ON THE GENERALIZATION OF TWO RESULTS OF
CAO AND ZHANG

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Received May 6, 2015. First published February 1, 2017.
Communicated by Stanislawa Kanas

Abstract. This paper studies the uniqueness of meromorphic functions

\[ f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} \quad \text{and} \quad g^n \prod_{i=1}^{k} (g^{(i)})^{n_i}, \]

that share two values, where \( n, n_k, k \in \mathbb{N}, n_i \in \mathbb{N} \cup \{0\}, \) \( i = 1, 2, \ldots, k-1. \) The results
significantly rectify, improve and generalize the results due to Cao and Zhang (2012).

Keywords: uniqueness; meromorphic function; weighted sharing; nonlinear differential polynomials

MSC 2010: 30D35

1. INTRODUCTION, DEFINITIONS AND RESULTS

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let \( f \) and \( g \) be two non-constant meromorphic functions and let \( a \in \mathbb{C}. \) We say that \( f \) and \( g \) share a CM (counting multiplicities) provided that \( f - a \) and \( g - a \) have the same zeros with the same multiplicities. Similarly, we say that \( f \) and \( g \) share a IM (ignoring multiplicities) provided that \( f - a \) and \( g - a \) have the same zeros ignoring multiplicities. In addition we say that \( f \) and \( g \) share \( \infty \) CM, if \( 1/f \) and \( 1/g \) share 0 CM, and we say that \( f \) and \( g \) share \( \infty \) IM, if \( 1/f \) and \( 1/g \) share 0 IM.

We adopt the standard notation of value distribution theory (see [8]). We denote by \( T(r) \) the maximum of \( T(r, f) \) and \( T(r, g) \). The symbol \( S(r) \) denotes any quantity satisfying \( S(r) = o(T(r)) \) as \( r \to \infty \), outside of a possible exceptional set of finite

DOI: 10.21136/MB.2017.0021-15
linear measure. A meromorphic function \( a(z) \) is called a small function with respect to \( f \) provided that \( T(r, a) = S(r, f) \).

Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions. Let \( a(z) \) be a small function with respect to \( f(z) \) and \( g(z) \). We say that \( f(z) \) and \( g(z) \) share \( a(z) \) CM if \( f(z) - a(z) \) and \( g(z) - a(z) \) have the same zeros with the same multiplicities, and we say that \( f(z), g(z) \) share \( a(z) \) IM if we do not consider the multiplicities. For the sake of simplicity we also use the notation

\[
\begin{align*}
n_i^* & := \begin{cases} 
0 & \text{if } n_i = 0, \\
1 & \text{if } n_i \neq 0, 
\end{cases} \\
n_i^{**} & = \begin{cases} 
0 & \text{if } n_i = 0, \\
n_i & \text{if } n_i \neq 0. 
\end{cases}
\end{align*}
\]

A finite value \( z_0 \) is called a fixed point of \( f \) if \( f(z_0) = z_0 \) or \( z_0 \) is a zero of \( f(z) - z \).

The following well known theorem in value distribution theory was posed by Hayman and settled by several authors almost at the same time ([2], [4]).

**Theorem A.** Let \( f(z) \) be a transcendental meromorphic function, \( n \in \mathbb{N} \). Then \( f^n f' = 1 \) has infinitely many solutions.

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua [5], Yang and Hua [16] obtained the following result.

**Theorem B.** Let \( f \) and \( g \) be two non-constant entire (meromorphic) functions, \( n \in \mathbb{N} \) such that \( n \geq 6 \) (\( n \geq 11 \)). If \( f^n f' \) and \( g^n g' \) share 1 CM, then either \( f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz} \), where \( c_1, c_2, c \in \mathbb{C} \) satisfy \( 4(c_1 c_2)^{n+1} c^2 = -1 \), or \( f \equiv tg \) for a constant \( t \) such that \( t^{n+1} = 1 \).

Considering the uniqueness question of entire or meromorphic functions having fixed points, Fang and Qiu [6] obtained the following theorem.

**Theorem C.** Let \( f \) and \( g \) be two non-constant meromorphic (entire) functions, \( n \in \mathbb{N} \) such that \( n \geq 11 \) (\( n \geq 6 \)). If \( f^n f' - z \) and \( g^n g' - z \) share 0 CM, then either \( f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2} \), where \( c_1, c_2, c \in \mathbb{C} \) satisfy \( 4(c_1 c_2)^{n+1} c^2 = -1 \), or \( f \equiv tg \) for a constant \( t \) such that \( t^{n+1} = 1 \).

