y)\|^2 + 2\|D(Y)\|^2 < \varepsilon + 2\|D(Y)\|^2 \\ \Rightarrow \|D^j(Y)\|^2 < \varepsilon + 2\|D(Y)\|^2. \end{aligned}$$

Therefore, the sequence $\{\|D^j(Y)\|^2\}_{j=1}^{\infty}$ is bounded, i.e., the upper limit exists. Let a random variable A be the upper limit of the sequence $\{\|D^j(Y)\|^2\}_{j=1}^{\infty}$. In other words, there exists a subsequence $\{\|D^{j_k}(Y)\|^2\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} \|D^{j_k}(Y)\| = A$.

Hence, for the same $\varepsilon > 0$ exists $k' \in \mathbb{N}$, such that $\|D^{j_{k'}}(Y) - A\| < \varepsilon$. Now, whenever D^j is a (δ, ε)-fine belated partial division then

$$\begin{aligned}\|D^j(Y) - A\|^2 &< \|D^j(Y) - D^{j_{k'}}(Y) + D^{j_{k'}}(Y) - A\|^2 \\ &< \|D^j(Y) - D^{j_{k'}}(Y)\|^2 + \|D^{j_{k'}}(Y) - A\|^2 < 2\varepsilon.\end{aligned}$$

That is, Y_t is SHB integrable. □

Proposition 3.5. *Let X_t and Y_t be adapted processes on $[0,1]$ which are* SHB *integrable with respect to W_t, and $\alpha \in \mathbb{R}$. Then $X_t + Y_t$, αX_t are* SHB *integrable with respect to W_t on $[0,1]$. Furthermore,*

(i) (SH) $\int_0^1 (X_t + Y_t) \circ \mathrm{d}W_t = (\mathrm{SH}) \int_0^1 X_t \circ \mathrm{d}W_t + (\mathrm{SH}) \int_0^1 Y_t \circ \mathrm{d}W_t$,
(ii) (SH) $\int_0^1 \alpha X_t \circ \mathrm{d}W_t = \alpha(\mathrm{SH}) \int_0^1 X_t \circ \mathrm{d}W_t$.

Proof. (i) Let (SH) $\int_0^1 X_t \circ \mathrm{d}W_t = A_1$, (SH) $\int_0^1 Y_t \circ \mathrm{d}W_t = A_2$.

By the definition of the SHB integral, for every $\varepsilon > 0$ there is a (δ_1, η_1)-fine belated partial division $D_1 = \{([\xi_i', t_i'], \xi_i')\colon\ i = 1, 2, \ldots, n\}$ of $[0,1]$ such that

$$E\Bigl|\sum_{i=1}^n \frac{X_{\xi_i'} + X_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'}) - A_1\Bigr|^2 < \frac{\varepsilon}{4}.$$

Similarly, for the same $\varepsilon > 0$, there is a (δ_2, η_2)-fine belated partial division $D_2 = \{([\xi_i'', t_i''], \xi_i'')\colon\ i = 1, 2, \ldots, n\}$ of $[0,1]$ such that

$$E\Bigl|\sum_{i=1}^n \frac{Y_{\xi_i''} + Y''_{\xi_{i+1}}}{2}(W_{t_i''} - W_{\xi_i''}) - A_2\Bigr|^2 < \frac{\varepsilon}{4}.$$

Now let $D \triangleq D_1 \cup D_2$, i.e., the division points and associated points of D are the union of those from D_1 and D_2. Then $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \ldots, n\}$ is a (δ, η)-fine belated partial division including the (δ, η)-fine belated partial division D_1 and the (δ, η)-fine belated partial division D_2. Hence, for the same $\varepsilon > 0$,

$$\begin{aligned}E\Bigl|\sum_{i=1}^n \Bigl[\frac{X_{\xi_i} + X_{\xi_{i+1}}}{2} + \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}\Bigr](W_{t_i} - W_{\xi_i}) - A_1 - A_2\Bigr|^2 \\ \leqslant 2E\Bigl|\sum_{i=1}^n \frac{X_{\xi_i} + X_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A_1\Bigr|^2 \\ + 2E\Bigl|\sum_{i=1}^n \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A_2\Bigr|^2 < \varepsilon.\end{aligned}$$

