GOLDIE EXTENDING ELEMENTS IN MODULAR LATTICES

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Abstract. The concept of a Goldie extending module is generalized to a Goldie extending element in a lattice. An element a of a lattice L with 0 is said to be a Goldie extending element if and only if for every $b \leqslant a$ there exists a direct summand c of a such that $b \wedge c$ is essential in both b and c. Some properties of such elements are obtained in the context of modular lattices. We give a necessary condition for the direct sum of Goldie extending elements to be Goldie extending. Some characterizations of a decomposition of a Goldie extending element in such a lattice are given. The concepts of an a-injective and an a-ejective element are introduced in a lattice and their properties related to extending elements are discussed.

Keywords: modular lattice; Goldie extending element

MSC 2010: 06B10, 06C05

1. Introduction

The notion of a ring in which every complement right ideal is a direct summand was introduced by Chatters and Hajarnavis [3]. They called such a ring a CS-ring (complements are summands). This notion has been studied by several researchers in the context of modules under the names an extending module or a module with C_1-property or a CS-module. These modules and their generalizations have been studied by several authors such as Harmanci and Smith [9], Akalan, Birkenmeier and Tercan [1], Dung et al. [5]. A module M is called *extending* if every submodule of M is a direct summand of M. In [6], Noyan Er studied the ring whose modules are direct sums of extending modules.

Călugăreanu [2] used lattice theory in module theory and studied several concepts from module theory in lattice theory. Keskin [11] obtained some properties of extending modules using modular lattices.

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In [10], Kamal and Sayed generalized the concept of an extending module by using the concept of an I-jective ideal. In [1], Akalan, Birkenmeier and Tercan defined a Goldie extending module over an associative ring. They defined two relations α and β on modules over associative rings and by using the relation β, formulated the concept of a Goldie extending module. They posed the following open problem.

Open problem. Determine necessary and/or sufficient conditions for a direct sum of Goldie extending modules to be Goldie extending.

In the present paper, we obtain a lattice theoretic analogue of the results of Akalan, Birkenmeier and Tercan [1] and answer this open problem in the context of certain modular lattices. We define a Goldie extending element in a lattice with 0 and obtain some properties of such elements in certain modular lattices by using the concept of ejectivity. The second section deals with preliminaries required in the subsequent sections. In the third section, we define an a-injective element in a lattice and prove some of its properties. In the last section we introduce the concept of a Goldie extending element and an ejective element in a lattice and give a characterization of a Goldie extending element by using the ejectivity of direct summands in certain modular lattices.

2. Preliminaries

The concepts from the lattice theory used in this paper are from Grätzer [7] and Crawley and Dilworth [4].

Throughout this paper L denotes a lattice with the least element 0.

The following remark can be proved by using modularity of L.

Remark 2.1. Let L be a modular lattice and $a, b, c \in L$. If $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$ then $a \wedge (b \vee c) = 0$.

Grzeszczuk and Puczyłowski, [8] developed the concept of Goldie dimension from the module theory, to modular lattices. In this context they defined the concept of an essential element in a lattice with the least element 0, see also Călugăreanu [2].

Definition 2.1 ([2], page 39)**.** An element $a \in L$ is called essential in $b \in L$ (or b is an essential extension of a), if there is no nonzero $c \leqslant b$ with $a \wedge c = 0$. We then write $a \leqslant_e b$. If $a \leqslant_e b$ and there is no $c > b$ such that $a \leqslant_e c$, then b is called a maximal essential extension of a. If $a \in L$ and there is no nonzero $c \in L$ such that $a \wedge c = 0$, then we say that a is *essential in* L.

Definition 2.2 ([2], page 39)**.** An element a is closed (or essentially closed) in b, if a has no proper essential extension in b. We denote this by $a \leqslant_{\mathrm{cl}} b$. If a does not have a proper essential extension in L, then we say that a is closed in L.

The concepts of a *semicomplement and a maximal semicomplement* are known in lattices with 0, see Szász [13], page 47. Let $a, b \in L$. We say that a, b are semicomplements of each other, if $a \wedge b = 0$.

Definition 2.3. If $a, b \in L$ and b is a maximal element in the set $\{x\colon x \in L, a \wedge x = 0\}$, then we say that b is a maximal semicomplement of a. In short we say that b is a *max-semicomplement* of a in L.

A max-semicomplement is called as a *pseudo-complement* by Călugăreanu [2], page 39, and Keskin [11]. However, in order to distinguish it from the concept of the pseudocomplement of an element in a lattice (a is called a pseudocomplement of b, if a is the largest element with the property $b \wedge a = 0$, see Grätzer [7], page 63), we use the term *max-semicomplement.*

In the lattice L shown in Figure 1, g, h are max-semicomplements of f but f does not have a pseudocomplement in L.

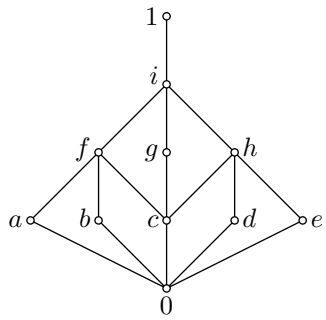

Figure 1.

In the theory of modules, (e.g. Lam [12], Proposition 6.24, page 215, Chatters and Hajarnavis [3], Proposition 2.2, and others), it is known that if A, B, C are modules of a ring R with $A \subseteq B \subseteq C$ and if A is closed in B, B is closed in C, then A is closed in C. The following proposition is an analog of this result for the elements of a modular lattice with 0, the proof of which is due to F. Wehrung.[1]

Proposition 2.1. *Let L be a modular lattice with 0. For $a, b, c \in L$, if $a \leqslant_{\mathrm{cl}} b$ and $b \leqslant_{\mathrm{cl}} c$, then $a \leqslant_{\mathrm{cl}} c$.*

Proof. Necessarily, $a < b < c$. Suppose that $a \nleqslant_{\mathrm{cl}} c$. By the definition of $\leqslant_{\mathrm{cl}}$, there exists x with $a < x \leqslant_{\mathrm{cl}} c$ and $a \leqslant_e x$. Since $a \leqslant_e x \wedge b \leqslant b$ and $a \leqslant_{\mathrm{cl}} b$, we get

$$a = x \wedge b. \tag{2.1}$$

[1] Friedrich Wehrung: A note on a partial ordering in modular lattices. Unpublished work (June, 2014), private communication. The authors are thankful to Professor Friedrich Wehrung for granting permission to include the proof.