We recall the following result by Xu, Yi and Zhang [13]
**Theorem D.** Let $f$ be a transcendental meromorphic function, $n \in \mathbb{N} \setminus \{1\}$, $k \in \mathbb{N}$. Then $f^n f^{(k)}$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.

Recently, Cao and Zhang [3] proved the following theorems.

**Theorem E.** Let $f$ and $g$ be two transcendental meromorphic functions, whose zeros are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $n > \max\{2k - 1, k + 4/k + 4\}$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then one of the following two conclusions holds:

(i) $f^n f^{(k)} \equiv g^n g^{(k)}$;
(ii) $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1, c_2, c \in \mathbb{C}$ such that $4(c_1 c_2)^n + 1 c^2 = -1$.

**Theorem F.** Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros are of multiplicities at least $k + 1$, where $k \in \mathbb{N}$ is such that $k \leq 5$. Let $n \in \mathbb{N}$ be such that $n \geq 10$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share $1$ CM, $f^{(k)}$ and $g^{(k)}$ share $0$ CM, $f$ and $g$ share $\infty$ IM, then one of the following two conclusions holds:

(i) $f \equiv tg$, where $t$ is a constant such that $t^{n+1} = 1$;
(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ are such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

**Remark 1.1.** Theorems E (Theorem 1.2 in [3]) and F (Theorem 1.3 in [3]) are new and seem fine. However, in the statements of both the Theorems E and F there are some contradiction. It is assumed that $f$ and $g$ have zeros of multiplicities at least $k$ in Theorem E and $k + 1$ in Theorem F. But further authors concluded that “$f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1, c_2, c \in \mathbb{C}$ are such that $4(c_1 c_2)^{n+1} c^2 = -1$” in Theorem E and “$f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ are such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$” in Theorem F. Here we see that $f$ and $g$ have no zeros, so their multiplicities are equal to $k = 0$. Furthermore, it is assumed that $k \in \mathbb{N}$, but in both Theorems E and F the case $k = 0$ is also considered, which is very strange.

The above discussion is sufficient enough to make oneself inquisitive to investigate the accurate forms of Theorems E and F. Also it is quite natural to ask the following questions.

**Question 1.2.** Can one remove the condition “zeros of $f$ and $g$ are of multiplicities at least $k(k + 1)$, where $k \in \mathbb{N}$” in Theorem E (Theorem F) keeping all the conclusions intact?

**Question 1.3.** Does Theorem F hold for $k \geq 6$?

We now explain the notation of weighted sharing as introduced in [10].
Definition 1.1 ([10]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f, g$ share the value $a$ with weight $k$.

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Clearly if $f, g$ share $(a, k)$, then $f, g$ share $(a, p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$, respectively.

2. Main results

In this paper, taking the possible answers of the above questions into background we obtain the following results which significantly rectify, improve and generalize Theorems E and F. Throughout this paper we use the following notation:

$$
\begin{align*}
  s &= \sum_{i=1}^{k} n_i^*, \\
  t &= \sum_{i=1}^{k} n_i^+, \\
  m &= \sum_{i=1}^{k} in_i^* \\
  m_1 &= \sum_{i=1}^{k} in_i^{**},
\end{align*}
$$

where $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$ and $n_k, k \in \mathbb{N}$. Also it is clear that $m_1 \leq sm$.

In this paper we always use $p(z)$ to denote a nonzero polynomial such that either $\deg(p) \leq n + s - 1$ or the zeros of $p(z)$ are of multiplicities at most $n - 1$, i.e.,

$$(2.1) \quad p(z) = a_n(z - z_1)^{l_1}(z - z_2)^{l_2} \ldots (z - z_t)^{l_t},$$

where $a_n \in \mathbb{C} \setminus \{0\}$, $z_i \in \mathbb{C}$, $i = 1, 2, \ldots, t$ are distinct and $l_1, l_2, \ldots, l_t \in \mathbb{N}$. Here we see that either $\sum_{i=1}^{t} l_i \leq n + s - 1$ or $l_i \leq n - 1$ for all $i = 1, 2, \ldots, t$.

