CHANGING OF THE DOMINATION NUMBER OF A GRAPH:
EDGE MULTISUBDIVISION AND EDGE REMOVAL

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Abstract. For a graphical property \( \mathcal{P} \) and a graph \( G \), a subset \( S \) of vertices of \( G \) is a \( \mathcal{P} \)-set if the subgraph induced by \( S \) has the property \( \mathcal{P} \). The domination number with respect to the property \( \mathcal{P} \), denoted by \( \gamma_{\mathcal{P}}(G) \), is the minimum cardinality of a dominating \( \mathcal{P} \)-set. We define the domination multisubdivision number with respect to \( \mathcal{P} \), denoted by \( \text{msd}_{\mathcal{P}}(G) \), as a minimum positive integer \( k \) such that there exists an edge which must be subdivided \( k \) times to change \( \gamma_{\mathcal{P}}(G) \). In this paper
(a) we present necessary and sufficient conditions for a change of \( \gamma_{\mathcal{P}}(G) \) after subdividing an edge of \( G \) once,
(b) we prove that if \( e \) is an edge of a graph \( G \) then \( \gamma_{\mathcal{P}}(G_{e,1}) < \gamma_{\mathcal{P}}(G) \) if and only if \( \gamma_{\mathcal{P}}(G-e) < \gamma_{\mathcal{P}}(G) \) (\( G_{e,t} \) denotes the graph obtained from \( G \) by subdivision of \( e \) with \( t \) vertices),
(c) we also prove that for every edge of a graph \( G \) we have \( \gamma_{\mathcal{P}}(G-e) \leq \gamma_{\mathcal{P}}(G_{e,3}) \leq \gamma_{\mathcal{P}}(G-e)+1 \), and
(d) we show that \( \text{msd}_{\mathcal{P}}(G) \leq 3 \), where \( \mathcal{P} \) is hereditary and closed under union with \( K_1 \).

Keywords: dominating set; edge subdivision; domination multisubdivision number; hereditary graph property

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1. Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. For the graph theory terminology not presented here, we follow Haynes et al. [14]. We denote the vertex set and the edge set of a graph \( G \) by \( V(G) \) and \( E(G) \), respectively. The subgraph induced by \( S \subseteq V(G) \) is denoted by \( \langle S, G \rangle \). For a vertex \( x \) of \( G \), \( N(x,G) \) denotes the set of all neighbors of \( x \) in \( G \), \( N[x,G] = N(x,G) \cup \{x\} \) and the degree of \( x \) is \( \text{deg}(x,G) = |N(x,G)| \). The maximum and minimum degrees of vertices in the graph \( G \) are denoted by \( \Delta(G) \) and \( \delta(G) \), respectively. For
a graph $G$, let $x \in X \subseteq V(G)$. A vertex $y$ is a **private neighbor of** $x$ with respect to $X$ if $N[y, G] \cap X = \{x\}$. The **private neighbor set of** $x$ with respect to $X$ is $pm_G(x, X) = \{y: N[y, G] \cap X = \{x\}\}$. For a graph $G$, the subdivision of the edge $e = uv \in E(G)$ with a vertex $x$ leads to a graph with the vertex set $V \cup \{x\}$ and the edge set $(E - \{uv\}) \cup \{ux, xv\}$. Let $G_{e, t}$ denote the graph obtained from $G$ by a subdivision of the edge $e$ with $t$ vertices (instead of the edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v)$). For $t = 1$ we write $G_e$.

Let $\mathcal{I}$ denote the set of all mutually non-isomorphic graphs. A **graph property** is any nonempty subset of $\mathcal{I}$. We say that a graph $G$ has the property $\mathcal{P}$ whenever there exists a graph $H \in \mathcal{P}$ which is isomorphic to $G$. For example, we list some graph properties:

- $\mathcal{O} = \{H \in \mathcal{I}: H$ is totally disconnected$\}$;
- $\mathcal{C} = \{H \in \mathcal{I}: H$ is connected$\}$;
- $\mathcal{T} = \{H \in \mathcal{I}: \delta(H) \geq 1\}$;
- $\mathcal{M} = \{H \in \mathcal{I}: H$ has a perfect matching$\}$;
- $\mathcal{F} = \{H \in \mathcal{I}: H$ is a forest$\}$;
- $\mathcal{UK} = \{H \in \mathcal{I}: \text{each component of } H \text{ is complete}\}$;
- $\mathcal{D}_k = \{H \in \mathcal{I}: \Delta(H) \leq k\}$.

A graph property $\mathcal{P}$ is called:

- (a) **hereditary (induced-hereditary)**, if the fact that a graph $G$ has property $\mathcal{P}$ implies that all subgraphs (induced subgraphs) of $G$ also belong to $\mathcal{P}$, and
- (b) **nondegenerate** if $\mathcal{O} \subseteq \mathcal{P}$. Any set $S \subseteq V(G)$ such that the induced subgraph $\langle S, G \rangle$ possesses the property $\mathcal{P}$ is called a $\mathcal{P}$-set.

Note that:

- (a) $\mathcal{I}$, $\mathcal{F}$ and $\mathcal{D}_k$ are nondegenerate and hereditary properties;
- (b) $\mathcal{UK}$ is nondegenerate, induced-hereditary and is not hereditary;
- (c) all $\mathcal{C}$, $\mathcal{T}$ and $\mathcal{M}$ are neither induced-hereditary nor nondegenerate. For a survey on this subject we refer to Borowiecki et al. [2].