In other words,

$$\text{(3.3)}\quad \text{(SH)}\int_0^1 (X_t+Y_t)\circ \mathrm{d}W_t = A_1 + A_2 = \text{(SH)}\int_0^1 X_t\circ \mathrm{d}W_t + \text{(SH)}\int_0^1 Y_t\circ \mathrm{d}W_t.$$

(ii) Assume that $\alpha \neq 0$ (since $\alpha = 0$ is trivial). Since the adapted process X_t on $[0,1]$ is SHB integrable, then by the definition, for every $\varepsilon > 0$ there is a (δ_3, η_3)-fine belated partial division $D_3 = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \ldots, n\}$ of $[0,1]$ such that

$$\text{(3.4)}\qquad E\Bigl|\sum_{i=1}^n \frac{X_{\xi_i} + X_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A_1\Bigr|^2 < \frac{\varepsilon}{\alpha^2}.$$

Then,

$$\begin{aligned} E\Bigl|\sum_{i=1}^n &\frac{\alpha X_{\xi_i} + \alpha X_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - \alpha A_1\Bigr|^2 \\ &= \alpha^2 E\Bigl|\sum_{i=1}^n \frac{X_{\xi_i} + X_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - A_1\Bigr|^2 < \varepsilon. \end{aligned}$$

By the definition of the SHB integral, we have $\text{(SH)}\int_0^1 \alpha X_t \circ \mathrm{d}W_t = \alpha A_1 = \alpha\text{(SH)}\int_0^1 X_t \circ \mathrm{d}W_t$. □

Proposition 3.6. *If an adapted process Y_t is* SHB *integrable with respect to W_t on $[0,1]$, then so it is on any subinterval $[c,d]$ of $[0,1]$.*

P r o o f. Take any two $(\delta, \eta/2)$-fine belated partial divisions $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \ldots, n\}$ of $[c,d]$ and $D' = \{([\xi_i', t_i'], \xi_i')\colon\ i = 1, 2, \ldots, n\}$ of $[c,d]$ and denote by A_1 and A_2, respectively, the Riemann sums over D and D'. That is,

$$\text{(3.5)}\qquad A_1 = \sum_D \frac{Y_{\xi_i} + Y_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}),$$

$$\text{(3.6)}\qquad A_2 = \sum_{D'} \frac{Y_{\xi_i'} + Y_{\xi_{i+1}'}}{2}(W_{t_i'} - W_{\xi_i'}).$$

Similarly, take another $(\delta, \eta/2)$-fine belated partial division $D'' = \{([\xi_i'', t_i''], \xi_i'')\colon\ i = 1, 2, \ldots, n\}$ of $[0,1] \setminus [c,d]$ and denote by A_3 the corresponding Riemann sum, i.e., $A_2 = \sum\limits_{D''} (Y_{\xi_i''} + Y_{\xi_{i+1}''})(W_{t_i''} - W_{\xi_i''})/2$. Then the union $D \cup D''$ forms a (δ, η)-fine belated partial division of $[0,1]$. Here, the division points and the associated points of $D \cup D''$ are the union of those from D and D''. The Riemann sum of Y_t over $D \cup D''$ is $A_1 + A_3$, and likewise, that over $D' \cup D''$ is $A_2 + A_3$. By Proposition 3.4, we have $E|A_1 - A_2|^2 \leqslant E|A_1 + A_3 - (A_2 + A_3)|^2 < \varepsilon$ for all $\varepsilon > 0$. Hence, the process Y_t is SHB integrable on the interval $[c,d]$ by Proposition 3.4. □

Now, we say that $Y_t \cdot \chi_{[c,d]}$ is SHB integrable on $[0,1]$ if Y_t is SHB integrable on $[c,d] \subset [0,1]$, where $\chi_{[c,d]}$ is the characteristic function of $[c,d]$ (see [8]).