Theorem 2.1. Let $f, g$ be two transcendental meromorphic functions, let $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$ be such that $n > 2s + m + 2t + 2$ and let $p(z)$ be defined as in (2.1). Let $f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} - p(z)$ and $g^n \prod_{i=1}^{k} (g^{(i)})^{n_i} - p(z)$ share $(0, k_1)$, where $k_1 = (3 + m_1 - s)/(n + s + m_1 - 2m - 1) + 3$, and $f$ and $g$ share $\infty$ IM.

Suppose $p(z)$ is not a constant. Then

(1) when each $l_i$ is a multiple of $n_1$, $i = 1, 2, \ldots, t$, where $l_i$ is defined as in (2.1),

then one of the following two conclusions holds:

$$
(1.1) \quad f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} \equiv g^n \prod_{i=1}^{k} (g^{(i)})^{n_i}. \quad \text{In particular, when } f, g \text{ share } 0 \text{ CM and } f(z)/g(z) \neq e^{az+b}, \quad \text{where } a, b \in \mathbb{C}, (a \neq 0), \quad \text{then } f \equiv tg, \quad \text{where } t \text{ is a constant such that } t^{n+s} = 1;
$$

(2) when each $l_i$ is not a multiple of $n_1$, $i = 1, 2, \ldots, t$, then one of the following two conclusions holds:

$$
(2.1) \quad f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} \equiv g^n \prod_{i=1}^{k} (g^{(i)})^{n_i}. \quad \text{In particular, when } f, g \text{ share } 0 \text{ IM and } f(z)/g(z) \neq e^{az+b}, \quad \text{where } a, b \in \mathbb{C}, (a \neq 0), \quad \text{then } f \equiv tg, \quad \text{where } t \text{ is a constant such that } t^{n+s} = 1;
$$

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\[(1.2) \ f(z) = c_1 e^{Q(z)}, \ g(z) = c_2 e^{-Q(z)}, \text{ where } Q(z) = \int_0^z p^{1/n_1}(t) \, dt, \ c_1, c_2, c \in \mathbb{C} \]\n
are such that \(c^{2n_1} (c_1 c_2)^{n+n_1} = (-1)^{n_1},\)

(2) when at least one of \(l_i\) is not a multiple of \(n_1, i = 1, 2, \ldots, t,\) then

\[f^n \prod_{i=1}^k (f^{(i)})^{n_i} \equiv g^n \prod_{i=1}^k (g^{(i)})^{n_i}.
\]

In particular, when \(f, g\) share 0 CM and \(f(z)/g(z) \neq e^{az+b},\) where \(a, b \in \mathbb{C}\) \((a \neq 0),\) then \(f \equiv tg,\) where \(t\) is a constant such that \(t^{n+s} = 1.\)

Suppose \(p(z) = b \in \mathbb{C} \setminus \{0\}.\) Then one of the following two conclusions holds:

(i) \(f^n \prod_{i=1}^k (f^{(i)})^{n_i} \equiv g^n \prod_{i=1}^k (g^{(i)})^{n_i}.\) In particular, when \(f, g\) share 0 CM and \(f(z)/g(z) \neq e^{az+b},\) where \(a, b \in \mathbb{C}\) \((a \neq 0),\) then \(f \equiv tg,\) where \(t\) is a constant such that \(t^{n+s} = 1;\)

(ii) \(f(z) = c_3 e^{cz}, \ g(z) = c_4 e^{-cz},\) where \(c_3, c_4, c \in \mathbb{C}\) are such that \((-1)^{m_1} (c_3 c_4)^{n+s} \times c^{2m_1} = b^2.\)

Remark 2.1. Instead of \(f\) and \(g\) share 0 CM, one can assume that \(f^{(k)}\) and \(g^{(k)}\) share 0 CM in Theorem 2.1 when \(n_i = 0, i = 1, 2, \ldots, k - 1.\)

We now explain some definitions and notation which are used in the paper.