A set of vertices $D \subseteq V(G)$ is a dominating set of a graph $G$ if every vertex not in $D$ is adjacent to a vertex in $D$. The domination number with respect to the property $\mathcal{P}$, denoted by $\gamma_\mathcal{P}(G)$, is the smallest cardinality of a dominating $\mathcal{P}$-set of $G$. A dominating $\mathcal{P}$-set of $G$ with cardinality $\gamma_\mathcal{P}(G)$ is called a $\gamma_\mathcal{P}$-**set of** $G$. If a property $\mathcal{P}$ is nondegenerate, then every maximal independent set is a $\mathcal{P}$-set and thus $\gamma_\mathcal{P}(G)$ exists. Note that $\gamma_\mathcal{I}(G)$, $\gamma_\mathcal{O}(G)$, $\gamma_\mathcal{C}(G)$, $\gamma_\mathcal{T}(G)$, $\gamma_\mathcal{M}(G)$, $\gamma_\mathcal{F}(G)$ and $\gamma_\mathcal{D}_k(G)$ are well known as the domination number $\gamma(G)$, the independent domination number $i(G)$ ([5]), the connected domination number $\gamma_c(G)$ ([24]), the total domination number $\gamma_t(G)$ ([3]), the paired-domination number $\gamma_{pr}(G)$ ([16]), the acyclic domination number $\gamma_a(G)$ ([17]) and the $k$-dependent domination number $\gamma^k(G)$ ([9]).

The concept of domination with respect to any graph property $\mathcal{P}$ was introduced by
Goddard et al. [10] and has been studied, for example, in [19], [20], [21], [22], [23] and elsewhere.

It is often of interest to know how the value of a graph parameter is affected when a small change is made in a graph. In [20], the present author began the study of the effects on $\gamma_P(G)$ when a graph $G$ is modified by deleting a vertex or by adding an edge ($P$ is nondegenerate). In this paper we concentrate on effects on $\gamma_P(G)$ when a graph is modified by deleting/subdividing an edge. An edge $e$ of a graph $G$ is called a $\gamma_P-ER^-$-critical edge of $G$ if $\gamma_P(G) > \gamma_P(G-e)$. Note that

(a) $\gamma-ER^-$-critical edges do not exist (see [13]),
(b) Grobler [11] was the first who began the investigation of $\gamma_P-ER^-$-critical edges when $P = O$, and
(c) necessary and sufficient conditions for an edge of a graph $G$ to be $\gamma_P-ER^-$-critical, where $P$ is hereditary, may be found in [20].

One measure of the stability of the domination number of $G$ under edge subdivision is the domination subdivision number with respect to the property $P$, denoted $\text{sd}_{\gamma_P}(G)$, which is the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase $\gamma_P(G)$. The following special cases for $\text{sd}_{\gamma_P}(G)$ have been investigated up to now:

(a) $\text{sd}_{\gamma_I}(G)$—the domination subdivision number defined by Velammal [25],
(b) $\text{sd}_{\gamma_T}(G)$—the total domination subdivision number introduced by Haynes et al. in [15],
(c) $\text{sd}_{\gamma_M}(G)$—the paired domination subdivision number introduced by Favaron et al. in [8],
(d) $\text{sd}_{\gamma_C}(G)$—the connected domination subdivision number introduced by Favaron et al. in [7], and
(e) $\text{sd}_{\gamma_P}(G)$—the domination subdivision number with respect to the nondegenerate property $P$ introduced by the present author in [23].

Here we focus on the existence of critical edges with respect to the subdivision/multisubdivision. Results in this direction, in the case when $P = I$, were recently obtained by Lemańska, Tey and Zuazua [18] and Dettlaff, Raczek and Topp [6].

For any nondegenerate property $P \subseteq I$ we define an edge $e$ of a graph $G$ to be

(i) a $\gamma_P-S^+$-critical edge of $G$ if $\gamma_P(G) < \gamma_P(G_e)$, and
(ii) a $\gamma_P-S^-$-critical edge of $G$ if $\gamma_P(G) > \gamma_P(G_e)$.

In Section 2:

(a) we present necessary and sufficient conditions for a change of $\gamma_P(G)$ after subdividing an edge of $G$ once, and
(b) we prove that an edge $e$ of a graph $G$ is $\gamma_H-S^-$-critical if and only if $e$ is $\gamma_H-ER^-$-critical, for any graph property $H \subseteq I$ which is induced-hereditary and closed under union with $K_1$. 11
In Section 3 we deal with changes of $\gamma_{\mathcal{P}}(G)$ when an edge of $G$ is multiple subdivided. To present our results we need the following definitions.