In particular, if $x \leqslant b$, then $a = x$, a contradiction; so $x \nleqslant b$, and so $b < b \vee x \leqslant c$. Since $b \leqslant_{\mathrm{cl}} c$, it follows that $b \nleqslant_e b \vee x$, thus there exists u such that $0 < u \leqslant b \vee x$ and $u \wedge b = 0$.

In particular, $u \wedge x \leqslant x$ and $a \wedge u \wedge x \leqslant b \wedge u = 0$, thus, since $a \leqslant_e x$, we get

$$u \wedge x = 0. \tag{2.2}$$

Suppose that $a = (u \vee x) \wedge b$. We compute

$$\begin{aligned} u &= u \wedge (b \vee x) = u \wedge (u \vee x) \wedge (b \vee x) = u \wedge (((u \vee x) \wedge b) \vee x) \quad \text{(by modularity)} \\ &= u \wedge (a \vee x) \quad \text{(by assumption)} = 0 \text{ (because } a \leqslant x \text{ and by (2.2))}; \end{aligned}$$

a contradiction. Hence, $a < (u \vee x) \wedge b \leqslant b$. Since $a \leqslant_{\mathrm{cl}} b$, we get $a \nleqslant_e (u \vee x) \wedge b$, thus there exists v such that

$$0 < v \leqslant (u \vee x) \wedge b \quad \text{and} \quad a \wedge v = 0. \tag{2.3}$$

Then it follows from (2.1) that $x \wedge v = x \wedge b \wedge v = a \wedge v = 0$. Thus, if $u \wedge (x \vee v) = 0$, then the triple (u, v, x) is independent, thus $v \wedge (x \vee u) = 0$, a contradiction since $0 < v \leqslant x \vee u$. This shows that the element $u' = u \wedge (x \vee v)$ is nonzero. Moreover, it follows from (2.3) together with modularity that

$$x \vee u' = (x \vee u) \wedge (x \vee v) \geqslant v,$$

so we may replace u by u' without changing the validity of (2.3). Therefore, we may assume that

$$0 < u \leqslant x \vee v \quad \text{and} \quad u \wedge b = 0. \tag{2.4}$$

One immediate consequence of (2.3) and (2.4) is that $x \vee u = x \vee v$.

From $a \vee v \leqslant b$ and $u \wedge b = 0$ it follows that $u \wedge (a \vee v) = 0$, thus, since $a \wedge v = 0$ and by modularity, the triple (a, u, v) is independent.

Moreover, since $a \leqslant x$ and by (2.2) together with modularity,

$$x \wedge (a \vee u) = a \vee (x \wedge u) = a.$$

A similar argument, now using the equality $x \wedge v = 0$, yields $x \wedge (a \vee v) = a$.

Set $x_1 = x \wedge (a \vee u \vee v)$.

We claim that the elements $a \vee u$, $a \vee v$, and x_1 are the atoms of an M_3, with bottom a and top $a \vee u \vee v$. The equality $(a \vee u) \wedge (a \vee v) = a$ follows from the fact that

the triple (a, u, v) is independent, while the equalities $x_1 \wedge (a \vee u) = x_1 \wedge (a \vee v) = a$ follow from the corresponding equalities with x_1 replaced by x, established above (note that $a \leqslant x_1 \leqslant x$). It is trivial that $x_1 \leqslant (a \vee u) \vee (a \vee v) = a \vee u \vee v$. Finally, by modularity and since $x \vee u = x \vee v \geqslant a \vee u \vee v$, we get

$$x_1 \vee (a \vee u) = (x \vee a \vee u) \wedge (a \vee u \vee v) = a \vee u \vee v.$$

Similarly, $x_1 \vee (a \vee v) = a \vee u \vee v$. This completes the proof of our claim. In particular, from $a \wedge v = 0$ and $v > 0$ it follows that $a < a \vee v$, thus $a < x_1$.

Now let $x_0 = x \wedge (u \vee v)$. Since $a \leqslant x$ and by modularity,

$$x_0 \vee a = x \wedge (a \vee u \vee v) = x_1,$$

thus $x_0 \vee a > a$, and thus $x_0 > 0$. However, $x_0 \wedge a = a \wedge (u \vee v) = 0$ by using the fact that the triple (a, u, v), is independent and $x_0 \leqslant x$. This contradicts our assumption that $a \leqslant_e x$. □

The following example shows that Proposition 2.1 need not hold in a nonmodular lattice.

Example 2.1. We note that in the lattice L shown in Figure 1, $c \leqslant_{\mathrm{cl}} f$ and $f \leqslant_{\mathrm{cl}} i$. But c is not closed in i, as $c \leqslant_e g \leqslant i$.

Definition 2.4. If $a, b, c \in L$ are such that $a \vee b = c$ and $a \wedge b = 0$, then a and b are called direct summands of c and we write $c = a \oplus b$. We say that c is a direct sum of a and b.

Lemma 2.1. *In a modular lattice L, if $a, b, c \in L$ are such that $c = a \oplus b$ then a is a max-semicomplement of b in c.*

Proof. Let $a, b, c \in L$ be such that $c = a \oplus b$. Let $d \in L$ be such that $a \leqslant d \leqslant c$ and $d \wedge b = 0$. Now, by modularity we get

$$d = c \wedge d = (a \vee b) \wedge d = a \vee (b \wedge d) = a.$$

Hence a is a max-semicomplement of b in c. □

From this we also conclude that the direct summands of c are closed in c.