**Definition 2.1** ([12]). Let \(p \in \mathbb{N}\) and \(a \in \mathbb{C} \cup \{\infty\}.\)

(i) \(N(r, a; f \geq p) (N(r, a; f \geq p))\) denotes the counting function (reduced counting function) of those \(a\)-points of \(f\) whose multiplicities are not less than \(p.\)

(ii) \(N(r, a; f \leq p) (N(r, a; f \leq p))\) denotes the counting function (reduced counting function) of those \(a\)-points of \(f\) whose multiplicities are not greater than \(p.\)

**Definition 2.2.** We denote by \(\overline{N}(r, a; f = k)\) the reduced counting function of those \(a\)-points of \(f\) whose multiplicities are exactly \(k,\) where \(k \in \mathbb{N}.\)

**Definition 2.3** ([19]). For \(a \in \mathbb{C} \cup \{\infty\}\) and \(p \in \mathbb{N}\) we denote by \(N_p(r, a; f)\) the sum \(\overline{N}(r, a; f) + \overline{N}(r, a; f \geq 2) + \ldots + \overline{N}(r, a; f \geq p).\) Clearly \(N_1(r, a; f) = \overline{N}(r, a; f).\)

**Definition 2.4** ([1]). Let \(f\) and \(g\) be two non-constant meromorphic functions such that \(f\) and \(g\) share the value 1 IM. Let \(z_0\) be a 1-point of \(f\) with multiplicity \(p,\) a 1-point of \(g\) with multiplicity \(q.\) We denote by \(\overline{N}_E(r, 1; f)\) the counting function of those 1-points of \(f\) and \(g\) where \(p > q,\) by \(N^{(1)}_E(r, 1; f)\) the counting function of those 1-points of \(f\) and \(g\) where \(p = q = 1\) and by \(\overline{N}_E^{(2)}(r, 1; f)\) the counting function of those 1-points of \(f\) and \(g\) where \(p = q \geq 2,\) each point in these counting functions being counted only once. In the same way we can define \(\overline{N}_L(r, 1; g), \ N^{(1)}_E(r, 1; g), \ \overline{N}_E^{(2)}(r, 1; g).\)

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**Definition 2.5 ([10])**. Let \( f, g \) share a value \( a \) IM. We denote by \( \overline{\mathcal{N}}_*(r, a; f, g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \). Clearly

\[
\overline{\mathcal{N}}_*(r, a; f, g) \equiv \overline{\mathcal{N}}_*(r, a; g, f) = \overline{\mathcal{N}}_L(r, a; f) + \overline{\mathcal{N}}_L(r, a; g).
\]

3. **Lemmas**

Let \( F, G \) be two non-constant meromorphic functions. Henceforth we shall denote by \( H \) and \( V \) the functions

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right)
\]

and

\[
V = \left( \frac{F'}{F-1} - \frac{F'}{F} \right) - \left( \frac{G'}{G-1} - \frac{G'}{G} \right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.
\]

**Lemma 3.1** ([20]). Let \( f \) be a non-constant meromorphic function and \( p, k \in \mathbb{N} \), then

\[
N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k \overline{N}(r, \infty; f) + S(r, f).
\]

**Lemma 3.2** ([11]). If \( N(r, 0; f^{(k)} | f \neq 0) \) denotes the counting function of those zeros of \( f^{(k)} \) which are not the zeros of \( f \), where a zero of \( f^{(k)} \) is counted according to its multiplicity, then

\[
N(r, 0; f^{(k)} | f \neq 0) \leq k \overline{N}(r, \infty; f) + N(r, 0; f | < k) + k \overline{N}(r, 0; f | \geq k) + S(r, f).
\]

**Lemma 3.3** ([8]). Suppose that \( f \) is a non-constant meromorphic function, \( k \in \mathbb{N} \setminus \{1\} \). If

\[
N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S\left(r, \frac{f'}{f}\right),
\]

then \( f(z) = e^{az+b} \), where \( a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C} \).

**Lemma 3.4** ([15]). Let \( f \) be a non-constant meromorphic function and \( P(f) = a_0 + a_1f + a_2f^2 + \ldots + a_nf^n \), where \( a_0, a_1, \ldots, a_{n-1} \in \mathbb{C} \) and \( a_n \in \mathbb{C} \setminus \{0\} \). Then

\[
T(r, P(f)) = nT(r, f) + O(1).
\]
Lemma 3.5. Let $f$ be a transcendental meromorphic function and $n, n_k, k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. Then $\varphi = f^n(f')^{n_1} \ldots (f^{(k)})^{n_k}$ is non-constant.

