CARDINALITIES OF DCCC NORMAL SPACES WITH
A RANK 2-DIAGONAL

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Abstract. A topological space $X$ has a rank 2-diagonal if there exists a diagonal sequence on $X$ of rank 2, that is, there is a countable family $\{U_n : n \in \omega\}$ of open covers of $X$ such that for each $x \in X$, $\{x\} = \bigcap\{\text{St}^2(x, U_n) : n \in \omega\}$. We say that a space $X$ satisfies the Discrete Countable Chain Condition (DCCC for short) if every discrete family of nonempty open subsets of $X$ is countable. We mainly prove that if $X$ is a DCCC normal space with a rank 2-diagonal, then the cardinality of $X$ is at most $\mathfrak{c}$. Moreover, we prove that if $X$ is a first countable DCCC normal space and has a $G_\delta$-diagonal, then the cardinality of $X$ is at most $\mathfrak{c}$.

Keywords: cardinality; Discrete Countable Chain Condition; normal space; rank 2-diagonal; $G_\delta$-diagonal

MSC 2010: 54D20, 54E35

1. Introduction

Diagonal properties are useful in estimating on the cardinality of a space. For example, Ginsburg and Woods in [4] proved that the cardinality of a space with countable extent and a $G_\delta$-diagonal is at most $\mathfrak{c}$. Therefore, if $X$ is Lindelöf and has a $G_\delta$-diagonal, then the cardinality of $X$ is at most $\mathfrak{c}$. However, the cardinality of a regular space with the countable Souslin number and a $G_\delta$-diagonal need not have an upper bound (see [7], [8]). Buzyakova in [2] proved that if a space $X$ with the countable Souslin number has a regular $G_\delta$-diagonal, then the cardinality of $X$ does not exceed $\mathfrak{c}$. Rank 3-diagonal is one type of diagonal property. Recently, we proved that if $X$ is a DCCC space (defined below) with a rank 3-diagonal, then the cardinality of $X$ is at most $\mathfrak{c}$ (see [10]). The following question is also asked in [10]:

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Question 1.1. Is the cardinality of a DCCC space with a rank 2-diagonal at most \(c\)?

In this paper, we prove that if \(X\) is a DCCC normal space with a rank 2-diagonal, then the cardinality of \(X\) is at most \(c\). We also prove that if \(X\) is a first countable DCCC normal space and has a \(G_\delta\)-diagonal, then the cardinality of \(X\) is at most \(c\).

2. Notation and terminology

All spaces are assumed to be Hausdorff unless otherwise stated.

The cardinality of a set \(X\) is denoted by \(|X|\), and \([X]^2\) denotes the set of two-element subsets of \(X\). We write \(\omega\) for the first infinite cardinal and \(c\) for the cardinality of the continuum.

Definition 2.1 ([9]). We say that a space \(X\) satisfies the Discrete Countable Chain Condition (DCCC for short) if every discrete family of nonempty open subsets of \(X\) is countable.

If \(A\) is a subset of \(X\) and \(\mathcal{U}\) is a family of subsets of \(X\), then \(\text{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}\). We also put \(\text{St}^0(A, \mathcal{U}) = A\) and for a nonnegative integer \(n\), \(\text{St}^{n+1}(A, \mathcal{U}) = \text{St}(\text{St}^n(A, \mathcal{U}), \mathcal{U})\). If \(A = \{x\}\) for some \(x \in X\), then we write \(\text{St}^n(x, \mathcal{U})\) instead of \(\text{St}^n(\{x\}, \mathcal{U})\).

Definition 2.2 ([1]). A diagonal sequence of rank \(k\) on a space \(X\), where \(k \in \omega\), is a countable family \(\{\mathcal{U}_n : n \in \omega\}\) of open coverings of \(X\) such that \(\{x\} = \bigcap\{\text{St}^k(x, \mathcal{U}_n) : n \in \omega\}\) for each \(x \in X\).

Definition 2.3 ([1]). A space \(X\) has a rank \(k\)-diagonal, where \(k \in \omega\), if there is a diagonal sequence \(\{\mathcal{U}_n : n \in \omega\}\) on \(X\) of rank \(k\).

