APPROXIMATE TRI-QUADRATIC FUNCTIONAL EQUATIONS
VIA LIPSCHITZ CONDITIONS

ISMAIL NIKOUFAR, Tehran

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Abstract. In this paper, we consider Lipschitz conditions for tri-quadratic functional equations. We introduce a new notion similar to that of the left invariant mean and prove that a family of functions with this property can be approximated by tri-quadratic functions via a Lipschitz norm.

Keywords: tri-quadratic functional equation; Lipschitz space; stability
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1. Introduction

A generalized stability problem for the quadratic functional equation

\[ Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \]

was proved by Skof in [11] for mappings from a normed space to a Banach space. Czerwik et al. in [1] verified the stability of the quadratic functional equations in Lipschitz spaces. The Lipschitz stability type problems for some functional equations were also studied by Tabor, see [12], [13]. In Lipschitz spaces we investigated the stability of cubic functional equations in [2] and the stability of quartic functional equations in [7] (see also [6], [5]). The stability problem for the quadratic and bi-quadratic functional equation has been studied by many mathematicians under various degrees of generality imposed on the equation or on the underlying space; see, for example, [3], [4], [10], [9] and the references therein. We obtained Lipschitz criteria for bi-quadratic functional equations in Lipschitz spaces in [8].

The algebra of Lipschitz functions on a complete metric space plays a role in noncommutative metric theory similar to that played by the algebra of continuous
functions on a compact space in noncommutative topology. Let \( \mathcal{H} \) be an abelian group and \( \mathcal{W} \) a real vector space. A function \( Q: \mathcal{H}^3 \rightarrow \mathcal{W} \) is called tri-quadratic if \( Q \) satisfies the system of equations

\[
Q(x + y, z, w) + Q(x - y, z, w) = 2Q(x, z, w) + 2Q(y, z, w),
Q(x, y + z, w) + Q(x, y - z, w) = 2Q(x, y, w) + 2Q(x, z, w),
Q(x, y, z + w) + Q(x, y, z - w) = 2Q(x, y, z) + 2Q(x, y, w)
\]

for all \( x, y, z \in \mathcal{H} \), that is, \( Q \) is quadratic in each variable. In this paper, we consider Lipschitz conditions for tri-quadratic functional equations. We prove that a family of functions satisfying tri-symmetric left invariant mean property can be approximated by tri-quadratic functions via a Lipschitz norm.

### 2. Lipschitz Conditions for Tri-Quadratic Functional Equations

In this section, we introduce the notion of tri-symmetric left invariant mean (TSLIM in brief) and prove that a family of functions with TSLIM property can be approximated by tri-quadratic functions via a Lipschitz norm.

A family \( S \) of subsets of \( \mathcal{W} \) is called linearly invariant if \( A + \alpha B \in S \) for \( A, B \in S \), \( \alpha \in \mathbb{R} \) and \( x + A \in S \) for \( A \in S \), \( x \in \mathcal{W} \). For example, the family of all closed balls with center at zero is a linearly invariant family in a normed vector space. We denote this family by \( CB(\mathcal{W}) \). Let \( \mathcal{L}(\mathcal{W}) \) be a linearly invariant family of subsets of \( \mathcal{W} \). By \( \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \) we denote the family of all functions \( Q: \mathcal{H} \rightarrow \mathcal{W} \) such that \( \text{Im} \ Q \subset B \) for some \( B \in \mathcal{L}(\mathcal{W}) \).

**Definition 2.1.** The function \( Q \) is called tri-symmetric if

\[
Q(x, y, z) = Q(y, z, x) = Q(z, x, y) = Q(x, z, y) = Q(x, y, z) = Q(y, x, z)
\]

for all \( x, y, z \in \mathcal{H} \).

**Definition 2.2.** We say that \( \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \) admits a tri-symmetric left invariant mean (briefly TSLIM), if the family of all functions \( Q: \mathcal{H} \rightarrow \mathcal{W} \) such that \( \text{Im} \ Q \subset B \) for some \( B \in \mathcal{L}(\mathcal{W}) \) is linearly invariant and there exists a linear operator \( \Gamma: \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \rightarrow \mathcal{W} \) such that

(i) if \( Q_{x,y,z} \in \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \) and \( x, y, z \in \mathcal{H} \), then

\[
\Gamma[Q_{x,y,z}] = \Gamma[Q_{y,z,x}] = \Gamma[Q_{z,x,y}] = \Gamma[Q_{x,z,y}] = \Gamma[Q_{y,x,z}],
\]