For every edge $e$ of a graph $G$ let
\begin{itemize}
  \item $\text{msd}_{\mathcal{P}}^+(e) = \min\{t: \gamma_{\mathcal{P}}(G_{e,t}) \neq \gamma_{\mathcal{P}}(G)\}$;
  \item $\text{msd}_{\mathcal{P}}^-(e) = \min\{t: \gamma_{\mathcal{P}}(G_{e,t}) > \gamma_{\mathcal{P}}(G)\}$;
  \item $\text{msd}_{\mathcal{P}}(e) = \min\{t: \gamma_{\mathcal{P}}(G_{e,t}) < \gamma_{\mathcal{P}}(G)\}$.
\end{itemize}

If $\gamma_{\mathcal{P}}(G_{e,t}) \geq \gamma_{\mathcal{P}}(G)$ for every $t \geq 1$, then we write $\text{msd}_{\mathcal{P}}^+(e) = \infty$. If $\gamma_{\mathcal{P}}(G_{e,t}) \leq \gamma_{\mathcal{P}}(G)$ for every $t \geq 1$, then we write $\text{msd}_{\mathcal{P}}^-(e) = \infty$.

**Definition 1.1.** For every graph $G$ with at least one edge and every nondegenerate property $\mathcal{P}$, we define the domination multisubdivision (plus domination multisubdivision, minus domination multisubdivision) number with respect to the property $\mathcal{P}$, denoted $\text{msd}_{\mathcal{P}}(G)$ ($\text{msd}_{\mathcal{P}}^+$, $\text{msd}_{\mathcal{P}}^-$, respectively) to be
\[
\begin{align*}
  &\text{msd}_{\mathcal{P}}(G) = \min\{\text{msd}_{\mathcal{P}}(e): e \in E(G)\}, \\
  &\text{msd}_{\mathcal{P}}^+(G) = \min\{\text{msd}_{\mathcal{P}}^+(e): e \in E(G)\}, \\
  &\text{msd}_{\mathcal{P}}^-(G) = \min\{\text{msd}_{\mathcal{P}}^-(e): e \in E(G)\},
\end{align*}
\]
respectively. If $\gamma_{\mathcal{P}}(G_{e,t}) \geq \gamma_{\mathcal{P}}(G)$ for every $t$ and every edge $e \in E(G)$, then we write $\text{msd}_{\mathcal{P}}^-(G) = \infty$.

The parameters $\text{msd}_{\mathcal{P}}^+(G)$ and $\text{msd}_{\mathcal{P}}^-(G)$ (in our designation) were introduced by Dettlaff, Raczek and Topp in [6] and by Avella-Alaminos, Dettlaff, Lemańska and Zuazua in [1], respectively. Note that in the case when $\mathcal{P} = \mathcal{I}$, clearly, $\text{msd}(G) = \text{msd}_{\mathcal{I}}^+(G)$, and $\text{msd}_{\mathcal{I}}^-(G) = \infty$. In Section 3 we prove that for every edge of a graph $G$ we have $\gamma_{\mathcal{P}}(G-e) \leq \gamma_{\mathcal{P}}(G_{e,3}) \leq \gamma_{\mathcal{P}}(G-e) + 1$ and we present necessary and sufficient conditions for the validity of $\gamma_{\mathcal{P}}(G-e) = \gamma_{\mathcal{P}}(G_{e,3})$. Our main result in that section is that $\text{msd}_{\mathcal{P}}(G) \leq 3$ for any graph $G$ and any graph property $\mathcal{P}$ which is hereditary and closed under union with $K_1$.

2. Single subdivision: critical edges

We begin this section with a characterization of $\gamma_{\mathcal{P}}-S^+$-critical edges of a graph. Note that if a property $\mathcal{P}$ is induced-hereditary and closed under union with $K_1$ then $\mathcal{P}$ is nondegenerate.

**Theorem 2.1.** Let $\mathcal{H} \subseteq \mathcal{I}$ be hereditary and closed under union with $K_1$. Let $G$ be a graph and $e = uv \in E(G)$. Then $\gamma_{\mathcal{H}}(G_e) \leq \gamma_{\mathcal{H}}(G) + 1$. If $e$ is a $\gamma_{\mathcal{H}}-S^+$-critical edge of $G$ then $\gamma_{\mathcal{H}}(G_e) = \gamma_{\mathcal{H}}(G) + 1$ and for each $\gamma_{\mathcal{H}}$-set $M$ of $G$ one of the following conditions holds:
\begin{enumerate}
  \item $u, v \in V(G) - M$;
\end{enumerate}
(ii) \( u \in M, v \in pm_G[u, M] \) and \( pm_G[u, M] \) is not a subset of \( \{u, v\} \);
(iii) \( v \in M, u \in pm_G[v, M] \) and \( pm_G[v, M] \) is not a subset of \( \{u, v\} \).

If \( e \) is not \( \gamma_H \)-critical and for each \( \gamma_H \)-set \( M \) of \( G \) one of (i), (ii) and (iii) holds then there is a dominating \( \mathcal{H} \)-set \( R \) of \( G - uv \) with \( u, v \in R \) and \( |R| \leq \gamma_H(G) \).