Proposition 3.7. *Let an adapted process Y_t be* SHB *integrable on $[0,a]$ and $[a,1]$ with respect to W_t. Then Y_t is* SHB *integrable on $[0,1]$ and furthermore,*

$$(\mathrm{SH})\int_0^1 Y_t \circ \mathrm{d}W_t = (\mathrm{SH})\int_0^a Y_t \circ \mathrm{d}W_t + (\mathrm{SH})\int_a^1 Y_t \circ \mathrm{d}W_t. \tag{3.7}$$

P r o o f. First, we have

$$(\mathrm{SH})\int_0^1 Y_t \cdot \chi_{[0,a]} \circ \mathrm{d}W_t = (\mathrm{SH})\int_0^a Y_t \circ \mathrm{d}W_t,$$
$$(\mathrm{SH})\int_0^1 Y_t \cdot \chi_{[a,1]} \circ \mathrm{d}W_t = (\mathrm{SH})\int_a^1 Y_t \circ \mathrm{d}W_t.$$

Then, by Proposition 3.5, we have

$$\begin{aligned}(\mathrm{SH})\int_0^1 Y_t \cdot \chi_{[0,a]} \circ \mathrm{d}W_t &+ (\mathrm{SH})\int_0^1 Y_t \cdot \chi_{[a,1]} \circ \mathrm{d}W_t \\ &= (\mathrm{SH})\int_0^1 (Y_t \cdot \chi_{[0,a]} + Y_t \cdot \chi_{[a,1]}) \circ \mathrm{d}W_t.\end{aligned}$$

Since, $Y_t \cdot \chi_{[0,a]} + Y_t \cdot \chi_{[a,1]} = Y_t \cdot \chi_{[0,1]}$, then we get

$$\begin{aligned}(\mathrm{SH})\int_0^1 Y_t \cdot \chi_{[0,1]} \circ \mathrm{d}W_t &= (\mathrm{SH})\int_0^1 Y_t \circ \mathrm{d}W_t \\ &= (\mathrm{SH})\int_0^a Y_t \circ \mathrm{d}W_t + (\mathrm{SH})\int_a^1 Y_t \circ \mathrm{d}W_t.\end{aligned}$$

□

4. Itô formula for SHB integral

As far as we know, one of the most important properties of the Stratonovich integral is that there is no second order term in the classical Stratonovich analogue of the Itô transformation formula (see [11], page 44). Therefore, in the following part we will verify the Itô formula for the SHB integral. Simply, we will prove the formula

$$f(W_t) = f(W_0) + (\mathrm{SH})\int_0^t f'(W_s) \circ \mathrm{d}W_s \tag{4.1}$$

where f is of class $C^2(\mathbb{R})$.

Lemma 4.1. *If $f(x) \in C^2(\mathbb{R})$ with bounded first and second derivatives, then the adapted process $f(W_t)$ is* SHB *integrable with respect to the Brownian motion to $\int_0^t f(W_s) \circ \mathrm{d}W_s$ on $[0,1]$.*

Proof. Let $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1,2,\dots,n\}$ be a belated partial division (see [15]) of $[0,1]$. By the mean value theorem, $(f(W_{\xi_i}) + f(W_{\xi_{i+1}}))/2 = f(W_{\xi_i}) + [f(W_{\xi_{i+1}}) - f(W_{\xi_i})]/2 = f(W_{\xi_i}) + \partial f/\partial x[(\widetilde{W_i})(W_{\xi_{i+1}} - W_{\xi_i})]/2$, where $\widetilde{W_i} \in [W_{\xi_i}, W_{\xi_{i+1}}]$. Since W_t is continuous with respect to t, there is a $\widehat{\xi_i} \in [\xi_i, \xi_{i+1}]$ such that $W_{\widehat{\xi_i}} = \widetilde{W_i}$ a.s. We have