However, Lemma 2.1 does not hold if L is a nonmodular lattice. Considering the lattice shown in Figure 1, we note that $i = a \oplus e$, but e is not a max-semicomplement of a in i.

R e m a r k 2.2. Let L be a modular lattice and let $a, b, c \in L$ be such that $a \leqslant b \leqslant c$. If a is a direct summand of c then a is also a direct summand of b. For, if $c = a \oplus d$, then by modularity $b = (a \vee d) \wedge b = a \vee (d \wedge b) = a \oplus (d \wedge b)$.

Now, we establish some properties of essential extensions and closed extensions, the proofs of which are similar to the module case.

Lemma 2.2. *In a lattice L the following statements hold.*

(1) *If $a, b, c \in L$, then $a \leqslant_e b$ implies $a \wedge c \leqslant_e b \wedge c$.*
(2) *If $a \leqslant b \leqslant c$, then $a \leqslant_e b$, $b \leqslant_e c$ if and only if $a \leqslant_e c$* ([8], *Lemma* 2).

Lemma 2.3 ([8], Lemma 3)**.** *Let L be a modular lattice. Suppose that $a, b, c, d \in L$ are such that $a \leqslant b$, $c \leqslant d$ and $b \wedge d = 0$. Then $a \leqslant_e b$, $c \leqslant_e d$ if and only if $a \oplus c \leqslant_e b \oplus d$.*

Lemma 2.4. *Let L be a modular lattice and $a, b, c \in L$, $a \leqslant b \leqslant c$. If d is a max-semicomplement of a in c then $d \wedge b$ is a max-semicomplement of a in b.*

Throughout this paper, wherever necessary, we assume that L satisfies one or more of the following conditions:

Condition (i): For any $a \leqslant b$ there exists a maximal essential extension of a in b.

Condition (ii): For any $a \leqslant b$ and for any $c \leqslant b$ with $c \wedge a = 0$, there exists a max-semicomplement $d \geqslant c$ of a in b.

Condition (iii): If the socle is involved $\mathrm{Soc}(a)$ exists for any $a \in L$.

A familiar and important class of lattices with these properties is that of upper continuous modular lattices, in particular, the lattice of ideals of a modular lattice with 0.

The following lemma is proved for upper continuous modular lattices by Călugăreanu [2], Corollary 4.3, page 42.

Lemma 2.5. *Let L be a modular lattice satisfying the Condition* (ii)*. Let $a, b \in L$ and $a \leqslant b$. Then a is closed in b if and only if a is a max-semicomplement of some $c \leqslant b$.*

P r o o f. Suppose that a is a max-semicomplement of some $c \leqslant b$ and $a \leqslant_e d \leqslant b$. Then $a \wedge c \wedge d = 0$ implies that $c \wedge d = 0$. Since a is a max-semicomplement of c, this implies that $a = d$. Thus a is closed in b.

Conversely, suppose that a is closed in b. Then there exists $c \leqslant b$ such that $a \wedge c = 0$. Since L satisfies the Condition (ii), there exists a max-semicomplement $d \geqslant c$ of a in b. Let f be such that $a \leqslant f \leqslant b$ and $f \wedge d = 0$. Let $g \leqslant f$ be such that

$a \wedge g = 0$. Clearly, $(a \vee g) \wedge d = 0$. Hence by Remark 2.1, $a \wedge (g \vee d) = 0$. Since d is a max-semicomplement of a, this implies that $g \vee d = d$. Since $g \leqslant f$ and $f \wedge d = 0$, we conclude that $g = 0$. Thus $a \leqslant_e f \leqslant b$. This implies that $a = f$. Thus a is a max-semicomplement of d in b. $\square$

The following example shows that Lemma 2.5 does not hold for a general modular lattice.

Example 2.2. Let $L = \{A \subseteq \mathbb{N}\colon\ A \text{ is finite}\} \cup \{\mathbb{N}\}$. Then L is a modular lattice. Any nonempty $A \in L$ is closed in $\mathbb{N}$ but A does not have a max-semicomplement in $\mathbb{N}$.

The following proposition is from Călugăreanu [2], Proposition 4.4, page 43.

Proposition 2.2. *Let L be a modular lattice satisfying the Condition* (ii) *and let $a, b \in L$ be such that $a \wedge b = 0$. Then a is a max-semicomplement of b in L if and only if a is closed in L and $a \vee b$ is essential in L.*

3. J-injective elements

The following properties are well known in the module theory, see Akalan et al. [1].

(1) C_3 property: If M_1, M_2 are direct summands of a module M, such that $M_1 \cap M_2 = \{0\}$, then $M_1 \oplus M_2$ is a direct summand of M.

(2) The summand intersection property: If M_1, M_2 are direct summands of a module M, then $M_1 \cap M_2$ is a direct summand of M.

We introduce these concepts in a lattice L. We denote the set of all direct summands of an element $a \in L$ by $\mathfrak{D}(a)$. That is, for every $b \in \mathfrak{D}(a)$ there exists $c \in \mathfrak{D}(a)$ such that $a = b \oplus c$.

Definition 3.1. Let $a \in L$. If for any two direct summands b, c of a, $b \vee c$ is a direct summand of a, then we say that a satisfies the summand sum property.

Example 3.1. Consider the lattice shown in Figure 2. We note that a and b are direct summands of i and $a \vee b = f$ is also a direct summand of i. Similarly, we can check for other direct summands of i. Hence i satisfies the summand sum property.

