CARDINALITIES OF DCCC NORMAL SPACES WITH A RANK 2-DIAGONAL

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Abstract. A topological space X has a rank 2-diagonal if there exists a diagonal sequence on X of rank 2, that is, there is a countable family $\{\mathcal{U}_n \colon n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\mathrm{St}^2(x, \mathcal{U}_n) \colon n \in \omega\}$. We say that a space X satisfies the Discrete Countable Chain Condition (DCCC for short) if every discrete family of nonempty open subsets of X is countable. We mainly prove that if X is a DCCC normal space with a rank 2-diagonal, then the cardinality of X is at most $\mathfrak{c}$. Moreover, we prove that if X is a first countable DCCC normal space and has a G_δ-diagonal, then the cardinality of X is at most $\mathfrak{c}$.

Keywords: cardinality; Discrete Countable Chain Condition; normal space; rank 2-diagonal; G_δ-diagonal

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1. Introduction

Diagonal properties are useful in estimating on the cardinality of a space. For example, Ginsburg and Woods in [4] proved that the cardinality of a space with countable extent and a G_δ-diagonal is at most $\mathfrak{c}$. Therefore, if X is Lindelöf and has a G_δ-diagonal, then the cardinality of X is at most $\mathfrak{c}$. However, the cardinality of a regular space with the countable Souslin number and a G_δ-diagonal need not have an upper bound (see [7], [8]). Buzyakova in [2] proved that if a space X with the countable Souslin number has a regular G_δ-diagonal, then the cardinality of X does not exceed $\mathfrak{c}$. Rank 3-diagonal is one type of diagonal property. Recently, we proved that if X is a DCCC space (defined below) with a rank 3-diagonal, then the cardinality of X is at most $\mathfrak{c}$ (see [10]). The following question is also asked in [10]:

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Question 1.1. Is the cardinality of a DCCC space with a rank 2-diagonal at most $\mathfrak{c}$?

In this paper, we prove that if X is a DCCC normal space with a rank 2-diagonal, then the cardinality of X is at most $\mathfrak{c}$. We also prove that if X is a first countable DCCC normal space and has a G_δ-diagonal, then the cardinality of X is at most $\mathfrak{c}$.

2. Notation and terminology

All spaces are assumed to be Hausdorff unless otherwise stated.

The cardinality of a set X is denoted by $|X|$, and $[X]^2$ denotes the set of two-element subsets of X. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

Definition 2.1 ([9]). We say that a space X satisfies the Discrete Countable Chain Condition (DCCC for short) if every discrete family of nonempty open subsets of X is countable.

If A is a subset of X and $\mathcal{U}$ is a family of subsets of X, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon U \cap A \neq \emptyset\}$. We also put $\mathrm{St}^0(A,\mathcal{U}) = A$ and for a nonnegative integer n, $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^n(A,\mathcal{U}),\mathcal{U})$. If $A = \{x\}$ for some $x \in X$, then we write $\mathrm{St}^n(x,\mathcal{U})$ instead of $\mathrm{St}^n(\{x\},\mathcal{U})$.

Definition 2.2 ([1]). A diagonal sequence of rank k on a space X, where $k \in \omega$, is a countable family $\{\mathcal{U}_n\colon n \in \omega\}$ of open coverings of X such that $\{x\} = \bigcap\{\mathrm{St}^k(x,\mathcal{U}_n)\colon n \in \omega\}$ for each $x \in X$.

Definition 2.3 ([1]). A space X has a rank k-diagonal, where $k \in \omega$, if there is a diagonal sequence $\{\mathcal{U}_n\colon n \in \omega\}$ on X of rank k.

Therefore, a space X has a rank 2-diagonal if there exists a diagonal sequence on X of rank 2, that is, there is a countable family $\{\mathcal{U}_n\colon n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\mathrm{St}^2(x,\mathcal{U}_n)\colon n \in \omega\}$.

All notation and terminology not explained here is given in [3].

3. Results

We will use the following countable version of a set-theoretic theorem due to Erdős and Radó.

Lemma 3.1 ([5], Theorem 2.3). *Let X be a set with $|X| > \mathfrak{c}$ and suppose $[X]^2 = \bigcup\{P_n\colon n \in \omega\}$. Then there exists $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subset P_{n_0}$.*

Lemma 3.2. *Let $\{\mathcal{U}_n\colon n \in \omega\}$ be a diagonal sequence on X of rank k, where $k \geqslant 1$. If $|X| > \mathfrak{c}$, then there exists an uncountable closed discrete subset S of X such that for any two distinct points $x, y \in S$ there exists $n_0 \in \omega$ such that $y \notin \mathrm{St}^k(x, \mathcal{U}_{n_0})$.*

P r o o f. Assume there exists a sequence $\{\mathcal{U}_n\colon n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^k(x, \mathcal{U}_n)\colon n \in \omega\}$ for every $x \in X$. We may suppose $\mathrm{St}^k(x, \mathcal{U}_{n+1}) \subset \mathrm{St}^k(x, \mathcal{U}_n)$ for any $n \in \omega$. For each $n \in \omega$ let

$$P_n = \big\{\{x, y\} \in [X]^2\colon\ x \notin \mathrm{St}^k(y, \mathcal{U}_n)\big\}.$$