**Proof.** Let \( x \in V(G_e) \) be the subdivision vertex and let \( M \) be a \( \gamma_H \)-set of \( G \). If \( u, v \notin M \) then \( M \cup \{x\} \) is a dominating \( \mathcal{H} \)-set of \( G_e \) (\( \mathcal{H} \) is closed under union with \( K_1 \)) and we have \( \gamma_H(G_e) \leq \gamma_H(G) + 1 \). If both \( u \) and \( v \) are in \( M \) then \( M \) is a dominating \( \mathcal{H} \)-set of \( G \) (\( \mathcal{H} \) is hereditary), which implies \( \gamma_H(G_e) \leq \gamma_H(G) \). If \( u \in M, v \notin M \) and \( v \notin pm_G[u, M] \) then again \( M \) is a dominating \( \mathcal{H} \)-set of \( G_e \) and hence \( \gamma_H(G_e) \leq \gamma_H(G) \). If neither \( pm_G[u, M] = \{u\} \) nor \( pm_G[u, M] = \{u, v\} \) then \( M \cup \{v\} \) is a dominating \( \mathcal{H} \)-set of \( G_e \) and we have \( \gamma_H(G_e) \leq \gamma_H(G) + 1 \). Thus \( \gamma_H(G_e) \leq \gamma_H(G) + 1 \) and if the equality is fulfilled then one of (i), (ii) and (iii) holds.

Now, let \( e \) for each \( \gamma_H \)-set \( M \) of \( G \) one of (i), (ii) and (iii) holds. Assume \( \gamma_H(G_e) \leq \gamma_H(G) \) and let \( R \) be a \( \gamma_H \)-set of \( G_e \).

**Case 1:** \( u, v \notin R \). Hence \( x \in R \). If \( u, v \notin pm_{G_e}[x, R] \) then \( R - \{x\} \) is a dominating \( \mathcal{H} \)-set of \( G \), a contradiction with \( \gamma_H(G_e) \leq \gamma_H(G) \). If \( u \in pm_{G_e}[x, R] \) and \( v \notin pm_{G_e}[x, R] \) then \( R_1 = (R - \{x\}) \cup \{u\} \) is a dominating \( \mathcal{H} \)-set of \( G \) of cardinality \( |R_1| = |R| = \gamma_H(G_e) \). Since \( \gamma_H(G_e) \leq \gamma_H(G) \), we have that \( R_1 \) is a \( \gamma_H \)-set of \( G \). But then \( u \in R_1, v \notin R_1 \) and \( v \notin pm_G[u, R_1] \), contradicting (ii). If \( u, v \in pm_G[x, R] \) then as above \( R_1 \) is a \( \gamma_H \)-set of \( G \) and since \( u \in R_1 \) and \( \{u, v\} = pm_G[u, R_1] \), again we arrive at a contradiction with (ii).

**Case 2:** \( u \in R \) and \( v \notin R \). Hence \( x \notin R \), otherwise \( R - \{x\} \) is a dominating \( \mathcal{H} \)-set of \( G \), contradicting \( \gamma_H(G_e) \leq \gamma_H(G) \). This implies that \( R \) is a \( \gamma_H \)-set of \( G \), \( u \in R \) and \( v \notin pm_G[u, R], \) a contradiction with (ii).

**Case 3:** \( u, v \in R \). Hence \( R \) is a dominating \( \mathcal{H} \)-set of \( G - uv \) and \( |R| = \gamma_H(G_e) \leq \gamma_H(G) \). \( \square \)

When we restrict our attention to the case where \( \mathcal{H} = I \), we can describe more precisely when an edge of a graph \( G \) is \( \gamma \)-critical.

**Corollary 2.2.** Let \( G \) be a graph and \( e = uv \in E(G) \). Then \( e \) is a \( \gamma \)-critical edge of \( G \) if and only if for each \( \gamma \)-set \( M \) of \( G \) one of (i), (ii) and (iii) stated in Theorem 2.1 holds.

**Proof.** **Necessity:** The result immediately follows by Theorem 2.1.

**Sufficiency:** Assume \( \gamma(G_e) \leq \gamma(G) \). Then by Theorem 2.1, there is a dominating set \( R \) of \( G - uv \) with \( u, v \in R \) and \( |R| \leq \gamma(G) \). But it is a well known fact that if \( f \)
is an edge of a graph \( G \) then always \( \gamma(G - f) \geq \gamma(G) \). Hence \( R \) is a \( \gamma \)-set of both \( G \) and \( G - e \) and \( u, v \in R \), contradicting all (i), (ii) and (iii). \( \square \)

**Theorem 2.3.** Let \( \mathcal{H} \subseteq \mathcal{I} \) be induced-hereditary and closed under union with \( K_1 \). An edge \( e \) of a graph \( G \) is \( \gamma_{\mathcal{H}} \)-critical if and only if \( e \) is \( \gamma_{\mathcal{H}}^{-ER} \)-critical.

**Proof.** As we have already shown, \( \mathcal{H} \) is nondegenerate and then all \( \gamma_{\mathcal{H}}(G - e) \), \( \gamma_{\mathcal{H}}(G_e) \) and \( \gamma_{\mathcal{H}}(G) \) exist. Let \( v \) be the subdivision vertex of \( G_e \).

**Sufficiency:** Let \( e = xy \) be a \( \gamma_{\mathcal{H}}^{-ER} \)-critical edge of \( G \) and \( M \) a \( \gamma_{\mathcal{H}} \)-set of \( G_e \). Hence \( \gamma_{\mathcal{H}}(G - e) < \gamma_{\mathcal{H}}(G) \) and \( x, y \in M \). But then \( M \) is a dominating \( \mathcal{H} \)-set of \( G_e \), which leads to \( \gamma_{\mathcal{H}}(G_e) \leq \gamma_{\mathcal{H}}(G - e) < \gamma_{\mathcal{H}}(G) \).