Therefore, a space \(X\) has a rank 2-diagonal if there exists a diagonal sequence on \(X\) of rank 2, that is, there is a countable family \(\{\mathcal{U}_n : n \in \omega\}\) of open covers of \(X\) such that for each \(x \in X\), \(\{x\} = \bigcap\{\text{St}^2(x, \mathcal{U}_n) : n \in \omega\}\).

All notation and terminology not explained here is given in [3].

3. Results

We will use the following countable version of a set-theoretic theorem due to Erdős and Radó.

Lemma 3.1 ([5], Theorem 2.3). Let \(X\) be a set with \(|X| > c\) and suppose \([X]^2 = \bigcup\{P_n : n \in \omega\}\). Then there exists \(n_0 < \omega\) and a subset \(S\) of \(X\) with \(|S| > \omega\) such that \([S]^2 \subset P_{n_0}\).
Lemma 3.2. Let \( \{U_n : n \in \omega \} \) be a diagonal sequence on \( X \) of rank \( k \), where \( k \geq 1 \). If \( |X| > \omega \), then there exists an uncountable closed discrete subset \( S \) of \( X \) such that for any two distinct points \( x, y \in S \) there exists \( n_0 \in \omega \) such that \( y \notin \text{St}^k(x, U_{n_0}) \).

Proof. Assume there exists a sequence \( \{U_n : n \in \omega \} \) of open covers of \( X \) such that \( \{x\} = \bigcap\{\text{St}^k(x, U_n) : n \in \omega \} \) for every \( x \in X \). We may suppose \( \text{St}^k(x, U_{n+1}) \subset \text{St}^k(x, U_n) \) for any \( n \in \omega \). For each \( n \in \omega \) let

\[
P_n = \{\{x, y\} \in [X]^2 : x \notin \text{St}^k(y, U_n)\}.
\]

Thus, \( [X]^2 = \bigcup \{P_n : n \in \omega \} \). Then by Lemma 3.1 there exists a subset \( S \) of \( X \) with \( |S| > \omega \) and \( |S|^2 \subset P_{n_0} \) for some \( n_0 \in \omega \). It is evident that for any two distinct points \( x, y \in S \), \( y \notin \text{St}^k(x, U_{n_0}) \). Now we show that \( S \) is closed and discrete. If not, let \( x \in X \) and suppose \( x \) was an accumulation point of \( S \). Since \( X \) is \( T_1 \), each neighborhood \( U \in U_{n_0} \) of \( x \) meets infinitely many members of \( S \). Therefore there exist distinct points \( y \) and \( z \) in \( S \cap U \). Thus, \( y \in U \subset \text{St}(z, U_{n_0}) \subset \text{St}^k(z, U_{n_0}) \). This is a contradiction. Thus, \( S \) has no accumulation points in \( X \); equivalently, \( S \) is a closed and discrete subset of \( X \). This completes the proof. \( \square \)

In Lemma 3.2, if the diagonal rank of \( X \) is at least 2, i.e., \( k \geq 2 \), then \( S \) has a disjoint open expansion \( \{\text{St}(x, U_n) : x \in S \} \). Indeed, if there exist distinct \( x, y \in S \) such that \( \text{St}(x, U_{n_0}) \cap \text{St}(y, U_{n_0}) \neq \emptyset \), then \( y \in \text{St}^2(x, U_{n_0}) \subset \text{St}^k(x, U_{n_0}) \). This is impossible.

Lemma 3.3. If \( S \) is a closed discrete set in a normal space \( X \) and \( U = \{U(x) : x \in S \} \) is a disjoint open expansion of \( S \), then there is a discrete open expansion \( \mathcal{V} = \{V(x) : x \in S\} \) of \( S \) such that \( x \in V(x) \subset U(x) \) for all \( x \in S \).