Proposition 3.1. *Let L be a modular lattice. If $a \in L$ satisfies the summand sum property, then every direct summand of a satisfies the summand sum property.*

Proof. Let $b \in \mathfrak{D}(a)$ and $c, d \in \mathfrak{D}(b)$. Since b is a direct summand of a, by Remark 2.2, c and d are also direct summands of a. As a satisfies the summand sum property, $c \vee d$ is a direct summand of a. Hence by Remark 2.2, $c \vee d$ is a direct summand of b. $\square$

Definition 3.2. Let $a \in L$ and $b, c \in \mathfrak{D}(a)$. If $b \wedge c \in \mathfrak{D}(a)$ whenever $b \wedge c \neq 0$, then we say that a satisfies the summand intersection property.

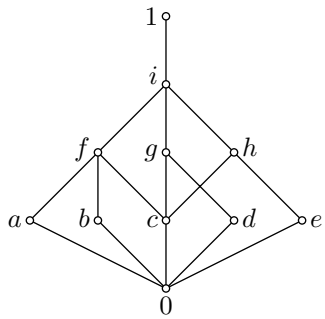

Figure 2.

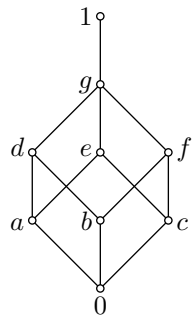

Figure 3.

E x a m p l e 3.2. Consider the lattice shown in Figure 3. We note that d and f are direct summands of g. Also, $d \wedge f = b$ is a direct summand of i. Similarly, we can check for other direct summands of i. Hence i satisfies the summand intersection property.

E x a m p l e 3.3. Consider the elements i, f, h in the lattice shown in Figure 2. Then f and h are direct summands of i. But $f \wedge h = c$ is not a direct summand of i. Hence i does not satisfy the summand intersection property.

Proposition 3.2. *Let L be a modular lattice. If $a \in L$ satisfies the summand intersection property, then every direct summand of a satisfies the summand intersection property.*

P r o o f. Let $a, b \in L$ and $b \in \mathfrak{D}(a)$. Let $c, d \in \mathfrak{D}(b)$ be such that $c \wedge d \neq 0$. Since b is a direct summand of a, by Remark 2.2, c and d are also direct summands of a. As a satisfies the summand intersection property, $c \wedge d$ is a direct summand of a. Then $c \wedge d$ is a direct summand of b. $\square$

Harmanci and Smith [9], Lemma 5, have proved the following lemma.

Lemma 3.1. *Let a module $M = M_1 \oplus M_2$ be the direct sum of submodules M_1, M_2. Then the following statements are equivalent.*

(1) *M_2 is M_1-injective.*
(2) *For each submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \subseteq M'$.*

We use this characterization to define a lattice formulation of the concept of c being b injective in $a = b \oplus c$.

Definition 3.3. Let $a, b, c \in L$ be such that $a = b \oplus c$. Then c is said to be b-injective in a if for every $d \leqslant a$ with $d \wedge c = 0$, there exists $e \leqslant a$ such that $a = e \oplus c$ and $d \leqslant e$.

If c is b-injective and b is c-injective in a, then we say that b and c are relatively injective.

Example 3.4. In the lattice shown in Figure 2, $i = a \oplus e$. For $b \leqslant i$ we have $b \leqslant f$ and $i = f \oplus e$. Similarly, we can verify that for every $x \leqslant i$ there exists $y \leqslant i$ such that $x \leqslant y$ and $i = y \oplus e$. Thus e is a-injective.

Lemma 3.2. *Let L be a modular lattice and $a, b, c \in L$. Let $a = b \oplus c$. If c is b-injective, then c is d-injective in $d \oplus c$ for any $d \leqslant b$.*

Proof. Let $d \leqslant b$. To show c is d-injective in $d \oplus c$, let $f \leqslant d \vee c$ be such that $f \wedge c = 0$. Since c is b-injective in a, there exists $g \leqslant a$ such that $a = g \oplus c$ and $f \leqslant g$. Put $h = g \wedge (d \vee c)$. Then $c \vee h = c \vee (g \wedge (d \vee c)) = (c \vee g) \wedge (d \vee c) = (b \vee c) \wedge (d \vee c) = d \vee c$. Thus c is d-injective in $d \oplus c$. $\square$

Now we define an analogue of the concept of an extending (or CS) module in the context of lattices.

Definition 3.4. An element a of a lattice L is called extending if every nonzero $b \leqslant a$ is essential in a direct summand of a.

Example 3.5. In the lattice shown in Figure 3, consider the element g. Every nonzero $x \leqslant g$ is a direct summand of g. Hence g is extending.

Consider the element i in the lattice L shown in Figure 4. We note that $b \leqslant i$ but b is neither a direct summand of i nor is it essential in a direct summand of i. Hence i is not extending.

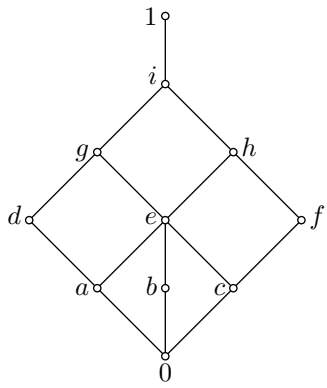

Figure 4.

We note that in a modular lattice L satisfying the Condition (i) stated in Section 2, every nonzero $a \leqslant b$ has a maximal essential extension in b and the maximal essential extension is closed. Hence a nonzero $a \in L$ is extending if every $b \leqslant a$ which is closed in a is a direct summand of a.

Let L be a modular lattice satisfying the Conditions (i) and (ii) and $a \leqslant b$. By Lemma 2.5, a is closed in b if and only if a is a max-semicomplement in b. Hence b is extending if every max-semicomplement in b is a direct summand of b.

The following result is an analogue of Corollary 2 from Harmanci and Smith [9] in the context of lattices.