(ii) if \( \text{Im} \ Q \subset B \) for some \( B \in \mathcal{L}(\mathcal{W}) \), then \( \Gamma[Q] \in B \),

(iii) if \( Q \in \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \) and \( a \in \mathcal{H} \), then \( \Gamma[Q^a] = \Gamma[Q] \), where \( Q^a(x) = Q(x+a) \).
Definition 2.3. Let \( \Delta : \mathcal{H}^3 \times \mathcal{H}^3 \to \mathcal{L}(\mathcal{W}) \) be a set-valued function such that

\[
\Delta((x + a, y + b, z + c), (u + a, v + b, w + c)) = \Delta((a + x, b + y, c + z), (a + u, b + v, c + w)) = \Delta((x, y, z), (u, v, w))
\]

for all \((a, b, c), (x, y, z), (u, v, w) \in \mathcal{H}^3\). A function \( Q : \mathcal{H}^3 \to \mathcal{W} \) is said to be \( \Delta \)-Lipschitz if

\[
Q(x, y, z) - Q(u, v, w) \in \Delta((x, y, z), (u, v, w))
\]

for all \((x, y, z), (u, v, w) \in \mathcal{H}^3\).

Let \( Q : \mathcal{H}^3 \to \mathcal{W} \) be a function. We consider its tri-quadratic difference as follows:

\[
TQ(x, y, z, w) := 2Q(x, z, w) + 2Q(y, z, w) - Q(x + y, z, w) - Q(x - y, z, w)
\]

for all \(x, y, z, w \in \mathcal{H}\).

Theorem 2.4. Let \( \mathcal{H} \) be an abelian group and let \( \mathcal{W} \) be a vector space. Assume that the family \( \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \) admits TSLIM. If \( F : \mathcal{H}^3 \to \mathcal{W} \) is a function and \( TF(t, \cdot, \cdot, \cdot) : \mathcal{H}^3 \to \mathcal{W} \) is \( \Delta \)-Lipschitz for every \( t \in \mathcal{H} \), then there exists a tri-quadratic function \( Q : \mathcal{H}^3 \to \mathcal{W} \) such that \( F - Q \) is \( \frac{1}{2} \Delta \)-Lipschitz. Moreover, if \( \text{Im}TF \subseteq A \) for some \( A \in \mathcal{L}(\mathcal{W}) \), then \( \text{Im}(F - Q) \subseteq \frac{1}{2}A \).

Proof. For every \((x, y, z) \in \mathcal{H}^3\) we define \( \varphi_x(\cdot, y, z) : \mathcal{H} \to \mathcal{W} \) by

\[
\varphi_x(\cdot, y, z) := \frac{1}{2}F(\cdot + x, y, z) + \frac{1}{2}F(\cdot - x, y, z) - F(\cdot, y, z).
\]

We prove that \( \text{Im} \varphi_x(\cdot, y, z) \subseteq A \) for some \( A \in \mathcal{L}(\mathcal{W}) \). We have for \((x, y, z) \in \mathcal{H}^3\),

\[
\varphi_x(\cdot, y, z) = \frac{1}{2}F(\cdot + x, y, z) + \frac{1}{2}F(\cdot - x, y, z) - F(\cdot, y, z)
\]

\[
- \frac{1}{2}F(\cdot, y, z) - \frac{1}{2}F(\cdot, y, z) + F(\cdot, y, z) + F(0, y, z)
\]

\[
+ F(x, y, z) - F(0, y, z)
\]

\[
= \frac{1}{2}TF(\cdot, 0, y, z) - \frac{1}{2}TF(\cdot, x, y, z) + F(x, y, z) - F(0, y, z).
\]

By assumption, since \( TF(t, \cdot, \cdot, \cdot) \) is \( \Delta \)-Lipschitz for every \( t \in \mathcal{H} \), \( \text{Im} \varphi_x(\cdot, y, z) \subseteq A \), where \( A := \frac{1}{2}\Delta((0, y, z), (x, y, z)) + F(x, y, z) - F(0, y, z) \). The family \( \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \) admits TSLIM, so there exists a linear operator \( \Gamma : \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W})) \to \mathcal{W} \) such that

(i) \( \Gamma[\varphi_x(\cdot, y, z)] = \Gamma[\varphi_y(\cdot, z, x)] = \Gamma[\varphi_z(\cdot, x, y)] = \Gamma[\varphi_y(\cdot, z, x)] = \Gamma[\varphi_z(\cdot, x, y)] = \Gamma[\varphi_z(\cdot, y, x)] \) for every \((x, y, z) \in \mathcal{H}^3\)
(ii) $\Gamma[\varphi_x(\cdot, y, z)] \in A$ for some $A \in \mathcal{L}(W)$ and every $(x, y, z) \in \mathcal{H}^3$,