**Necessity:** Let \( e = xy \) be a \( \gamma_{\mathcal{H}}^{-S} \)-critical edge of \( G \) and \( M \) a \( \gamma_{\mathcal{H}} \)-set of \( G_e \). Hence \( \gamma_{\mathcal{H}}(G_e) < \gamma_{\mathcal{H}}(G) \). Assume \( v \notin M \). Hence at least one of \( x \) and \( y \) is in \( M \). If both \( x, y \in M \) then \( M \) is a dominating \( \mathcal{H} \)-set of \( G - e \) and the result follows. If \( x \notin M \) and \( y \in M \) then \( M \) is a dominating \( \mathcal{H} \)-set of \( G \), a contradiction. Thus we may assume \( v \) is in all \( \gamma_{\mathcal{H}} \)-sets of \( G_e \). Since \( \mathcal{H} \) is induced-hereditary, at least one of \( x \) and \( y \) is not in \( M \). First let \( x \in M \) and \( y \notin M \). Then \( y \in pn_{G_e}[v, M] \), which implies \( M - \{v\} \) is a dominating \( \mathcal{H} \)-set of \( G \), a contradiction. Hence neither \( x \) nor \( y \) are in \( M \). If \( x, y \notin pn_{G_e}[v, M] \) then \( M - \{v\} \) is a dominating \( \mathcal{H} \)-set of \( G \), a contradiction. Hence at least one of \( x \) and \( y \), say \( y \), is in \( pn_{G_e}[v, M] \). But then \( (M - \{v\}) \cup \{y\} \) is a dominating \( \mathcal{H} \)-set of \( G \), a contradiction. \( \square \)

Note that
(a) there do not exist \( \gamma \)-\( ER^{-} \)-critical edges (see [13]), and
(b) necessary and sufficient conditions for an edge of a graph \( G \) to be \( \gamma_{\mathcal{H}}^{-ER} \)-critical may be found in [20].

Now we define the following classes of graphs:
\( (CS^{-}_P) \gamma_P(G) > \gamma_P(G_e) \) for every edge \( e \) of \( G \), and
\( (CER^{-}_P) \gamma_P(G) > \gamma_P(G - e) \) for every edge \( e \) of \( G \).

As an immediate consequence of Theorem 2.3 we obtain the next result.

**Corollary 2.4.** If \( \mathcal{H} \subseteq \mathcal{I} \) is induced-hereditary and closed under union with \( K_1 \) then the classes of graphs \( CS^{-}_P \) and \( CER^{-}_P \) coincide.

Note that the class \( CER^{-}_P \) in the case when \( P = \mathcal{O} \) was introduced by Grobler [11] and also considered in [12], [13], [4].
3. Multiple subdivision

We first state our theorems, then we pose a problem they generate, and after that we give the proofs.

Recall that $G_{e,t}$ denotes the graph obtained from a graph $G$ by the subdivision of the edge $e \in E(G)$ with $t$ vertices (instead of edge $e = uv$ we put a path $(u, x_1, x_2, \ldots, x_t, v)$). For any graph $G$ and any nondegenerate property $P$ let us denote by $V_P(G)$ the set \{ $v \in V(G)\colon \gamma_P(G - v) < \gamma_P(G)$\}. Our first result shows that the value of the difference $\gamma_P(G_{e,3}) - \gamma_P(G - e)$ is either 0 or 1.

**Theorem 3.1.** Let $H \subseteq \mathcal{I}$ be induced-hereditary and closed under union with $K_1$. If $e = uv$ is an edge of a graph $G$ then $\gamma_H(G - e) \leq \gamma_H(G_{e,3}) \leq \gamma_H(G - e) + 1$. Moreover, the following conditions are equivalent:

- (A$_1$) $\gamma_H(G - e) = \gamma_H(G_{e,3})$;
- (A$_2$) at least one of the following holds:
  - (i) $u \in V_H(G - e)$ and $v$ belongs to some $\gamma_H$-set of $G - u$;
  - (ii) $v \in V_H(G - e)$ and $u$ belongs to some $\gamma_H$-set of $G - v$.

If in addition $H$ is hereditary then (A$_1$) and (A$_2$) are equivalent to

- (A$_3$) $\gamma_H(G - e) = 1 + \gamma_H(G)$.

The main result in this section is the following.

**Theorem 3.2.** Let $e$ be an edge of a graph $G$ and let $H \subseteq \mathcal{I}$ be hereditary and closed under union with $K_1$.

- (i) Then $\gamma_H(G) = \gamma_H(G_{e,3})$ if and only if $\gamma_H(G) = \gamma_H(G - e) + 1$.
- (ii) If $\gamma_H(G) = \gamma_H(G - e) + 1$ then $\gamma_H(G) = \gamma_H(G_{e,1}) + 1 = \gamma_H(G_{e,2}) + 1 = \gamma_H(G_{e,3}) = \gamma_H(G_{e,4}) = \gamma_H(G_{e,5}) = \gamma_H(G_{e,6}) - 1$.
- (iii) Then $\gamma_H(G) \leq 3$. In particular (Dettlaff, Raczek and Topp [6] when $H = \mathcal{I}$), $\gamma_H(G) \leq 3$.