$$
\begin{aligned}
E\bigg|\sum_{i=1}^n \frac{1}{2}(f(W_{\xi_i}) + f(W_{\xi_{i+1}}))(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \circ \mathrm{d}W_t\bigg|^2 \\
&= E\bigg|\sum_{i=1}^n \Big[f(W_{\xi_i}) + \frac{1}{2}\frac{\partial f(W_{\widehat{\xi_i}})}{\partial x}(W_{\xi_{i+1}} - W_{\xi_i})\Big](W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \circ \mathrm{d}W_t\bigg|^2 \\
&= E\bigg|\sum_{i=1}^n \Big[f(W_{\xi_i}) + \frac{1}{2}\frac{\partial f(W_{\widehat{\xi_i}})}{\partial x}(W_{\xi_{i+1}} - W_{\xi_i})\Big](W_{t_i} - W_{\xi_i}) \\
&\quad - \int_0^1 f(W_t)\,\mathrm{d}W_t - \frac{1}{2}\int_0^1 \frac{\partial f}{\partial x}(W_t)\,\mathrm{d}t\bigg|^2 \\
&\leqslant 2E\bigg|\sum_{i=1}^n f(W_{\xi_i})(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t)\,\mathrm{d}W_t\bigg|^2 \\
&\quad + E\bigg|\sum_{i=1}^n \frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i} - W_{\xi_i})(W_{\xi_{i+1}} - W_{\xi_i}) - \int_0^1 \frac{\partial f}{\partial x}(W_t)\,\mathrm{d}t\bigg|^2 = 2R_1 + R_2
\end{aligned}
$$

where

$$
R_1 = E\bigg|\sum_{i=1}^n f(W_{\xi_i})(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t)\,\mathrm{d}W_t\bigg|^2
$$

and

$$
R_2 = E\bigg|\sum_{i=1}^n \frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i} - W_{\xi_i})(W_{\xi_{i+1}} - W_{\xi_i}) - \int_0^1 \frac{\partial f}{\partial x}(W_t)\,\mathrm{d}t\bigg|^2.
$$

In the previous paper [15], we have already proved that for every $\varepsilon > 0$ there exists a (δ_1, η_1)-fine belated partial division $D_1 = \{([\xi_i, t_i], \xi_i)\colon\ i = 1,2,\dots,n\}$ of $[0,1]$ such that $R_1 < \varepsilon$. We now consider R_2:

$$
R_2 = E\bigg|\sum_{i=1}^n \frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i} - W_{\xi_i})(W_{\xi_{i+1}} - W_{\xi_i}) - \int_0^1 \frac{\partial f}{\partial x}(W_t)\,\mathrm{d}t\bigg|^2
$$

$$\leqslant 2E\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i}-W_{\xi_i})(W_{\xi_{i+1}}-W_{t_i})\right|^2$$

$$+\,2E\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i}-W_{\xi_i})(W_{t_i}-W_{\xi_i})-\int_0^1\frac{\partial f}{\partial x}(W_t)\,\mathrm{d}t\right|^2=2b_1+2b_2$$

where

$$b_1=E\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i}-W_{\xi_i})(W_{\xi_{i+1}}-W_{t_i})\right|^2$$

$$=\sum_{i=1}^{n}M\times E(W_{t_i}-W_{\xi_i})^2\times E(W_{\xi_{i+1}}-W_{t_i})^2,$$

$$b_2=E\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i}-W_{\xi_i})(W_{t_i}-W_{\xi_i})-\int_0^1\frac{\partial f}{\partial x}(W_t)\,\mathrm{d}t\right|^2$$

where $M\triangleq\sup\{f'(W_t)\colon\ 0\leqslant t\leqslant 1;\ \omega\in\Omega\}$. Since $E(W_{\xi_{i+1}}-W_{t_i})^2=\xi_{i+1}-t_i$ and $\left|[0,1]\setminus\bigcup_{i=1}^{n}[\xi_i,t_i]\right|<\eta$, we have $b_1<M\eta$. Further,