Proof. Suppose $\varphi$ is constant. Then $\overline{N}(r, 0; f) = 0$ and $\overline{N}(r, \infty; f) = 0$. Also we see that
\[
\left(\frac{1}{f}\right)^{n+s} = \frac{(f')^{n_1} \ldots (f^{(k)})^{n_k}}{f^s} \frac{1}{\varphi}.
\]
Then by Lemma 3.4 we have
\[
(n + s)T(r, f) \leq \sum_{i=1}^{k} n_i^* T\left(r, \frac{f(i)}{f}\right) + T\left(r, \frac{1}{\varphi}\right) + O(1)
\]
\[
\leq \sum_{i=1}^{k} n_i^* N\left(r, \infty; \frac{f(i)}{f}\right) + S(r, f)
\]
\[
\leq \sum_{i=1}^{k} n_i^* (N_i(r, 0; f) + i\overline{N}(r, \infty; f)) + S(r, f) = S(r, f),
\]
which is impossible. Hence $\varphi$ is non-constant. This completes the proof. \qed

Lemma 3.6 ([17]). Let $f_j$, $j = 1, 2, 3$ be meromorphic and $f_1$ non-constant. Suppose that
\[
\sum_{j=1}^{3} f_j = 1
\]
and
\[
\sum_{j=1}^{3} N(r, 0; f_j) + 2 \sum_{j=1}^{3} \overline{N}(r, \infty; f_j) < (\lambda + o(1))T(r),
\]
as $r \to \infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1 \leq j \leq 3} T(r, f_j)$. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 3.7 ([17], Theorem 1.24). Let $f$ be a non-constant meromorphic function and let $k \in \mathbb{N}$. Suppose that $f^{(k)} \not\equiv 0$, then
\[
N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).
\]

Lemma 3.8 ([7]). Let $f(z)$ be a non-constant entire function and let $k \in \mathbb{N} \setminus \{1\}$. If $f(z)f^{(k)}(z) \not\equiv 0$, then $f(z) = e^{az+b}$, where $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$.

Lemma 3.9 ([8], [18]). Let $f$ be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then
\[
T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).
\]
Lemma 3.10. Let $f$, $g$ be two non-constant meromorphic functions and $F = f^n \prod_{i=1}^{k} (f^{(i)})^{n_i}$, $G = g^n \prod_{i=1}^{k} (g^{(i)})^{n_i}$, where $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. Suppose $H \neq 0$. If $F, G$ share $(1, k_1)$, $f$, $g$ share $(\infty, p)$, where $k_1 \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $p \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, then

\[(n+s)(p+1)+m_1-1)N(r,\infty;f|\geq p+1) \leq N(r,0;F)+N(r,0;G)+\mathcal{N}_s(r,1;F,G)+S(r).
\]

Proof. Suppose $\infty$ is an e.v.P of both $f$ and $g$, then the lemma follows immediately.

Next suppose $\infty$ is not an e.v.P of $f$ and $g$. We assert that $V \equiv 0$. If not, suppose $V \equiv 0$. Then by integration we obtain

\[1 - \frac{1}{F} \equiv A\left(1 - \frac{1}{G}\right).
\]

It means that if $z_0$ is a pole of $f$ then it is a pole of $g$. Hence from the definition of $F$ and $G$ we have $1/F(z_0) = 0$ and $1/G(z_0) = 0$. So $A = 1$ and hence $F \equiv G$. Consequently $H \equiv 0$, which contradicts our assumption. Hence $V \equiv 0$. Let $z_0$ be a pole of $f$ with multiplicity $q$ and a pole of $g$ with multiplicity $r$. If both $q$ and $r$ are $\leq p$, then $q = r$ but when both $q$ and $r$ are $\geq p+1$, they may or may not be equal. Clearly $z_0$ is a pole of $F$ with multiplicity $(n+s)q + m_1$ and a pole of $G$ with multiplicity $(n+s)r + m_1$. We note that there is no pole of $F$ and $G$ of order $t_1$ satisfying $(n+s)p + m_1 + 1 \leq t_1 \leq (n+s)(p+1) + m_1 - 1$. Since $f$ and $g$ share $(\infty,p)$, from the definition of $V$ it is clear that $z_0$ is a zero of $V$ with multiplicity at least $(n+s)(p+1) + m_1 - 1$.