**Example 3.3.** It is easy to see that if $G = K_{3n_2\ldots n_m}$, where $m \geq 2$ and $n_i \geq 3$ for $2 \leq i \leq m$, then $\gamma_O(G) = \gamma_O(G_{e,3}) = \gamma_O(G - e) + 1 = 3$ for every edge $e$ of $G$. Hence by Theorem 3.2, $\gamma_O(G) = \gamma_O(G_{e,5}) = 1$ and $\gamma_O(G_{e,6}) = 6$.

In view of Theorem 3.2 (iii), we can split the family of all graphs $G$ into three classes with respect to the value of $\gamma_P(G)$, where $P \subseteq \mathcal{I}$ is hereditary and closed under union with $K_1$. We define that a graph $G$ belongs to the class $S_P^+$ whenever $\gamma_P(G) = i$, $i \in \{1, 2, 3\}$. It is straightforward to verify that if $k \geq 1$ and $O \subseteq P \subseteq \mathcal{I}$ then
\[ P_{3k}, C_{3k} \in S^1; P_{3k+2}, C_{3k+2} \in S^2; \text{ and } P_{3k+1}, C_{3k+1} \in S^3. \]

Thus, none of \( S^1, S^2 \) and \( S^3 \) is empty.

We conclude this part with an open problem.

**Problem 3.4.** Characterize the graphs belonging to \( S^i_p \), or find further properties of such graphs.

Remark that Dettlaff, Raczek and Topp recently characterized all trees belonging to \( S^1 \) and \( S^3 \) (see [6]).

### 3.1. Proofs.

For the proofs of Theorems 3.1 and 3.2, we need the following results.

**Theorem A ([20]).** Let \( \mathcal{H} \subseteq \mathcal{I} \) be nondegenerate and closed under union with \( K_1 \). Let \( G \) be a graph and \( v \in V(G) \).

(i) If \( v \) belongs to no \( \gamma_{\mathcal{H}} \)-set of \( G \) then \( \gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) \).

(ii) If \( \gamma_{\mathcal{H}}(G - v) < \gamma_{\mathcal{H}}(G) \) then \( \gamma_{\mathcal{H}}(G - v) = \gamma_{\mathcal{H}}(G) - 1 \). Moreover, if \( M \) is a \( \gamma_{\mathcal{H}} \)-set of \( G - v \) then \( M \cup \{v\} \) is a \( \gamma_{\mathcal{H}} \)-set of \( G \) and \( \{v\} = pm_G[v, M \cup \{v\}] \).

**Theorem B ([20]).** Let \( \mathcal{H} \subseteq \mathcal{I} \) be hereditary and closed under union with \( K_1 \). Let \( e = uv \) be an edge of a graph \( G \). If \( \gamma_{\mathcal{H}}(G) < \gamma_{\mathcal{H}}(G - e) \) then \( \gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) - 1 \). Moreover, \( \gamma_{\mathcal{H}}(G) = \gamma_{\mathcal{H}}(G - e) - 1 \) if and only if at least one of the conditions (i) and (ii) stated in Theorem 3.1 holds.

**Theorem C ([20]).** Let \( e = xy \) be an edge of a graph \( G \) and let \( \mathcal{H} \subseteq \mathcal{I} \) be hereditary and closed under union with \( K_1 \). If \( \gamma_{\mathcal{H}}(G) > \gamma_{\mathcal{H}}(G - e) \) then:

(i) no \( \gamma_{\mathcal{H}} \)-set of \( G - e \) is an \( \mathcal{H} \)-set of \( G \);

(ii) both \( x \) and \( y \) are in all \( \gamma_{\mathcal{H}} \)-sets of \( G - e \);

(iii) \( \gamma_{\mathcal{H}}(G - x) \geq \gamma_{\mathcal{H}}(G - e) \) and \( \gamma_{\mathcal{H}}(G - y) \geq \gamma_{\mathcal{H}}(G - e) \);

(iv) if \( \gamma_{\mathcal{H}}(G - x) = \gamma_{\mathcal{H}}(G - e) \) then \( y \) belongs to no \( \gamma_{\mathcal{H}} \)-set of \( G - x \);

(v) if \( \gamma_{\mathcal{H}}(G - y) = \gamma_{\mathcal{H}}(G - e) \) then \( x \) belongs to no \( \gamma_{\mathcal{H}} \)-set of \( G - y \).

**Proof of Theorem 3.1.** Let \( D \) be a \( \gamma_{\mathcal{H}} \)-set of \( G - e \). Then since \( \mathcal{H} \) is closed under union with \( K_1 \), \( D \cup \{x_2\} \) is a dominating \( \mathcal{H} \)-set of \( G_{e,3} \). Hence \( \gamma_{\mathcal{H}}(G_{e,3}) \leq |D \cup \{y\}| \leq \gamma_{\mathcal{H}}(G - e) + 1 \).

For the left-hand side inequality, let \( \tilde{D} \) be a \( \gamma_{\mathcal{H}} \)-set of \( G_{e,3} \) and \( S = \tilde{D} \cap \{x_1, x_2, x_3\} \).