Lemma 3.3. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii). *Suppose that $a \in L$ is extending. Then every direct summand of a is extending.*

The following result, from Keskin [11], Proposition 3.2, is a lattice theoretic analogue of Lemma 7.9, page 59, from Dung et al. [5].

Lemma 3.4. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii). *Let $a, b, c \in L$ be such that $a = b \oplus c$ and let b and c be extending. Then a is extending if and only if every closed element $d \leqslant a$ with either $d \wedge b = 0$ or $d \wedge c = 0$ is a direct summand of a.*

Lemma 3.5. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *and $a \in L$. Suppose that any two direct summands of a whose join is a are relatively injective. If $a = a_1 \oplus a_2$, then a is extending if and only if a_1, a_2 are extending.*

Proof. If a is extending, then by Lemma 3.3, a_1, a_2 are extending.

To prove the converse, let $c \leqslant_{\mathrm{cl}} a$. Suppose that $c \wedge a_1 = 0$. Since a_1, a_2 are relatively injective, there is $d \geqslant c$ such that $d \oplus a_1 = a$. Thus by modularity, the interval sublattices $[0, d]$ and $[0, a_2]$ are isomorphic and it follows that d is extending as a_2 is extending. Therefore, $c \leqslant_e x$ for some $x \in \mathfrak{D}(d) \subseteq \mathfrak{D}(a)$. As c is closed, we get $c = x$.

The case $c \wedge a_2 = 0$ follows by symmetry. Hence by Lemma 3.4, a is extending. $\square$

Lemma 3.6. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *and let $a, b, c \in L$ be such that $a = b \oplus c$. Suppose that a is extending and satisfies the summand sum property. Then b and c are relatively injective.*

Proof. Let $a = b \oplus c$ and let d be a max-semicomplement of b in a. By Proposition 2.2, $d \oplus b \leqslant_e a$ and d is closed in a. Now, since a is extending, d is a direct summand of a. Thus we have obtained two direct summands b and d of a such that $b \wedge d = 0$. By the summand sum property, $b \oplus d$ is a direct summand of a. But then $d \oplus b \leqslant_e a$ implies $a = d \oplus b$. Hence b is c-injective.

Similarly, we can show that c is b-injective. $\square$

4. Goldie extending elements

In this section, as a generalization of an extending element, we define a Goldie extending element in a lattice. We also define an a-ejective element in a lattice and discuss some of its properties.

Akalan et al. [1] defined two relations α and β on the set of submodules of a module M and also the concept of a Goldie extending module. We define these relations and the concept of a Goldie extending element in a lattice.

Definition 4.1. Let $a, b \in L$. Then

(1) $a \,\alpha\, b$ if and only if there exists $c \in L$ such that $a \leqslant_e c$ and $b \leqslant_e c$,
(2) $a \,\beta\, b$ if and only if $a \wedge b \leqslant_e a$ and $a \wedge b \leqslant_e b$.

Example 4.1. In the lattice shown in Figure 4, consider the elements g, h and i. We note that $g \leqslant_e i$ and $h \leqslant_e i$. Hence $g \,\alpha\, h$ holds. Also, the elements g and h satisfy $g \wedge h \leqslant_e g$ and $g \wedge h \leqslant_e h$. Hence $g \,\beta\, h$ holds.

Remark 4.1. $a \,\alpha\, b$ implies $a \,\beta\, b$. But the converse need not hold.

Proof. Let $c \in L$ be such that $a \leqslant_e c$ and $b \leqslant_e c$. By Lemma 2.2, it follows that $a \wedge b \leqslant_e a$ and $a \wedge b \leqslant_e b$. Thus $a \,\alpha\, b$ implies $a \,\beta\, b$.

For the converse, consider the elements d, f in the lattice shown in Figure 5. Then $d \wedge f = b \leqslant_e d$ and $d \wedge f = b \leqslant_e f$. Thus $d \,\beta\, f$ holds. But there is no x such that $x \geqslant_e d$ and $x \geqslant_e f$. Hence $d \,\alpha\, f$ does not hold. □

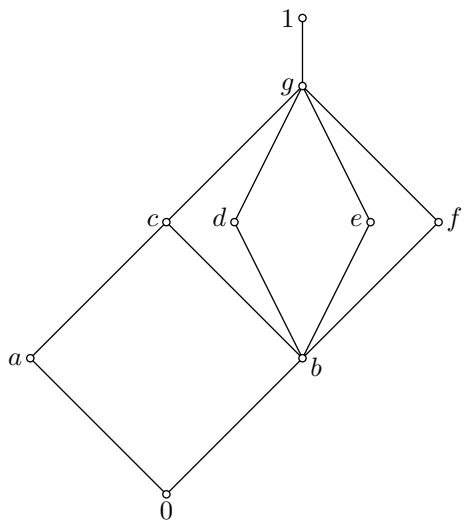

Figure 5.

Remark 4.2. $a \beta b$ is an equivalence relation on L.

We give a characterization of an extending element.

Proposition 4.1. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii). *An element $a \in L$ is extending if and only if for every $b \leqslant a$ there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \alpha c$.*

Proof. Let a be an extending element in L. Let $b \leqslant a$. Since a is extending, there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \leqslant_e c$. Now using $c \leqslant_e c$ we get $b \alpha c$.

Conversely, suppose that for every $b \leqslant a$ there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \alpha c$. Then there exists $d \in L$ such that $b \leqslant_e d$ and $c \leqslant_e d$. It follows that $b \leqslant_e a \wedge d$ and $c \leqslant_e a \wedge d$. Being a direct summand, c is closed in a. Hence $c = a \wedge d$ and $b \leqslant_e c$. Thus, a is extending. □

By using the condition β we give the definition of a Goldie extending element in a lattice L with 0. We also give some of its characterizations.