(iii) if $u \in \mathcal{H}$ and $\varphi^u_x(\cdot, y, z): \mathcal{H} \to W$ defined by $\varphi^u_x(\cdot, y, z) := \varphi_x(\cdot + u, y, z)$ for every $(x, y, z) \in \mathcal{H}^3$, then $\varphi^u_x(\cdot, y, z) \in \mathcal{M}(\mathcal{H}, \mathcal{L}(W))$ and $\Gamma[\varphi^u_x(\cdot, y, z)] = \Gamma[\varphi_x(\cdot, y, z)]$.

Define the function $Q: \mathcal{H}^3 \to W$ by $Q(x, y, z) := \Gamma[\varphi_x(\cdot, y, z)]$. In view of property (ii) of $\Gamma$, $Q$ is tri-symmetric. We prove that $F - Q$ is $\frac{1}{2}\Delta$-Lipschitz. Since $T\mathcal{F}(t, \cdot, \cdot, \cdot) = \Delta$-Lipschitz for $t \in \mathcal{H}$,

\[(2.1)\]

\[T\mathcal{F}(t, x, y, z) - T\mathcal{F}(t, u, v, w) \in \Delta((x, y, z), (u, v, w))\]

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$ and so

\[\text{Im}\left(\frac{1}{2}T\mathcal{F}(\cdot, x, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, u, v, w)\right) \subseteq \frac{1}{2}\Delta((x, y, z), (u, v, w)).\]

In view of property (ii) of $\Gamma$, we find that

\[\Gamma\left[\frac{1}{2}T\mathcal{F}(\cdot, x, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, u, v, w)\right] \in \frac{1}{2}\Delta((x, y, z), (u, v, w))\]

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$. Note that $\mathcal{M}(\mathcal{H}, \mathcal{L}(W))$ contains constant functions. Property (ii) of $\Gamma$ entails that for a constant function $C: \mathcal{H} \to W$, $\Gamma[C] = C$. For every $(x, y, z) \in \mathcal{H}^3$ we define the constant function $C_{x,y,z}: \mathcal{H} \to W$ by $C_{x,y,z}(\cdot) := F(x, y, z)$. We see that

\[(F(x, y, z) - Q(x, y, z)) - (F(u, v, w) - Q(u, v, w)) = (\Gamma[C_{x,y,z}(\cdot)] - (\Gamma[C_{u,v,w}(\cdot)] - \Gamma[\varphi_u(\cdot, v, w)])\]

\[= \Gamma[\varphi_{x,y,z}(\cdot) - \varphi_x(\cdot, y, z)] - \Gamma[C_{u,v,w}(\cdot) - \varphi_u(\cdot, v, w)]\]

\[= \Gamma\left[\frac{1}{2}T\mathcal{F}(\cdot, x, y, z) - \frac{1}{2}T\mathcal{F}(\cdot, u, v, w)\right]\]

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$. This shows that

\[(F(x, y, z) - Q(x, y, z)) - (F(u, v, w) - Q(u, v, w)) \in \frac{1}{2}\Delta((x, y, z), (u, v, w))\]

for all $(x, y, z), (u, v, w) \in \mathcal{H}^3$, i.e., $F - Q$ is a $\frac{1}{2}\Delta$-Lipschitz function. Applying property (iii) of $\Gamma$ and the definition of $\Gamma$, we find that

\[(2.2)\]

\[2Q(x, z, w) + 2Q(y, z, w) = 2\Gamma[\varphi_x(\cdot, z, w)] + 2\Gamma[\varphi_y(\cdot, z, w)]\]

\[= \Gamma[\varphi^y_x(\cdot, z, w)] + \Gamma[\varphi^y_x(\cdot, z, w)] + 2\Gamma[\varphi_y(\cdot, z, w)].\]
On the other hand, we have

\begin{equation}
\Gamma [\phi^y(\cdot, z, w)] + \Gamma [\phi^{-y}(\cdot, z, w)] + 2\Gamma [\phi(\cdot, z, w)] \\
= \Gamma \left[ \frac{1}{2} F(\cdot + x + y, z, w) + \frac{1}{2} F(\cdot - x + y, z, w) - F(\cdot + y, z, w) \right] \\
+ \Gamma \left[ \frac{1}{2} F(\cdot + x - y, z, w) + \frac{1}{2} F(\cdot - x - y, z, w) - F(\cdot - y, z, w) \right] \\
+ \Gamma [F(\cdot + y, z, w) - F(\cdot - y, z, w) - C(\cdot, z, w)] \\
= \Gamma (x + y, z, w) + \Gamma (x - y, z, w) \\
= Q(x + y, z, w) + Q(x - y, z, w)
\end{equation}