So from the definition of $V$ we have

\[(n+s)(p+1)+m_1-1)N(r,\infty;f|\geq p+1) \leq N(r,0;V) \leq N(r,\infty;V) + S(r,f) + S(r,g) \leq N(r,0;F) + N(r,0;G) + \mathcal{N}_s(r,1;F,G) + S(r).
\]

This completes the proof. \[\Box\]

Lemma 3.11. Let $f$, $g$ be two non-constant meromorphic functions, $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. Suppose $H \neq 0$. If $F, G$ share $(1, k_1)$ and $f$, $g$ share $(\infty, 0)$, where $F$ and $G$ are given as in Lemma 3.10, $k_1 \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, then

\[N(r,\infty;f) \leq \frac{2(t+1)}{n+s+m_1-2m-1}T(r) + \frac{1}{n+s+m_1-2m-1}\mathcal{N}_s(r,1;F,G) + S(r).
\]
Proof. Using Lemmas 3.2 and 3.10 for \( p = 0 \) we get

\[
(n + s + m_1 - 1)\overline{N}(r, \infty; f) \\
\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq \overline{N}(r, 0; f) + \sum_{i=1}^{k} n_i \overline{N}(r, 0; f^{(i)} | f \neq 0) + \overline{N}(r, 0; g) \\
+ \sum_{i=1}^{k} n_i \overline{N}(r, 0; g^{(i)} | g \neq 0) + \overline{N}(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq \overline{N}(r, 0; f) + m\overline{N}(r, \infty; f) + t\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + m\overline{N}(r, \infty; g) \\
+ t\overline{N}(r, 0; g) + \overline{N}(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq 2(t + 1)T(r) + 2m\overline{N}(r, \infty; f) + \overline{N}(r, 1; F, G) + S(r).
\]

Hence the lemma follows. \( \square \)

**Lemma 3.12.** Let \( f \) be a non-constant meromorphic function, \( F = f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} \), where \( n, n_k, k \in \mathbb{N} \) and \( n_i \in \mathbb{N} \cup \{0\}, i = 1, 2, \ldots, k - 1 \) are such that \( n > s \). Then

\[
(n - s)T(r, f) \leq T(r, F) - sN(r, \infty; f) - N\left(r, 0; \prod_{i=1}^{k} (f^{(i)})^{n_i}\right) + S(r, f).
\]

Proof. Note that

\[
N(r, \infty; F) = N(r, \infty; f^n) + N\left(r, \infty; \prod_{i=1}^{k} (f^{(i)})^{n_i}\right) \\
= N(r, \infty; f^n) + sN(r, \infty; f) + m_1\overline{N}(r, \infty; f) + S(r, f).
\]

That is,

\[
N(r, \infty; f^n) = N(r, \infty; F) - sN(r, \infty; f) - m_1\overline{N}(r, \infty; f) + S(r, f).
\]

Also

\[
m(r, f^n) = m\left(r, \frac{F}{\prod_{i=1}^{k} (f^{(i)})^{n_i}}\right) \leq m(r, F) + m\left(r, \frac{1}{\prod_{i=1}^{k} (f^{(i)})^{n_i}}\right) + S(r, f) \\
= m(r, F) + T\left(r, \prod_{i=1}^{k} (f^{(k)})^{n_k}\right) - N\left(r, 0; \prod_{i=1}^{k} (f^{(k)})^{n_i}\right) + S(r, f)
\]

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second theorem for small functions (see [14]) we get
\[ n > s + 1 \]
This completes the proof.

\[ N = m(r, F) + N\bigg( r, \infty; \prod_{i=1}^{k} (f^{(i)})^{n_i} \bigg) + m(r, 0; \prod_{i=1}^{k} (f^{(i)})^{n_i}) \]
\[ = m(r, F) + sN(r, \infty; f) + m_1 \overline{N}(r, \infty; f) + m\bigg( r, 0; \prod_{i=1}^{k} (f^{(i)})^{n_i} \bigg) \]
\[ + m(r, f^s) - N\bigg( r, 0; \prod_{i=1}^{k} (f^{(i)})^{n_i} \bigg) + S(r, f) \]
\[ = m(r, F) + sT(r, f) + m_1 \overline{N}(r, \infty; f) - N\bigg( r, 0; \prod_{i=1}^{k} (f^{(i)})^{n_i} \bigg) + S(r, f). \]