If \( S = \{x_2\} \) then \( \tilde{D} - \{x_2\} \) is a dominating \( \mathcal{H} \)-set of \( G - e \) and \( \gamma_{\mathcal{H}}(G - e) \leq |\tilde{D} - \{x_2\}| = \gamma_{\mathcal{H}}(G_{e,3}) - 1 \). If \( S = \{x_1, x_2\} \) then \( pm_{G_{e,3}}[x_1, \tilde{D}] = \{u\} \) and hence \( D_1 = (\tilde{D} - \{x_1, x_2\}) \cup \{u\} \) is a dominating \( \mathcal{H} \)-set of \( G - e \), which implies \( \gamma_{\mathcal{H}}(G - e) \leq |D_1| < |\tilde{D}| = \gamma_{\mathcal{H}}(G_{e,3}) \).
Let $S = \{x_1\}$. If $u \in pn[x_1, \bar{D}]$ then $\bar{D}_2 = (\bar{D} - \{x_1\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of $G - e$ and hence $\gamma_\mathcal{H}(G - e) \leq |\bar{D}_2| = |\bar{D}| = \gamma_\mathcal{H}(G_{e,3})$. If $u \notin pn[x_1, \bar{D}]$ then $\bar{D} - \{x_1\}$ is a dominating $\mathcal{H}$-set of $G - e$ and $\gamma_\mathcal{H}(G - e) \leq |\bar{D}| - 1 = \gamma_\mathcal{H}(G_{e,3}) - 1$.

If $S = \{x_1, x_3\}$ then at least one of $pn_{G_{e,3}}[x_1, \bar{D}] = \{x_1, u\}$ and $pn_{G_{e,3}}[x_3, \bar{D}] = \{x_3, v\}$ holds, otherwise $(\bar{D} - \{x_1, x_3\}) \cup \{x_2\}$ would be a dominating $\mathcal{H}$-set of $G_{e,3}$, contradicting the choice of $\bar{D}$. Say, without loss of generality, $pn_{G_{e,3}}[x_3, \bar{D}] = \{x_3, v\}$. Then $\bar{D}_3 = (\bar{D} - \{x_3\}) \cup \{v\}$ is a $\gamma_\mathcal{H}$-set of $G_{e,3}$ and $\bar{D}_3 \cap \{x_1, x_2, x_3\} = \{x_1\}$. As above we obtain $\gamma_\mathcal{H}(G - e) < \gamma_\mathcal{H}(G_{e,3})$. By reason of symmetry, the left-hand side inequality is proved.

$(A_2) \Rightarrow (A_1)$ Let us assume without loss of generality that (i) holds. Let $D$ be a $\gamma_\mathcal{H}(G - u)$-set and $v \in D$. By Theorem A, $D \cup \{u\}$ is a $\gamma_\mathcal{H}$-set of $G - e$ and $pn_{G - e}[u, D \cup \{u\}] = \{u\}$. Hence $D \cup \{x_1\}$ is a dominating $\mathcal{H}$-set of $G_{e,3}$ and $\gamma_\mathcal{H}(G_{e,3}) \leq |D \cup \{x_1\}| = \gamma_\mathcal{H}(G - e)$. But we have already shown that $\gamma_\mathcal{H}(G_{e,3}) \geq \gamma_\mathcal{H}(G - e)$. Therefore $\gamma_\mathcal{H}(G_{e,3}) = \gamma_\mathcal{H}(G - e)$.

$(A_1) \Rightarrow (A_2)$ Suppose $\gamma_\mathcal{H}(G - e) = \gamma_\mathcal{H}(G_{e,3})$. Let $\bar{D}$ be a $\gamma_\mathcal{H}(G_{e,3})$-set and $S = D \cap \{x_1, x_2, x_3\}$. If $S = \{x_2\}$ then $\bar{D} - \{x_2\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. If $S = \{x_1, x_2\}$ then clearly $pn_{G_{e,3}}[x_1, \bar{D}] = \{u\}$, which implies that $(\bar{D} - \{x_1, x_2\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction.

Let $S = \{x_1\}$. Hence $v \in \bar{D}$. If $u \notin pn_{G_{e,3}}[x_1, \bar{D}]$ then $\bar{D} - \{x_1\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. If $u \in pn_{G_{e,3}}[x_1, \bar{D}]$ then $D_1 = (\bar{D} - \{x_1\}) \cup \{u\}$ is a $\gamma_\mathcal{H}$-set of $G - e$, $u, v \in D_1$, $D_1 - \{u\}$ is a $\gamma_\mathcal{H}$-set of $G - u$ (by Theorem A) and $v \in D_1 - \{u\}$. In addition it follows that $u \in V_{\mathcal{H}}(G - e)$. Thus, (i) holds.

By symmetry we still have the case when $S = \{x_1, x_3\}$. If $u \notin pn_{G_{e,3}}[x_1, \bar{D}]$ and $v \notin pn_{G_{e,3}}[x_3, \bar{D}]$ then $\bar{D} - \{x_1, x_3\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. If $u \in pn_{G_{e,3}}[x_1, \bar{D}]$ and $v \notin pn_{G_{e,3}}[x_3, \bar{D}]$ then $(\bar{D} - \{x_1, x_3\}) \cup \{u\}$ is a dominating $\mathcal{H}$-set of $G - e$, a contradiction. So, $u \in pn_{G_{e,3}}[x_1, \bar{D}]$ and $v \in pn_{G_{e,3}}[x_3, \bar{D}]$. Then $D_2 = (\bar{D} - \{x_1, x_3\}) \cup \{u, v\}$ is a $\gamma_\mathcal{H}$-set of $G - e$ and both $\{u\} = pn_{G - e}[x_1, D_2]$ and $\{v\} = pn_{G - e}[x_3, D_2]$ hold. Thus both (i) and (ii) are fulfilled.

