ON THE EQUIVALENCE OF DIFFERENTIAL OPERATORS OF INFINITE ORDER WITH CONSTANT COEFFICIENTS

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Received January 18, 2016. First published November 25, 2016.
Communicated by Vladimír Müller

Abstract. We investigate the conditions of equivalence of a differential operator of infinite order with constant coefficients to the operator of differentiation in one space of analytic functions. We also study the conditions of continuity of a differential operator of infinite order with variable coefficients in such space.

Keywords: space of analytic functions; operator of differentiation of infinite order; equivalence of operators; commutant

MSC 2010: 47B38

1. Introduction

Two linear continuous operators A and B acting in a topological vector space H are called equivalent if there exists an isomorphism T of H such that $TA = BT$. The conditions of equivalence of differential operators from different classes that act in functional spaces are frequent objects of investigation in functional analysis. These studies were initiated by Delsarte [1] for spaces of functions of a real variable and by Delsarte and Lions in [2] for spaces of entire functions. In the space of analytic functions the conditions of equivalence of different classes of differential operators were obtained by many mathematicians (see e.g. [3]–[6]). In particular, conditions of equivalence of a differential operator of infinite order with constant coefficients to the operator of multiple differentiation in spaces of analytic functions in circular domains were studied in [5]. Maldonado, Prada, Senosiain in [7] investigated the conditions of equivalence of a differential operator of infinite order with constant coefficients to the operator of differentiation in the space of functions which is isomorphic to the space s of rapidly decreasing sequences. In this paper we study some properties

of differential operators of infinite order in the space s. We also fix the mistaken statement of Theorem 3 from [7].

We denote by s the space of all functions of the form $f(z) = \sum\limits_{n=0}^{\infty} f_n z^n$ of complex variable, where $f_n \in \mathbb{C}$, $n = 0, 1, \ldots$, and

$$\|f\|_k = |f_0| + \sum_{n=1}^{\infty} |f_n| n^k < \infty$$

for all $k \in \mathbb{N}$. The topology on the space s is generated by the system of norms $\{\|\cdot\|_k \colon\ k \in \mathbb{N}\}$ (see [7], [8]). The symbol $\mathcal{L}(s)$ stands for the set of all linear continuous operators in s. Note that any f of s is an analytic function in the unit disc $|z| < 1$. We start with auxiliary properties of differential operators of infinite order in the space s.

2. Applicability of differential operators of infinite order

Note that the product of any f, g of s belongs to the space s. Herewith, $M_g \in \mathcal{L}(s)$, where $M_g f = gf$.

Theorem 2.1. *Let $(\psi_n)_{n=0}^{\infty}$ be a sequence of functions of s such that the operator $L\colon\ s \to s$ defined by the formula*

$$(Lf)(z) = \sum_{n=0}^{\infty} \psi_n(z) f^{(n)}(z) \tag{1}$$

is continuous and linear, and the series on the right-hand side of (1) *is convergent in s for any $f \in s$. Then there exists $N \in \mathbb{N}$ such that $\psi_n = 0$, $n > N$.*

Proof. For each $n = 0, 1, \ldots$ the formula $(L_n f)(z) = \psi_n(z) f^{(n)}(z)$ defines a linear continuous operator on s. The condition of the theorem implies that the sequence $(L_n f)_{n=0}^{\infty}$ is bounded in s. Therefore the sequence of operators $(L_n)_{n=0}^{\infty}$ is pointwisely bounded in s. Since s is the Fréchet space [8], according to the Uniform Boundedness Principle this sequence of operators is equicontinuous. Then we have

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \exists\, C > 0\ \forall\, n \geqslant 0\ \forall\, f \in s\colon\ \|L_n f\|_p \leqslant C \|f\|_q. \tag{2}$$

Setting $f(z) = z^k$ in (2), we get

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \exists\, C > 0\ \forall\, n \geqslant 0\ \forall\, k \geqslant n\colon\ \|\psi_n(z) z^{k-n}\|_p \leqslant C \frac{k^q (k-n)!}{k!}. \tag{3}$$

Fix any $p \in \mathbb{N}$ and take q and C such that (3) holds. We show that $\psi_n = 0$ for all $n > q - p$. Fix any $n > q - p$. Let $\psi_n(z) = \sum\limits_{j=0}^{\infty} \psi_{j,n} z^j$. Then for all $k > n$ we get

$$\|\psi_n(z) z^{k-n}\|_p = \sum_{j=0}^{\infty} |\psi_{j,n}| (j + k - n)^p. \tag{4}$$

Since $(k-n)!/k! < 1/(k-n)^n$ for $k > n$, using (3) and (4) we have

$$|\psi_{j,n}| \leqslant \frac{\|\psi_n(z) z^{k-n}\|_p}{(j+k-n)^p} \leqslant C \frac{k^q (k-n)!}{(j+k-n)^p k!} \leqslant C \frac{k^q}{(k-n)^{n+p}}$$

for all $j = 0, 1, \ldots$, $k > n$. Thus,

$$|\psi_{j,n}| \leqslant C \frac{k^q}{(k-n)^q} \frac{1}{(k-n)^{n+p-q}} \tag{5}$$

for all $j = 0, 1, \ldots$ and $k > n$. Fix an arbitrary $j \geqslant 0$. Passing to the limit when $k \to \infty$ in (5) we get $\psi_{j,n} = 0$. Therefore $\psi_n = 0$. $\square$

3. Commutant of D in s

Theorem 3.1. *In order that the operator T belongs to the class $\mathcal{L}(s)$ and commutes with the operator of differentiation D, it is necessary and sufficient that T has the form*

$$T = \sum_{j=0}^{m} c_j D^j, \tag{6}$$

where m is a positive integer, c_j, $j = \overline{0, m}$ are complex numbers.