Thus, $[X]^2 = \bigcup\{P_n\colon n \in \omega\}$. Then by Lemma 3.1 there exists a subset S of X with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 \in \omega$. It is evident that for any two distinct points $x, y \in S$, $y \notin \mathrm{St}^k(x, \mathcal{U}_{n_0})$. Now we show that S is closed and discrete. If not, let $x \in X$ and suppose x were an accumulation point of S. Since X is T_1, each neighborhood $U \in \mathcal{U}_{n_0}$ of x meets infinitely many members of S. Therefore there exist distinct points y and z in $S \cap U$. Thus, $y \in U \subset \mathrm{St}(z, \mathcal{U}_{n_0}) \subset \mathrm{St}^k(z, \mathcal{U}_{n_0})$. This is a contradiction. Thus, S has no accumulation points in X; equivalently, S is a closed and discrete subset of X. This completes the proof. □

In Lemma 3.2, if the diagonal rank of X is at least 2, i.e., $k \geqslant 2$, then S has a disjoint open expansion $\{\mathrm{St}(x, \mathcal{U}_{n_0})\colon x \in S\}$. Indeed, if there exist distinct $x, y \in S$ such that $\mathrm{St}(x, \mathcal{U}_{n_0}) \cap \mathrm{St}(y, \mathcal{U}_{n_0}) \neq \emptyset$, then $y \in \mathrm{St}^2(x, \mathcal{U}_{n_0}) \subset \mathrm{St}^k(x, \mathcal{U}_{n_0})$. This is impossible.

Lemma 3.3. *If S is a closed discrete set in a normal space X and $\mathcal{U} = \{U(x)\colon x \in S\}$ is a disjoint open expansion of S, then there is a discrete open expansion $\mathcal{V} = \{V(x)\colon x \in S\}$ of S with $x \in V(x) \subset U(x)$ for all $x \in S$.*

P r o o f. By normality there exists an open set W in X such that $S \subset W \subset \overline{W} \subset \bigcup\mathcal{U}$. For all $x \in S$ let $V(x) = U(x) \cap W$. It is easily verified that $\mathcal{V} = \{V(x)\colon x \in S\}$ is a discrete open collection of cardinality $|S|$. □

Theorem 3.4. *If X is a DCCC normal space and if it has a rank 2-diagonal, then the cardinality of X does not exceed $\mathfrak{c}$.*

P r o o f. Assume the contrary. It follows from Lemma 3.2 that $\{\mathrm{St}(x, \mathcal{U}_{n_0})\colon x \in S\}$ is an uncountable pairwise disjoint family of nonempty open sets of X. Since X is normal, by Lemma 3.3 there is a discrete open expansion $\mathcal{V} = \{V(x)\colon x \in S\}$ of S with $x \in V(x) \subset \mathrm{St}(x, \mathcal{U}_{n_0})$, for all $x \in S$. This contradicts the fact that X is DCCC. This proves that $|X| \leqslant \mathfrak{c}$. □

Recall that a space X is star countable if whenever $\mathcal{U}$ is an open cover of X, there is a countable subset A of X such that $\mathrm{St}(A,\mathcal{U}) = X$. In [10], the authors have proved that every star countable space is DCCC. Moreover, the cardinality of every star countable space with a rank 2-diagonal is at most $\mathfrak{c}$ (see [11]). Therefore, by the above observations, it is natural to ask whether a DCCC normal space is star countable. However, the answer is negative (see [6], page 99).

We say that a topological space X has a G_δ-diagonal if there exists a sequence $\{G_n\colon n \in \omega\}$ of open sets in X^2 such that $\Delta_X = \bigcap\{G_n\colon n < \omega\}$, where $\Delta_X = \{(x,x)\colon x \in X\}$. A space X has a G_δ-diagonal if and only if X has a rank 1-diagonal.

Theorem 3.5. *If X is a first countable DCCC normal space and if it has a G_δ-diagonal, then the cardinality of X does not exceed $\mathfrak{c}$.*

Proof. By the assumption, there exists a sequence $\{\mathcal{U}_n\colon n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}(x,\mathcal{U}_n)\colon n \in \omega\}$ for every $x \in X$. We may suppose $\mathrm{St}(x,\mathcal{U}_{n+1}) \subset \mathrm{St}(x,\mathcal{U}_n)$ for any $n \in \omega$. Let $\mathcal{B}(x) = \{B_m(x)\colon m \in \omega\}$ be a local base for x. Assume $B_{m+1}(x) \subset B_m(x)$ for any $m \in \omega$. For each $n \in \omega$ let

$$P_n = \big\{\{x,y\} \in [X]^2\colon x \notin \mathrm{St}(y,\mathcal{U}_n); B_n(x) \cap B_n(y) = \emptyset\}\big\}.$$

Thus, $[X]^2 = \bigcup\{P_n\colon n \in \omega\}$. Suppose that $|X| > \mathfrak{c}$. Then by Lemma 3.1 there exists a subset S of X with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 \in \omega$. As in the proof of Lemma 3.2, one easily sees that S is closed and discrete. Besides, it is evident that for any two distinct points $x, y \in S$, $B_{n_0}(x) \cap B_{n_0}(y) = \emptyset$.

Since X is normal, by Lemma 3.3 there is a discrete open expansion $\mathcal{V} = \{V(x)\colon x \in S\}$ of S with $x \in V(x) \subset B_{n_0}(x)$, for all $x \in S$. This contradicts the fact that X is DCCC. This proves that $|X| \leqslant \mathfrak{c}$. □

Theorem 3.5 suggests the following question.

Question 3.6. Let X be a DCCC normal space with a G_δ-diagonal. Is X CCC?

It is well known that the cardinality of a first countable CCC space is at most $\mathfrak{c}$. Therefore, a positive answer to Question 3.6 would imply a trivial proof of Theorem 3.5.

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