Definition 4.2. Let $a \in L$. If for every $b \leqslant a$ there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \beta c$ then a is said to be a Goldie extending element. In short, we say that a is a G-*extending element*.

We note that whenever $b \leqslant_e a$, then a is the only direct summand of a such that $b \beta a$. In a modular lattice L, we can equivalently define a G-extending element as follows.

An element $a \in L$ is called a *Goldie extending element*, if for every closed element $b \leqslant a$ there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \beta c$ holds.

Example 4.2. In the lattice shown in Figure 5, the element g is G-extending. Here the direct summands of g are a, d, e and f. Also $a \beta a$, $b \beta d$, $c \beta f$, $d \beta f$ $e \beta d$ hold.

In the following result we give a necessary and sufficient condition for an element to be Goldie extending.

Lemma 4.1. *Let $a \in L$. Then the following statements are equivalent.*

(1) *a is a G-extending element.*
(2) *For every $b \leqslant a$ there exists $c \leqslant a$ and $d \in \mathfrak{D}(a)$ such that $c \leqslant_e b$ and $c \leqslant_e d$.*

Proof. (1) $\Rightarrow$ (2): Let a be G-extending. Then for every $b \leqslant a$ there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \beta c$. Then $b \wedge c \leqslant_e b$ and $b \wedge c \leqslant_e c$. Putting $d = b \wedge c$, it follows that $d \leqslant_e b$ and $d \leqslant_e c$.

(2) $\Rightarrow$ (1): Let $a, b \in L$ be such that $b \leqslant a$. Then by (2) there exist $c \leqslant a$ and a direct summand d of a such that $c \leqslant_e b$ and $c \leqslant_e d$.

Put $f = b \wedge d$. Let $g \leqslant b$ be such that $f \wedge g = 0$. This implies $0 = b \wedge d \wedge g = d \wedge g$ and so $c \wedge g = 0$. Now $c \leqslant_e b$ implies $g = 0$. Thus $b \wedge d \leqslant_e b$. Similarly, $b \wedge d \leqslant_e d$ Thus, $b \; \beta \; d$ holds and a is G-extending. $\square$

The concept of the socle of a module is well known. It was generalized for lattices, see Călugăreanu [2], page 47.

Definition 4.3. For $x \in L$, we denote the *socle* of x by $\mathrm{Soc}(x)$ and define it as

$$\mathrm{Soc}(x) = \bigvee \{a\colon\ a \leqslant x \text{ where } a \text{ is an atom in } L\}.$$

If $a \leqslant b$, then it is clear that $\mathrm{Soc}(a) \leqslant \mathrm{Soc}(b)$.

Example 4.3. In the lattice shown in Figure 4, for $g \in L$, $\mathrm{Soc}(g) = e$ and for the element e, $\mathrm{Soc}(e) = e$.

Lemma 4.2. *Let $x, y \in L$. Suppose that for any $x \in L$, $\mathrm{Soc}(x)$ exists, then $x \leqslant_e y$ implies $\mathrm{Soc}(x) = \mathrm{Soc}(y)$.*

Proof. Let $x \leqslant_e y$. It is clear that $\mathrm{Soc}(x) \leqslant \mathrm{Soc}(y)$. Since $x \leqslant_e y$, for no atom $a \leqslant y$, $x \wedge a = 0$ holds. Hence each atom $a \leqslant y$ satisfies $a \leqslant x$. Thus $\mathrm{Soc}(y) \leqslant \mathrm{Soc}(x)$. $\square$

We define a weak extending element in a lattice L with 0 and give its relationship with a G-extending element.

Definition 4.4. An element a of a lattice L with 0 is called weak extending, if for every $b \leqslant a$, $\mathrm{Soc}(b) \leqslant_e d$ for some $d \in \mathfrak{D}(a)$.

Definition 4.5. An element a of a lattice L is said to satisfy the condition

($*$) if for every $b \leqslant a$ there exists a decomposition $a = a_1 \oplus a_2$ of a such that $b \wedge a_2 = 0$ and $b \oplus a_2 \leqslant_e a$.

Example 4.4. Consider the element i in the lattice shown in Figure 4. Then $a \leqslant i$ and b and f are max-semicomplements of a. Also, $a \vee b = e \leqslant_e i$ and $a \vee f = h \leqslant_e i$. Similarly, we can check for all $x \leqslant i$. Hence i satisfies the condition ($*$).

Also, we note that the element b is neither a direct summand of i, nor is it essential in a direct summand of i. This shows that an element satisfying the condition ($*$), need not be extending.

Now, consider the elements g and c in the lattice shown in Figure 6. We note that $c \leqslant g$ and there exists no direct summand k of g such that $c \vee k \leqslant_e g$. Hence g does not satisfy the condition ($*$).

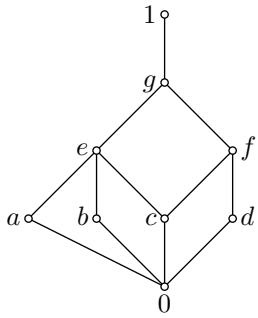

Figure 6.

It is clear that if a is extending then a satisfies the condition $(*)$.

Proposition 4.2. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *in which* $\mathrm{Soc}(x)$ *exists for every $x \in L$. Consider the following statements for $a \in L$.*

(1) *a is extending.*
(2) *a is* G*-extending.*
(3) *a satisfies the condition* $(*)$.
(4) *a is weak extending.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *and* (2) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2): Let a be extending and $b \leqslant a$. By Proposition 4.1 there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \,\alpha\, c$ and so by Remark 4.1, $b \,\beta\, c$. Hence a is G-extending.

(2) $\Rightarrow$ (3): Let $b \leqslant a$. Then by (2) there exists a direct summand $c \in \mathfrak{D}(a)$ such that $b \,\beta\, c$. Then $b \wedge c \leqslant_e b$, $b \wedge c \leqslant_e c$. Let $a = c \oplus d$ for some $d \in \mathfrak{D}(a)$.