$(A_3) \iff (A_3)$ By Theorem B.

Proof of Theorem 3.2. (i) Necessity: Let $\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G_{e,3})$. By Theorem 3.1 we know that $\gamma_\mathcal{H}(G - e) \leq \gamma_\mathcal{H}(G_{e,3}) \leq \gamma_\mathcal{H}(G - e) + 1$ and if $\gamma_\mathcal{H}(G - e) = \gamma_\mathcal{H}(G_{e,3})$ then $\gamma_\mathcal{H}(G_{e,3}) = \gamma_\mathcal{H}(G) + 1$. Thus $\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G_{e,3}) = \gamma_\mathcal{H}(G - e) + 1$.

Sufficiency: Let $\gamma_\mathcal{H}(G - e) + 1 = \gamma_\mathcal{H}(G)$. Assume $\gamma_\mathcal{H}(G) \neq \gamma_\mathcal{H}(G_{e,3})$. Now by Theorem 3.1, $\gamma_\mathcal{H}(G_{e,3}) = \gamma_\mathcal{H}(G - e)$. Applying again Theorem 3.1 we obtain $\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G - e) - 1$, a contradiction. Thus, $\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G_{e,3})$.

(ii) By (i), $\gamma_\mathcal{H}(G) = \gamma_\mathcal{H}(G_{e,3})$. Let $M$ be a $\gamma_\mathcal{H}$-set of $G - e$ and $e = uv$. By Theorem C (ii), both $u$ and $v$ are in $M$. Then
(a) $M$ is a dominating $H$-set of $G_{e,1}$ and $G_{e,2}$.
(b) $M \cup \{x_3\}$ is a dominating $H$-set of $G_{e,4}$ and $G_{e,5}$, and
(c) $M \cup \{x_3, x_5\}$ is a dominating $H$-set of $G_{e,6}$. Hence

(A) $\gamma_H(G_{e,i}) \leq \gamma_H(G - e) = \gamma_H(G) - 1$ for $i = 1, 2$; $\gamma_H(G_{e,j}) \leq \gamma_H(G - e) + 1 = \gamma_H(G)$ for $i = 4, 5$; $\gamma_H(G_{e,6}) \leq \gamma_H(G - e) + 2 = \gamma_H(G) + 1$.

By Theorem C, $\min\{\gamma_H(G - u), \gamma_H(G - v)\} \geq \gamma_H(G - e)$ and by Theorem A we have $\gamma_H(G - \{u, v\}) = \gamma_H((G - u) - v) \geq \gamma_H(G - u) - 1 \geq \gamma_H(G - e) - 1$. Suppose that $\gamma_H(G - \{u, v\}) = \gamma_H(G - e) - 1$. Then both $\gamma_H(G - u) = \gamma_H(G - e)$ and $\gamma_H((G - u) - v) = \gamma_H(G - u) - 1$ hold. By the second equality and Theorem A we deduce that $v$ belongs to some $\gamma_H$-set of $G - u$. On the other hand, since $\gamma_H(G) = \gamma_H(G - e) + 1 > \gamma_H(G - u), v$ belongs to no $\gamma_H$-set of $G - u$, a contradiction. Thus,

(B) $\min\{\gamma_H(G - u), \gamma_H(G - v), \gamma_H(G - \{u, v\})\} \geq \gamma_H(G - e)$.

Let $D_t$ be a $\gamma_H$-set of $G_{e,t}$ and $U_t = D_t \cap \{x_1, \ldots, x_t\}$, where $t = 1, \ldots, 6$.

Case 1: $t \in \{1, 2\}$. Assume $U_t \neq \emptyset$. Then $D_t - U_t$ is a dominating $H$-set for at least one of the graphs $G - e$, $G - u$, $G - v$ and $G - \{u, v\}$. Using (B) we have

$$\gamma_H(G) = \gamma_H(G - e) + 1 \leq |D_t - U_t| + 1 = \gamma_H(G_{e,t}) - |U_t| + 1 \leq \gamma_H(G_{e,t}),$$

contradicting (A). Thus $U_t$ is empty. But then $D_t$ is a dominating $H$-set of $G - e$, which leads to $\gamma_H(G_{e,t}) \geq \gamma_H(G - e)$. Now by (A) the equality $\gamma_H(G_{e,t}) = \gamma_H(G - e)$ follows.

Case 2: $t \in \{4, 5\}$. Obviously $U_t \neq \emptyset$. As in Case 1 we obtain $\gamma_H(G) \leq \gamma_H(G_{e,t})$. Since by (A) $\gamma_H(G_{e,t}) \leq \gamma_H(G)$, we have $\gamma_H(G_{e,t}) = \gamma_H(G)$.

Case 3: $t = 6$. Clearly $|U_6| \geq 2$. As in Case 1 we obtain $\gamma_H(G) \leq \gamma_H(G_{e,6}) - |U_6| + 1$. Since $|U_6| \geq 2$, we have $\gamma_H(G) \leq \gamma_H(G_{e,6}) - 1$. Now by (A) we deduce that $\gamma_H(G) = \gamma_H(G_{e,6}) - 1$.

(iii) Immediately by (i) and (ii). \(\square\)

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