$$b_2=E\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i}-W_{\xi_i})(W_{t_i}-W_{\xi_i})-\int_0^1\frac{\partial f}{\partial x}(W_t)\,\mathrm{d}t\right|^2$$

$$\leqslant E\left|\sum_{i=1}^{n}\frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})(W_{t_i}-W_{\xi_i})^2-\frac{\partial f}{\partial x}(W_{\bar{\xi_i}})(W_{t_i}-W_{\xi_i})^2\right|^2+M\eta$$

$$\leqslant\left\|\frac{\partial f}{\partial x}(W_{\widehat{\xi_i}})-\frac{\partial f}{\partial x}(W_{\bar{\xi_i}})\right\|_\infty^2\sum_{i=1}^{n}E|W_{t_i}-W_{\xi_i}|^4+M\eta.$$

Since $f'(x)$ is bounded, $\|\partial f/\partial x(W_{\widehat{\xi_i}})-\partial f/\partial x(W_{\bar{\xi_i}})\|_\infty^2$ is also bounded by a constant $4M^2$. We choose a positive function $\delta<\varepsilon$, that is, $\max_i|t_i-\xi_i|<\varepsilon$. In addition, $E|W_{t_i}-W_{\xi_i}|^4=3(t_i-\xi_i)^2$. Then

$$b_2\leqslant 4M^2\sum_{i=1}^{n}C(t_i-\xi_i)^2+M\eta\leqslant 4M^2\varepsilon\sum_{i=1}^{n}C(t_i-\xi_i)+M\eta<C'\varepsilon+M\eta.$$

In particular, C' is a constant number. Hence, for the same $\varepsilon>0$, let $\delta=\min\{\delta_1,\varepsilon\}$ and $\eta=\eta_1<\varepsilon$. Whenever $D=\{([\xi_i,t_i],\xi_i)\}_n^{i=1}$ is a (δ,η)-fine belated partial division of $[0,1]$ then

$$E\left|\sum_{i=1}^{n}\frac{f(W_{\xi_i})+f(W_{\xi_{i+1}})}{2}(W_{t_i}-W_{\xi_i})-\int_0^1 f(W_t)\circ\mathrm{d}W_t\right|^2<R_1+2b_1+2b_2<\overline{M}\varepsilon$$

where $\overline{M}$ is a constant. That is, $f(X_t)$ is SHB integrable to $\int_0^t f(W_s)\circ\mathrm{d}W_s$ on $[0,1]$. $\square$

Theorem 4.2. *If $f(x) \in C^2(\mathbb{R})$ then the adapted process $f(W_t)$ is* SHB *integrable with respect to the Brownian motion to $\int_0^t f(W_s) \circ \mathrm{d}W_s$ on $[0,1]$.*

P r o o f. We first let $g(x) = f(x) \cdot 1_{\{|x| \leqslant m\}}$, where m is an integer. The function $g(x)$ has bounded first and second derivatives on the compact set $\{|x| \leqslant m\}$. By Lemma 4.1, we get for all $\varepsilon > 0$,

$$E\bigg|\sum_{i=1}^{n} \frac{g(W_{\xi_i}) + g(W_{\xi_{i+1}})}{2}(W_{t_i} - W_{\xi_i}) - \int_0^1 g(W_t) \circ \mathrm{d}W_t\bigg|^2 < \varepsilon \tag{4.2}$$

for every (δ, η)-fine belated partial division $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \ldots, n\}$ of $[0,1]$. Substituting $g(x)$ with $f(x) \cdot 1_{\{|x| \leqslant m\}}$ in (4.2), then

$$\begin{aligned}
&E\bigg|\sum_{i=1}^{n} \frac{f(W_{\xi_i}) + f(W_{\xi_{i+1}})}{2} \cdot 1_{\{|W_t| \leqslant m\}}(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \cdot 1_{\{|W_t| \leqslant m\}} \circ \mathrm{d}W_t\bigg|^2 \\
&\quad = E\bigg[\bigg|\sum_{i=1}^{n} \frac{f(W_{\xi_i}) + f(W_{\xi_{i+1}})}{2}(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \circ \mathrm{d}W_t\bigg|^2 \cdot 1_{\{|W_t| \leqslant m\}}\bigg] < \varepsilon.
\end{aligned}$$