Now
\[ nT(r, f) = N(r, \infty; f^n) + m(r, f^n) \]
\[ \leq T(r, F) + sT(r, f) - sN(r, \infty; f) - N\bigg( r, 0; \prod_{i=1}^{k} (f^{(i)})^{n_i} \bigg) + S(r, f), \]
i.e.,
\[ (n - s)T(r, f) \leq T(r, F) - sN(r, \infty; f) - N\bigg( r, 0; \prod_{i=1}^{k} (f^{(i)})^{n_i} \bigg) + S(r, f). \]
This completes the proof. \( \square \)

**Lemma 3.13.** Let \( f \) be a transcendental meromorphic function, \( n, n_k, k \in \mathbb{N}, n_i \in \mathbb{N} \cup \{0\}, i = 1, 2, \ldots, k - 1 \) and let \( a(z) \) (\( \not\equiv 0, \infty \)) be a small function of \( f \). If \( n > s + 1 \), then \( f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - a(z) \) has infinitely many zeros.

**Proof.** Let \( F = f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} \). Now in view of Lemma 3.12 and the second theorem for small functions (see [14]) we get
\[ (n - s)T(r, f) \leq T(r, F) - sN(r, \infty; f) - N\bigg( r, 0; (f')^{n_1} \cdots (f^{(k)})^{n_k} \bigg) + S(r, f) \]
\[ \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, a(z); F) - sN(r, \infty; f) \]
\[ - N(r, 0; (f')^{n_1} \cdots (f^{(k)})^{n_k}) + (\varepsilon + o(1))T(r, f) \]
\[ \leq \overline{N}(r, 0; f) + \overline{N}(r, a(z); F) + (\varepsilon + o(1))T(r, f) \]
\[ \leq T(r, f) + \overline{N}(r, a(z); F) + (\varepsilon + o(1))T(r, f) \]
for all \( \varepsilon > 0 \). Take \( \varepsilon < 1 \). Since \( n > s + 1 \), from the above one can easily see that \( F - a(z) \) has infinitely many zeros. This completes the proof. \( \square \)
Lemma 3.14 ([9]). Let $f$ and $g$ be two non-constant meromorphic functions. Suppose that $f$ and $g$ share $0$ and $\infty$ CM, $f^{(k)}$ and $g^{(k)}$ share $0$ CM for $k = 1, 2, \ldots, 6$. Then $f$ and $g$ satisfy one of the following conditions:

(i) $f \equiv tg$, where $t \neq 0$ is a constant,
(ii) $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, where $a, b, c$ and $d$ are constants such that $ac \neq 0$,
(iii) $f(z) = a/(1 - be^{\alpha(z)})$, $g(z) = a/(e^{-\alpha(z)} - b)$, where $a, b$ are nonzero constants and $\alpha(z)$ is a non-constant entire function,
(iv) $f(z) = a(1-be^{cz})$, $g(z) = d(e^{-cz} - b)$, where $a, b, c$ and $d$ are nonzero constants.

Lemma 3.15. Let $f$ and $g$ be two transcendental meromorphic functions such that $f(z)/g(z) \neq e^{az+b}$, where $a, b \in \mathbb{C}$ ($a \neq 0$) and let $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k - 1$ be such that $n \geq 2$. Suppose $f$ and $g$ share $0$ CM and $\infty$ IM. If \[ f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} \equiv g^n \prod_{i=1}^{k} (g^{(i)})^{n_i} \]
then $f \equiv tg$, where $t$ is a constant such that $t^{n+s} = 1$.