From (2.2) and (2.3) it follows that $Q$ is quadratic in its first variable. Since $Q$ is tri-symmetric, $Q$ is quadratic in its second and third variables and hence $Q$ is tri-quadratic. Moreover, if $\text{Im} TF \subset A$, then

$$
\text{Im} \left( \frac{1}{2} TF(\cdot, x, y, z) \right) \subset \text{Im} \left( \frac{1}{2} T \right) \subset \frac{1}{2} A.
$$

In other words, $\frac{1}{2} TF(\cdot, x, y, z) \in \mathcal{M}(\mathcal{H}, \mathcal{L}(\mathcal{W}))$ for all $(x, y, z) \in \mathcal{H}^3$. Thus, property (ii) of $\Gamma$ implies

$$
F(x, y, z) - Q(x, y, z) = \Gamma \left[ \frac{1}{2} TF(\cdot, x, y, z) \right] \in \frac{1}{2} A
$$

for all $(x, y, z) \in \mathcal{H}^3$. Therefore, $\text{Im}(F - Q) \subset \frac{1}{2} A$. \hfill \Box

**Definition 2.5.** Let $(\mathcal{H}^3, \varrho)$ be a metric group and $\mathcal{W}$ a normed space. A function $m_F: \mathbb{R}^+ \to \mathbb{R}^+$ is a module of continuity of $F: \mathcal{H}^3 \to \mathcal{W}$ if $\varrho((x, y, z), (u, v, w)) \leq \delta$ implies $\|F(x, y, z) - F(u, v, w)\| \leq m_F(\delta)$ for every $\delta > 0$ and $(x, y, z), (u, v, w) \in \mathcal{H}^3$.

**Definition 2.6.** A function $F: \mathcal{H}^3 \to \mathcal{W}$ is called a Lipschitz function of order $\alpha > 0$ if there exists a constant $L > 0$ such that

\begin{equation}
\|F(x, y, z) - F(u, v, w)\| \leq L \varrho^\alpha((x, y, z), (u, v, w))
\end{equation}

for every $(x, y, z), (u, v, w) \in \mathcal{H}^3$.

For a metric group $(\mathcal{H}^3, \varrho)$, a normed space $\mathcal{W}$, and $\alpha \in (0, 1]$, let $\text{Lip}_\alpha(\mathcal{H}^3, \mathcal{W})$ be the Lipschitz space consisting of all bounded Lipschitz functions of order $\alpha > 0$ with the norm

$$
\|F\|_\alpha := \|F\|_{\sup} + P_\alpha(F),
$$
where \( \| \cdot \|_{\text{sup}} \) is the supremum norm and

\[
P_\alpha(\mathcal{F}) = \sup \left\{ \frac{\| \mathcal{F}(x, y, z) - \mathcal{F}(u, v, w) \|}{\varrho^\alpha((x, y, z), (u, v, w))} : (x, y, z), (u, v, w) \in \mathcal{H}^3, (x, y, z) \neq (u, v, w) \right\}.
\]

**Definition 2.7.** Consider an abelian group \((\mathcal{H}^3, +)\) with a metric \(\varrho\) invariant under translation, i.e., satisfying the condition

\[
\varrho((x + a, y + b, z + c), (u + a, v + b, w + c)) = \varrho((x, y, z), (u, v, w))
\]

for all \((a, b, c), (x, y, z), (u, v, w) \in \mathcal{H}^3\). A metric \(D\) on \(\mathcal{H}^3 \times \mathcal{H}\) is called a metric pair if it is invariant under translation and the following condition holds:

\[
D((x, y, z, a), (u, v, w, a)) = D((a, x, y, z), (a, u, v, w)) = \varrho((x, y, z), (u, v, w))
\]

for all \(a \in \mathcal{H}, (x, y, z), (u, v, w) \in \mathcal{H}^3\).