Proof. Necessity. Let $T \in \mathcal{L}(s)$ commute with the operator of differentiation, i.e.

$$TD = DT. \tag{7}$$

Denote $Tz^n = t_n(z)$, $n = 0, 1, \ldots$ We show that there is a sequence of complex numbers $(c_n)_{n=0}^{\infty}$ such that

$$t_n(z) = \sum_{j=0}^{n} \frac{n!}{(n-j)!} c_j z^{n-j}, \tag{8}$$

$n = 0, 1, \ldots$ Applying (7) to z^n, $n = 0, 1, \ldots$, we get $t_0(z) = c_0$, where $c_0 \in \mathbb{C}$ and

$$t_n'(z) = n t_{n-1}(z), \tag{9}$$

$n = 1, 2, \ldots$ The validity of (8) is proved by induction using (9).

We show now that the sequence $(c_n)_{n=0}^{\infty}$ is finite. Since $T \in \mathcal{L}(s)$, we have

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \exists\, C > 0\ \forall\, f \in s\colon\ \|Tf\|_p \leqslant C\|f\|_q. \tag{10}$$

Fix an arbitrary $p \in \mathbb{N}$ and find a $q \in \mathbb{N}$ and $C > 0$ using (10). Setting $f(z) = z^n$ in (10), we get $\|t_n\|_p \leqslant Cn^q$, $n \in \mathbb{N}$. Since

$$\|t_n\|_p = n!|c_n| + \sum_{j=0}^{n-1} \frac{n!}{(n-j)!}|c_j|(n-j)^p \geqslant \frac{n!}{(n-j)!}|c_j|(n-j)^p,$$
$$|c_j| \leqslant \frac{(n-j)!}{(n-j)^p n!}\|t_n\|_p \leqslant C\frac{n^q}{(n-j)^p}\frac{(n-j)!}{n!} \leqslant C\frac{n^q}{(n-j)^{p+j}}$$

for all $n \in \mathbb{N}$ and $j = \overline{0, n-1}$. Thus,

$$|c_j| \leqslant C\frac{n^q}{(n-j)^{p+j}} \tag{11}$$

for all $n \in \mathbb{N}$ and $j = \overline{0, n-1}$. We choose an arbitrary positive integer m such that $m > q - p$. Fix an arbitrary $j > m$. Then $j > q - p$. Using (11) for $n > j + 1$ and letting n tend to infinity in (11), we get that $|c_j| = 0$. Thus, $c_j = 0$ for all $j > m$. Then (8) implies that

$$t_n(z) = \sum_{j=0}^{\min\{m,n\}} \frac{n!}{(n-j)!}c_j z^{n-j} = \sum_{j=0}^{\min\{m,n\}} c_j D^j(z^n) = \left(\sum_{j=0}^{m} c_j D^j\right)(z^n)$$

for all $n = 0, 1, \ldots$ Since the system $(z^n)_{n=0}^{\infty}$ forms a basis in s, the continuity of T implies that for any $f(z) = \sum\limits_{j=0}^{\infty} f_n z^n$ of s the equality

$$\begin{aligned}(Tf)(z) &= \sum_{n=0}^{\infty} f_n \left(\sum_{j=0}^{m} c_j D^j\right)(z^n) = \left(\sum_{j=0}^{m} c_j D^j\right)\left(\sum_{j=0}^{\infty} f_n z^n\right)\\ &= \sum_{j=0}^{m} c_j (D^j f)(z)\end{aligned}$$

holds. Thus, T can be represented in the form (6). The necessity of conditions of Theorem 3.1 is proved. The sufficiency of conditions of the theorem is obvious. $\square$

Since the set of operators from $\mathcal{L}(s)$ which are commuting with the operator of differentiation coincides with the set of polynomials with respect to the differentiation, the operator of differentiation is the minimally commuting operator in the space s.

4. The main result

We proceed now to solve the main problem of studying conditions of equivalence of the operator of differentiation of infinite order with constant coefficients to the operator of differentiation in s.