By Lemma 2.3, $(b \wedge c) \oplus d \leqslant_e c \oplus d = a$. Hence by (2) of Lemma 2.2, $b \oplus d \leqslant_e a$. Since L is modular, the direct summand d of a is closed in a. It follows from Proposition 2.2 that the direct summand d is a max-semicomplement of b in a. Thus, a satisfies the condition $(*)$.

(2) $\Rightarrow$ (4): Let a be G-extending. Let $b \leqslant a$. Clearly, $\mathrm{Soc}(b) \leqslant a$. By (2), there exists $c \in \mathfrak{D}(a)$ such that $\mathrm{Soc}(b) \,\beta\, c$. That is, $\mathrm{Soc}(b) \wedge c \leqslant_e c$ and $\mathrm{Soc}(b) \wedge c \leqslant_e \mathrm{Soc}(b)$. Let x be an atom such that $x \leqslant b$. Then $x \leqslant \mathrm{Soc}(b)$. As $\mathrm{Soc}(b) \wedge c \leqslant_e \mathrm{Soc}(b)$, we have $x \wedge (\mathrm{Soc}(b) \wedge c) \neq 0$. Thus $x \leqslant \mathrm{Soc}(b) \wedge c$. This implies $\mathrm{Soc}(b) \leqslant \mathrm{Soc}(b) \wedge c$. Thus $\mathrm{Soc}(b) \wedge c = \mathrm{Soc}(b)$. Hence $\mathrm{Soc}(b) \leqslant_e c$. Thus, a is weak extending. $\square$

Now, we give a characterization of a G-extending element.

Theorem 4.1. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *and $a \in L$.*

(1) *Suppose that every $b \leqslant a$ has a unique maximal essential extension in a. Then a is* G*-extending if and only if a is extending.*

(2) *If $\mathrm{Soc}(a) \leqslant_e a$, then a is* G*-extending. If L is atomic and if a is weak extending, then a is* G*-extending.*

Proof. (1) Let a be G-extending. Let $c \in L$ be such that c is closed in a. By Lemma 4.1, there exist a $d \leqslant a$ and a direct summand $f \in \mathfrak{D}(a)$ such that $d \leqslant_e f$ and $d \leqslant_e c$. Since the direct summand f is closed in a and c is also closed in a, the elements c and f are two maximal essential extensions of b in a. By (1), $c = f$. Hence every closed element $x \leqslant a$ is a direct summand of a. Thus a is extending.

The converse follows from Proposition 4.2.

(2) Let $\mathrm{Soc}(a) \leqslant_e a$. The if part follows from Proposition 4.2.

Conversely, let a be weak extending. Let $b \leqslant a$. Then $\mathrm{Soc}(b) \leqslant_e c$ for some $c \in \mathfrak{D}(a)$.

Let $t \leqslant b$ be nonzero. Since L is atomic, there exists an atom $x \leqslant t$. Then $\mathrm{Soc}(b) \wedge x = x$ implies $\mathrm{Soc}(b) \wedge t \neq 0$. Hence $\mathrm{Soc}(b) \leqslant_e b$. Thus by Lemma 4.1, a is G-extending. □

The notion of a module N being M-injective is generalized to M-ejective by Akalan et al. [1], Definition 2.1. The concept of ejectivity is a generalization of injectivity. We use the characterization of ejectivity given in Theorem 2.7 from Akalan et al. [1] to define an a-ejective element in a lattice L. We also prove a relation between an extending element and a G-extending element by using ejectivity of direct summands.

Definition 4.6. Let $a, b, c \in L$ be such that $a = b \oplus c$. Then b is said to be c-ejective in a, if for every $d \leqslant a$ such that $d \wedge b = 0$ there exists an $f \leqslant a$ such that $a = b \oplus f$ and $d \wedge f \leqslant_e d$.

If b is c-ejective and c is b-ejective then we say that b and c are relatively ejective.

Example 4.5. In the lattice shown in Figure 5, $g = a \oplus e$. Also, for $b \leqslant g$, $b \wedge a = 0$ there exists $f \leqslant g$ such that $g = a \oplus f$ and $b \wedge f \leqslant_e b$. Similarly, we can show that for all $x \leqslant g$ satisfying $x \wedge a = 0$ there exists $y \leqslant g$ such that $g = a \oplus y$ and $x \wedge y \leqslant_e x$. Hence a is e-ejective.

Lemma 4.3. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *and let $a, b, c \in L$ be such that $a = b \oplus c$. If b is c-injective then b is c-ejective. Also, the converse holds if every $d \leqslant a$ is closed in a.*

P r o o f. Let $a = b \oplus c$ and let b be c-injective. Let $d \leqslant a$ be such that $d \wedge b = 0$. Since b in c-injective, there exists $f \leqslant a$ such that $a = b \oplus f$ and $d \leqslant f$. Clearly $d \wedge f = d \leqslant_e d$. Hence b is c-ejective.

For the converse, let b be c-ejective and let every $x \leqslant a$ be closed in a. Let $d \leqslant a$ be such that $d \wedge b = 0$. Since b in c-ejective, there exists an $f \in L$ such that $a = b \oplus f$ and $d \wedge f \leqslant_e d$. As every $x \leqslant a$ is closed in a, $d \wedge f = d$, that is $d \leqslant f$. Hence b is c-injective. □

Proposition 4.3. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *and let $a \in L$ be such that a is* G*-extending. If a has the summand intersection property, then every direct summand of a is* G*-extending.*

P r o o f. Let $b \in \mathfrak{D}(a)$ be a direct summand of a and let $c \leqslant b$. Since a is G-extending, by Lemma 4.1 there exist a $d \leqslant a$ and a direct summand $f \in \mathfrak{D}(a)$ such that $d \leqslant_e c$ and $d \leqslant_e f$. By Lemma 2.2, $d \wedge b \leqslant_e c \wedge b$ and $d \wedge b \leqslant_e f \wedge b$. That is, $d \wedge b \leqslant_e c$ and $d \wedge b \leqslant_e f \wedge b$. Since a has the summand intersection property, $f \wedge b$ is a direct summand of a. As $f \wedge b \leqslant b$ we conclude that $f \wedge b$ is a direct summand of b. Hence $c \, \beta \, (f \wedge b)$ where $(f \wedge b) \in \mathfrak{D}(b)$. Thus, b is G-extending. □

Now we formulate and answer the open problem posed by Akalan, Birkenmeier and Tercan [1], mentioned in the introduction in the context of certain modular lattice.