**Theorem 2.8.** Let \((\mathcal{H}^3, +, \varrho, D)\) be a metric pair, \(W\) a normed space such that \(\mathcal{M}(\mathcal{H}, CB(W))\) admits TSLIM, and \(\mathcal{F}: \mathcal{H}^3 \to W\) a function. If \(T\mathcal{F} \in \text{Lip}_\alpha(\mathcal{H} \times \mathcal{H}^3, W)\), then there exists a tri-quadratic function \(Q\) such that

\[
\| \mathcal{F} - Q \|_\alpha \leq \frac{1}{2} \| T\mathcal{F} \|_\alpha.
\]

**Proof.** Assume that \(m_{T\mathcal{F}}: \mathbb{R}^+ \to \mathbb{R}^+\) is the module of continuity of \(T\mathcal{F}\) with the metric pair \(D\). Define the set-valued function \(\Delta: \mathcal{H}^3 \times \mathcal{H}^3 \to CB(W)\) by

\[
\Delta((x, y, z), (u, v, w)) := \inf_{\varrho((x, y, z), (u, v, w)) \leq \delta} m_{T\mathcal{F}}(\delta) B(0, 1),
\]

where \(B(0, 1)\) is the closed unit ball with center at zero. The following inequality shows that \(T\mathcal{F}(t, \cdot, \cdot, \cdot)\) is \(\Delta\)-Lipschitz:

\[
\| T\mathcal{F}(t, x, y, z) - T\mathcal{F}(t, u, v, w) \| \leq \inf_{D((t, x, y, z), (t, u, v, w)) \leq \delta} m_{T\mathcal{F}}(\delta)
= \inf_{\varrho((x, y, z), (u, v, w)) \leq \delta} m_{T\mathcal{F}}(\delta)
\]

for all \(t \in \mathcal{H}, (x, y, z), (u, v, w) \in \mathcal{H}^3\). Thus, there exists a tri-quadratic function \(Q\) such that \(\mathcal{F} - Q\) is \(\frac{1}{2}\Delta\)-Lipschitz by Theorem 2.4. Hence,

\[
\| (\mathcal{F} - Q)(x, y, z) - (\mathcal{F} - Q)(u, v, w) \| \leq \inf_{\varrho((x, y, z), (u, v, w)) \leq \delta} \frac{1}{2} m_{T\mathcal{F}}(\delta),
\]

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which entails that $m_{\mathcal{F}-Q} = \frac{1}{2}m_{\mathcal{T}\mathcal{F}}$. Moreover, $\|T\mathcal{F}\|_{\sup} < \infty$ and clearly $\text{Im} \, T\mathcal{F} \subset \|T\mathcal{F}\|_{\sup}B(0,1)$. Using the last part of Theorem 2.4 we get

$$(2.5) \quad \|\mathcal{F} - Q\|_{\sup} \leq \frac{1}{2}\|T\mathcal{F}\|_{\sup}.$$ 

Define the function $\omega: \mathbb{R}^+ \to \mathbb{R}^+$ by $\omega(t) := P_\alpha(T\mathcal{F})t^\alpha$. In view of $T\mathcal{F} \in \text{Lip}_\alpha(\mathcal{H} \times \mathcal{H}^3, \mathcal{W})$, we have

$$\|T\mathcal{F}(t, x, y, z) - T\mathcal{F}(t, u, v, w)\| \leq \omega(D(((t, x, y, z), (t, u, v, w))), \omega((x, y, z), (u, v, w))).$$

which ensures that $\omega$ is the module of continuity of the function $T\mathcal{F}$ and consequently $m_{\mathcal{F}-Q} = \frac{1}{2}\omega$. Then,

$$\|(\mathcal{F} - Q)(x, y, z) - (\mathcal{F} - Q)(u, v, w)\| \leq \frac{1}{2}\omega(g((x, y, z), (u, v, w)))$$

$$= \frac{1}{2}P_\alpha(T\mathcal{F})g^\alpha((x, y, z), (u, v, w)).$$

This inequality implies that $\mathcal{F} - Q$ is a Lipschitz function of order $\alpha$ and $P_\alpha(\mathcal{F} - Q) \leq \frac{1}{2}P_\alpha(T\mathcal{F})$. From inequality (2.5) it follows that

$$\|\mathcal{F} - Q\|_\alpha = \|\mathcal{F} - Q\|_{\sup} + P_\alpha(\mathcal{F} - Q)$$

$$\leq \frac{1}{2}\|T\mathcal{F}\|_{\sup} + \frac{1}{2}P_\alpha(T\mathcal{F}) = \frac{1}{2}\|T\mathcal{F}\|_\alpha.$$ 

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References


Author’s address: Ismail Nikoufar, Department of Mathematics, Payame Noor University, P.O. Box 19395-3697 Tehran, Iran, e-mail: nikoufar@pnu.ac.ir.