Proof. By normality there exists an open set \( W \) in \( X \) such that \( S \subset W \subset \overline{W} \subset \bigcup U \). For all \( x \in S \) let \( V(x) = U(x) \cap W \). It is easily verified that \( \mathcal{V} = \{V(x) : x \in S\} \) is a discrete open collection of cardinality \( |S| \). \( \square \)

Theorem 3.4. If \( X \) is a DCCC normal space and if it has a rank 2-diagonal, then the cardinality of \( X \) does not exceed \( \omega \).

Proof. Assume the contrary. It follows from Lemma 3.2 that \( \{\text{St}(x, U_{n_0}) : x \in S\} \) is an uncountable pairwise disjoint family of nonempty open sets of \( X \). Since \( X \) is normal, by Lemma 3.3 there is a discrete open expansion \( \mathcal{V} = \{V(x) : x \in S\} \) of \( S \) with \( x \in V(x) \subset \text{St}(x, U_{n_0}) \), for all \( x \in S \). This contradicts the fact that \( X \) is DCCC. This proves that \( |X| \leq \omega \). \( \square \)
Recall that a space $X$ is star countable if whenever $\mathcal{U}$ is an open cover of $X$, there is a countable subset $A$ of $X$ such that $\text{St}(A, \mathcal{U}) = X$. In [10], the authors have proved that every star countable space is DCCC. Moreover, the cardinality of every star countable space with a rank 2-diagonal is at most $c$ (see [11]). Therefore, by the above observations, it is natural to ask whether a DCCC normal space is star countable. However, the answer is negative (see [6], page 99).

We say that a topological space $X$ has a $G_\delta$-diagonal if there exists a sequence $\{G_n: n \in \omega\}$ of open sets in $X^2$ such that $\Delta_X = \bigcap\{G_n: n < \omega\}$, where $\Delta_X = \{ (x, x): x \in X\}$. A space $X$ has a $G_\delta$-diagonal if and only if $X$ has a rank 1-diagonal.

**Theorem 3.5.** If $X$ is a first countable DCCC normal space and if it has a $G_\delta$-diagonal, then the cardinality of $X$ does not exceed $c$.

**Proof.** By the assumption, there exists a sequence $\{\mathcal{U}_n: n \in \omega\}$ of open covers of $X$ such that $\{x\} = \bigcap\{\text{St}(x, \mathcal{U}_n): n \in \omega\}$ for every $x \in X$. We may suppose $\text{St}(x, \mathcal{U}_{n+1}) \subset \text{St}(x, \mathcal{U}_n)$ for any $n \in \omega$. Let $\mathcal{B}(x) = \{B_m(x): m \in \omega\}$ be a local base for $x$. Assume $B_{m+1}(x) \subset B_m(x)$ for any $m \in \omega$. For each $n \in \omega$ let

$$P_n = \{ \{x, y\} \in [X]^2: x \notin \text{St}(y, \mathcal{U}_n); B_n(x) \cap B_n(y) = \emptyset\}.$$ 

Thus, $[X]^2 = \bigcup\{P_n: n \in \omega\}$. Suppose that $|X| > c$. Then by Lemma 3.1 there exists a subset $S$ of $X$ with $|S| > \omega$ and $|S|^2 \subset P_{n_0}$ for some $n_0 \in \omega$. As in the proof of Lemma 3.2, one easily sees that $S$ is closed and discrete. Besides, it is evident that for any two distinct points $x, y \in S$, $B_{n_0}(x) \cap B_{n_0}(y) = \emptyset$.

Since $X$ is normal, by Lemma 3.3 there is a discrete open expansion $\mathcal{V} = \{V(x): x \in S\}$ of $S$ with $x \in V(x) \subset B_{n_0}(x)$, for all $x \in S$. This contradicts the fact that $X$ is DCCC. This proves that $|X| \leq c$. \hfill $\square$

Theorem 3.5 suggests the following question.

**Question 3.6.** Let $X$ be a DCCC normal space with a $G_\delta$-diagonal. Is $X$ CCC?

It is well known that the cardinality of a first countable CCC space is at most $c$. Therefore, a positive answer to Question 3.6 would imply a trivial proof of Theorem 3.5.

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References


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