Open problem. Give necessary and/or sufficient condition for the direct sum of G-extending elements in a modular lattice with 0, to be G-extending.

In the following we give a sufficient condition for the direct sum of G-extending elements to be G-extending.

Theorem 4.2. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii)*. Suppose that $a, b, c \in L$ are such that $a = b \oplus c$. Also suppose that b and every x such that $x \oplus b = a$ are* G*-extending. If b is c-ejective then a is* G*-extending.*

P r o o f. Let b be c-ejective. Let $d \leqslant a$.

If $d \wedge b = 0$ then there exists $f \in L$ such that $a = b \oplus f$ and $d \wedge f \leqslant_e d$. Since f is G-extending, there exist $g \leqslant f$ and a direct summand $h \in \mathfrak{D}(f)$ such that $g \leqslant_e d \wedge f$ and $g \leqslant_e h$. As $f \in \mathfrak{D}(a)$, it follows that $h \in \mathfrak{D}(a)$. Then $g \leqslant_e d \wedge f \leqslant_e d$ and $g \leqslant_e h$ imply that $d \, \beta \, h$.

If $d \wedge b \neq 0$, then $d \wedge b$ has a max-semicomplement $i \leqslant d$ such that $(d \wedge f) \oplus i \leqslant_e d$. Now, $(d \wedge f) \wedge i = 0$ implies $f \wedge i = 0$ and since b is c-ejective, there exists $j \in L$ such that $a = b \oplus j$ and $i \wedge j \leqslant_e i$. Again, j is G-extending and $i \wedge j \leqslant j$ imply that there exist $k \in L$ such that $k \leqslant j$ and a direct summand $l \in \mathfrak{D}(j)$ such that $k \leqslant_e i \wedge j$ and $k \leqslant_e l$.

Also, as b is G-extending and $d \wedge b \leqslant b$ there exist $m \in L$ such that $m \leqslant b$ and a direct summand $n \in \mathfrak{D}(b)$ such that $m \leqslant_e d \wedge b$ and $m \leqslant_e n$.

Now, $k \leqslant_e i \wedge j$, $m \leqslant_e d \wedge b$ and $(i \wedge j) \wedge (d \wedge b) = 0$ imply that

$$k \oplus m \leqslant_e (i \wedge j) \oplus (d \wedge b).$$

Using $i \wedge j \leqslant_e i$ we get $k \oplus m \leqslant_e i \oplus (d \wedge b)$. That is, $k \oplus m \leqslant_e d$.

Also, $k \oplus m \leqslant_e l \oplus n$. Since l is a direct summand of j, n is a direct summand of b and $a = b \oplus j$, we have that $l \oplus n$ is a direct summand of a. Hence $k \oplus m \leqslant_e d$ and $k \oplus m \leqslant_e p = l \oplus n$. Thus $d \,\beta\, p$ holds. It follows that a is G-extending. □

Lemma 4.4. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *and $a \in L$. Suppose that every $b \leqslant a$ has a unique maximal essential extension in a. If a is* G-*extending then a has the summand intersection property.*

Proof. Let $x, y \in \mathfrak{D}(a)$ be such that $x \wedge y \neq 0$. Let p and q be maximal essential extensions of $x \wedge y$ in x and y, respectively. But, our assumption implies that $p = q$. Thus $x \wedge y \leqslant_e p \leqslant x$ and $x \wedge y \leqslant_e p \leqslant y$. This implies $x \wedge y = p$. Thus $x \wedge y$ is closed in x. As x, being a direct summand of a, is closed in a, by Proposition 2.1 we conclude that $x \wedge y$ is closed in a. Now, by Theorem 4.1 (1), a is extending, hence $x \wedge y$ a direct summand of a. Thus a has the summand intersection property. □

Lemma 4.5. *Let L be a modular lattice satisfying the Conditions* (i) *and* (ii) *and let $a, b, c \in L$ be such that $a = b \oplus c$ and* $\mathrm{Soc}(a) \leqslant_e c$. *Suppose that a has the summand intersection property and a is weak extending. Then every direct summand of c is* G-*extending.*

Proof. It is clear that c has the summand intersection property. So, $\mathrm{Soc}(a) \leqslant_e c$ implies $\mathrm{Soc}(a) \wedge c \leqslant_e c$, that is $\mathrm{Soc}(c) \leqslant_e c$.

Now, we show that c is weak extending. Let $d \leqslant c$. Since a is weak extending, there exists $f \in \mathfrak{D}(a)$ such that $\mathrm{Soc}(d) \leqslant_e f$.

It follows that $\mathrm{Soc}(d) \wedge c \leqslant_e f \wedge c$, that is $\mathrm{Soc}(d) \leqslant_e f \wedge c$. By the summand intersection property, $f \wedge c$ is a direct summand of a. Then $f \wedge c \leqslant c$ implies $f \wedge c$ is a direct summand of c. Hence c is weak extending. By (2) of Theorem 4.1, c is G-extending.

Now, c is G-extending and satisfies the summand intersection property. Hence by Proposition 4.3 every direct summand of c is G-extending. □

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