Given that $\lim\limits_{m \to \infty} f(W_t) \cdot 1_{\{|W_t| \leqslant m\}} = 1$ a.s., we have

$$\begin{aligned}
&E\bigg[\bigg|\sum_{i=1}^{n} \frac{f(W_{\xi_i}) + f(W_{\xi_{i+1}})}{2}(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \circ \mathrm{d}W_t\bigg|^2 \cdot 1_{\{|W_t| \leqslant m\}}\bigg] < \varepsilon \\
&\Rightarrow \liminf_{m \to \infty} E\bigg[\bigg|\sum_{i=1}^{n} \frac{f(W_{\xi_i}) + f(W_{\xi_{i+1}})}{2}(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \circ \mathrm{d}W_t\bigg|^2 \cdot 1_{\{|W_t| \leqslant m\}}\bigg] < \varepsilon \\
&\Rightarrow E\bigg[\liminf_{m \to \infty}\bigg|\sum_{i=1}^{n} \frac{f(W_{\xi_i}) + f(W_{\xi_{i+1}})}{2}(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \circ \mathrm{d}W_t\bigg|^2 \cdot 1_{\{|W_t| \leqslant m\}}\bigg] < \varepsilon \\
&\Rightarrow E\bigg|\sum_{i=1}^{n} \frac{f(W_{\xi_i}) + f(W_{\xi_{i+1}})}{2}(W_{t_i} - W_{\xi_i}) - \int_0^1 f(W_t) \circ \mathrm{d}W_t\bigg|^2 < \varepsilon.
\end{aligned}$$

Altogether, $f(W_t)$ is SHB integrable on $[0,1]$. In addition,

$$(\mathrm{SH})\int_0^1 f(W_t) \circ \mathrm{d}W_t = \int_0^1 f(W_t) \circ \mathrm{d}W_t \quad \text{a.s.} \tag{4.3}$$

□

E x a m p l e 4.3. Using the definition of the SHB integral, verify

$$(\mathrm{SH})\int_0^1 W_t \circ \mathrm{d}W_t = \frac{1}{2}W_1^2.$$

P r o o f. Let $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \dots, n\}$ be a (δ, η)-fine belated partial division of $[0, 1]$.