Lemma 4.1. *Let $(\varphi_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that the operator $\varphi(D) = \sum\limits_{n=0}^{\infty} \varphi_n D^n$ acts in s linearly and continuously. If $\varphi(D)$ is equivalent to D^m in s for some $m \in \mathbb{N}$, then $\varphi(D) = \sum\limits_{n=0}^{m} \varphi_n D^n$ and $\varphi_m \neq 0$.*

Proof. Since the operator of differentiation of infinite order with constant coefficients $\varphi(D) = \sum\limits_{n=0}^{\infty} \varphi_n D^n$ acts in the space s linearly and continuously, by Theorem 2.1 this operator has a finite order. Then there exists $N \in \mathbb{N}$ such that $\varphi(D) = \sum\limits_{n=0}^{N} \varphi_n D^n$ and $\varphi_N \neq 0$. Note that the dimension of the kernel of the operator of differentiation of finite order acting in the space s is equal to the order of this operator. Since equivalent operators have equal dimensions of kernels, $N = m$ and $\varphi_m \neq 0$. $\square$

Remark 4.1. The statement of Lemma 4.1 implies the correctness of the hypothesis, which is formulated in Remark 1 of [7].

Theorem 4.1. *Let $(\varphi_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that the operator $\varphi(D) = \sum\limits_{n=0}^{\infty} \varphi_n D^n$ acts in s linearly and continuously. In order that $\varphi(D)$ is equivalent to D in s, it is necessary and sufficient that $\varphi(D) = \varphi_0 I + \varphi_1 D$ and $|\varphi_1| = 1$.*

Proof. Necessity. Assume that the operator $\varphi(D)$ is equivalent to D in the space s. Then Lemma 4.1 implies that $\varphi(D) = \varphi_0 I + \varphi_1 D$, $\varphi_1 \neq 0$. It remains to prove that $|\varphi_1| = 1$. The operator T_0, which acts according to the rule $(T_0 f)(z) = \exp(\varphi_0 z/\varphi_1) f(z)$, is an isomorphism of s such that the equality $T_0(\varphi_0 I + \varphi_1 D) = (\varphi_1 D) T_0$ holds. Then the operator $\varphi_0 I + \varphi_1 D$ is equivalent to $\varphi_1 D$ in s. As a consequence of transitivity of equivalence we obtain that the operator $\varphi_1 D$ is equivalent to the operator D in the space s. Then there exists an isomorphism T of s such that the equality $T(\varphi_1 D) = DT$ holds. Let $Tz^n = t_n(z)$, $n = 0, 1, \ldots$ Then there exists a sequence of complex numbers $(c_n)_{n=0}^{\infty}$, such that

$t_n(z) = \sum\limits_{k=0}^{n} (n!/k!)\varphi_1^k c_{n-k} z^k$, $n = 0, 1, \ldots$ These equalities can be proved by induction on n. Herewith $T1 = t_0(z) = c_0 \neq 0$, because the operator T is an isomorphism of S. Since $T \in \mathcal{L}(s)$,

$$\forall\, p \in \mathbb{N}\ \exists\, q \in \mathbb{N}\ \exists\, C > 0\ \forall\, f \in s\colon\ \|Tf\|_p \leqslant C\|f\|_q. \tag{12}$$

Fix an arbitrary $p \in \mathbb{N}$ and find $q \in \mathbb{N}$, $C > 0$ according to (12). Setting in (12) $f(z) = z^n$, we get $\|t_n\|_p \leqslant Cn^q$ for $n \in \mathbb{N}$. Since

$$\|t_n\|_p = n!|c_n| + \sum_{k=1}^{n} \frac{n!}{k!}|\varphi_1|^k |c_{n-k}| k^p \geqslant |\varphi_1|^n |c_0| n^p,$$
$$|\varphi_1|^n \leqslant \frac{\|t_n\|_p}{|c_0| n^p} \leqslant \frac{C}{|c_0|} n^{q-p}$$

for all $n \in \mathbb{N}$. It follows that $|\varphi_1| \leqslant \sqrt[n]{C/|c_0|}(\sqrt[n]{n})^{q-p}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $|\varphi_1| \leqslant 1$. Since $\varphi_1 D$ and D are equivalent in s, $(1/\varphi_1)D$ and D are equivalent in s. Then, according to the above-proved, $|1/\varphi_1| \leqslant 1$, i.e. $|\varphi_1| \geqslant 1$. Thus, $|\varphi_1| = 1$ and the necessity of conditions of Theorem 4.1 is proved.

Sufficiency. Let $\varphi(D) = \varphi_0 I + \varphi_1 D$ and $|\varphi_1| = 1$. The operator T, which acts according to the rule $(Tf)(z) = \exp(\varphi_0 z) f(\varphi_1 z)$, is an isomorphism of s such that $T(\varphi_0 I + \varphi_1 D) = DT$. Thus, the operator $\varphi_0 I + \varphi_1 D$ is equivalent to D in s. $\square$

Remark 4.2. Theorem 3 in [7] asserts that the operator $\varphi(D) = \sum\limits_{n=0}^{\infty} \varphi_n D^n$ is equivalent to D in the space s if and only if $\varphi(D) = \varphi_0 I + \varphi_1 D$, with $\varphi_1 \neq 0$. It follows from the theorem we have proved above that this statement is not correct, because the condition $|\varphi_1| = 1$ is missed in Theorem 3 from [7].

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