$$\begin{aligned}
&E\Bigl|\sum_{i=1}^{n} \frac{W_{\xi_i} + W_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - \frac{1}{2}W_1^2\Bigr|^2 \\
&\quad = E\Bigl|\sum_{i=1}^{n} \frac{W_{\xi_i} + W_{\xi_{i+1}}}{2}(W_{\xi_{i+1}} - W_{\xi_i}) - \frac{1}{2}W_1^2 - \frac{W_{\xi_i} + W_{\xi_{i+1}}}{2}(W_{\xi_{i+1}} - W_{t_i})\Bigr|^2 \\
&\quad = E\Bigl|\sum_{i=1}^{n} \frac{W_{\xi_i} + W_{\xi_{i+1}}}{2}(W_{\xi_{i+1}} - W_{t_i}) - \frac{W_{t_i} - W_{\xi_i}}{2}(W_{\xi_{i+1}} - W_{t_i}) \\
&\qquad + \frac{W_{t_i} - W_{\xi_i}}{2}(W_{\xi_{i+1}} - W_{t_i})\Bigr|^2 \\
&\quad = E\Bigl|\sum_{i=1}^{n} W_{\xi_i}(W_{\xi_{i+1}} - W_{t_i}) + \frac{W_{\xi_{i+1}} - W_{t_i}}{2}(W_{\xi_{i+1}} - W_{t_i}) \\
&\qquad + \frac{W_{t_i} - W_{\xi_i}}{2}(W_{\xi_{i+1}} - W_{t_i})\Bigr|^2 \\
&\quad \leqslant 3E\Bigl|\sum_{i=1}^{n} W_{\xi_i}(W_{\xi_{i+1}} - W_{t_i})\Bigr|^2 + 3E\Bigl|\sum_{i=1}^{n} \frac{W_{\xi_{i+1}} - W_{t_i}}{2}(W_{\xi_{i+1}} - W_{t_i})\Bigr|^2 \\
&\qquad + 3E\Bigl|\sum_{i=1}^{n} \frac{W_{t_i} - W_{\xi_i}}{2}(W_{\xi_{i+1}} - W_{t_i})\Bigr|^2 \\
&\quad = 3\sum_{i=1}^{n} EW_{\xi_i}^2(W_{\xi_{i+1}} - W_{t_i})^2 + \frac{3}{2}\sum_{i=1}^{n} E|(W_{\xi_{i+1}} - W_{t_i})|^4 \\
&\qquad + \frac{3}{2}\sum_{i=1}^{n} E(W_{\xi_{i+1}} - W_{t_i})^2(W_{t_i} - W_{\xi_i})^2 \\
&\quad \leqslant 3\sum_{i=1}^{n}(\xi_{i+1} - t_i) + \frac{3}{2}\sum_{i=1}^{n} E(W_1)^4(\xi_i - t_i)^2 + \frac{3}{2}\sum_{i=1}^{n}(\xi_{i+1} - t_i)(t_i - \xi_i).
\end{aligned}$$

Since $D = \{([\xi_i, t_i], \xi_i)\colon\ i = 1, 2, \dots, n\}$ is a (δ, η)-fine belated partial division of $[0, 1]$, we have $\sum\limits_{i=1}^{n}(\xi_{i+1} - t_i) < \eta$ and in addition $\sum\limits_{i=1}^{n}(\xi_{i+1} - t_i)^2 < \eta$. Also, by the Itô formula, $E(W_1)^4 = 4 \cdot \frac{3}{2}\int_0^1 EW_t^2\,\mathrm{d}t = 3$.

Hence, for $\varepsilon > 0$, let $\eta < \varepsilon/9$ and thus

$$E\Bigl|\sum_{i=1}^{n} \frac{W_{\xi_i} + W_{\xi_{i+1}}}{2}(W_{t_i} - W_{\xi_i}) - \frac{1}{2}W_1^2\Bigr|^2 < 3\eta + \frac{9}{2}\eta + \frac{3}{2}\eta < \varepsilon.$$

That is, (SH) $\int_0^1 W_t \circ \mathrm{d}W_t = \frac{1}{2}W_1^2$. $\square$

We note that from Theorem 4.2, we obtain that if f is of class $C^2(\mathbb{R})$, $f(W_t)$ is SHB integrable to $\int_0^1 f(W_t)\circ \mathrm{d}W_t$. Now we consider the Itô formula for Stratonovich integral: if $f\in C^3$, then

$$f(W_t)=f(W_0)+\int_0^1 f'(W_t)\circ \mathrm{d}W_t \tag{4.4}$$

and we substitute (SH) $\int_0^1 f'(W_t)\circ \mathrm{d}W_t$ for $\int_0^1 f'(W_t)\circ \mathrm{d}W_t$. Then

$$f(W_t)=f(W_0)+(\mathrm{SH})\int_0^1 f'(W_t)\circ \mathrm{d}W_t. \tag{4.5}$$

As shown in Example 4.3 and (4.5), the SHB integral is more "natural" than the Itô integral. There is no second order term in (4.5). This matches our intuitive sense of the integration formula. With this slight modification of integral, we keep the form of the fundamental theorem of calculus.

5. Conclusion

In conclusion, from the definition of the Stratonovich integral using the Henstock-Kurzweil method, we not only keep the important properties of the classical Stratonovich integral, but also probably enlarge the scope of the integrands which satisfy the ideal "Itô formula".

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