Proof. Suppose \[ f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} = g^n \prod_{i=1}^{k} (g^{(i)})^{n_i}, \]
i.e.,
\[ \frac{f^n}{g^n} = \prod_{i=1}^{k} (g^{(i)})^{n_i} / \prod_{i=1}^{k} (f^{(i)})^{n_i}. \]
Since $f$ and $g$ share $\infty$ IM, it follows from (3.3) that $f$ and $g$ share $\infty$ CM and so $f^{(i)}$ and $g^{(i)}$ share $\infty$ CM, where $i = 1, 2, \ldots, k$. Let $h_1 = f/g$ and $h_2 = \frac{\prod_{i=1}^{k} (f^{(i)})^{n_i}}{\prod_{i=1}^{k} (g^{(i)})^{n_i}}$. Since $f$ and $g$ share $0$ CM, it follows that $h_1 \neq 0, \infty$ and $h_2 \neq 0, \infty$. From (3.4) we see that
\[ h_1^n h_2 \equiv 1. \]
First we suppose $h_1$ is a non-constant entire function. Clearly $h_2$ is also a non-constant entire function. Let $F_1 = h_1^n$ and $G_1 = h$. From (3.5) we get
\[ F_1 G_1 \equiv 1. \]
Clearly $F_1 \neq dG_1$, where $d$ is a nonzero constant, otherwise $F_1$ would be a constant and so $h_1$ would be a constant. Since $F_1 \neq 0, \infty$ and $G_1 \neq 0, \infty$ there exist two non-constant entire functions $\alpha$ and $\beta$ such that $F_1 = e^\alpha$ and $G_1 = e^\beta$. Now from (3.6)
we see that $\alpha + \beta = C$, where $C \in \mathbb{C}$. Therefore $\alpha' = -\beta'$. Note that $F_1' = \alpha' e^{\alpha}$ and $G_1' = \beta' e^{\beta}$. This shows that $F_1'$ and $G_1'$ share 0 CM. Note that if $F_1 \neq 0, \infty, G_1 \neq 0, \infty$ and $F_1 \neq dG_1$, where $d$ is a nonzero constant. Now in view of Lemma 3.14 we have to consider the case

$$F_1(z) = c_1 e^{az} \quad \text{and} \quad G_1(z) = c_2 e^{-az},$$

where $a, c_1, c_2$ are nonzero constants such that $c_1 c_2 = 1$. Since $(f(z)/g(z))^n = c_1 e^{az}$, it follows that

$$(3.7) \quad \frac{f(z)}{g(z)} = t_1 e^{a/nz} = t_1 e^{cz},$$

where $c, t_1$ are nonzero constants such that $t_1^n = c_1$ and $c = a/n$.

Now from (3.7) we arrive at a contradiction. Hence $h_1$ is constant. Then from (3.3) we get $h_1^{n+s} = 1$. Therefore we have $f \equiv tg$, where $t$ is a constant such that $t^{n+s} = 1$. This completes the proof. \hfill \Box

Remark 3.1. Instead of $f$ and $g$ share 0 CM, one can assume that $f^{(k)}$ and $g^{(k)}$ share 0 CM in Lemma 3.15 when $n_i = 0, i = 1, 2, \ldots, k - 1$.

**Lemma 3.16.** Let $f$, $g$ be two transcendental meromorphic functions and let $f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} - p(z)$ and $g^n \prod_{i=1}^{k} (g^{(i)})^{n_i} - p(z)$ share 0 CM and $f$, $g$ share $\infty$ IM, where $p(z)$ is defined as in (2.1) and $n, n_1, \ldots, n_k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$. Suppose $f^n (f')^{n_1} \ldots (f^{(k)})^{n_k} g^n (g')^{n_1} \ldots (g^{(k)})^{n_k} \equiv p^2$.

(i) If $p(z)$ is not a constant and $l_i$ is a multiple of $n_i$ for all $i = 1, 2, \ldots, t$, where $l_i$ is defined as in (2.1), then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p^{1/n_1} (t) dt$, $c_1, c_2, c \in \mathbb{C}$ are such that $c^{n_1} (c_1 c_2)^{n_1 + n_1} = (-1)^{n_1}$.

(ii) If $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ are such that $(-1)^{n_1} (c_3 c_4)^{n_1 + s} d^{2m_1} = b^2$.

**Proof.** Suppose

$$(3.8) \quad f^n (f')^{n_1} \ldots (f^{(k)})^{n_k} g^n (g')^{n_1} \ldots (g^{(k)})^{n_k} \equiv p^2.$$ 

Since $f$ and $g$ share $\infty$ IM, from (3.8) one can easily see that $f$ and $g$ are transcendental entire functions. We now consider the following cases.

Case 1. Let $\deg(p(z)) = l \in \mathbb{N}$. From (3.8) it follows that $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$. Let

$$(3.9) \quad F_1 = \frac{f^n (f')^{n_1} \ldots (f^{(k)})^{n_k}}{p} \quad \text{and} \quad G_1 = \frac{g^n (g')^{n_1} \ldots (g^{(k)})^{n_k}}{p}.$$
From (3.8) we get

\[(3.10) \quad F_1 G_1 \equiv 1.\]

By Lemma 3.5, we have \(F_1 \not\equiv cG_1\), where \(c \in \mathbb{C} \setminus \{0\}\). Let

\[(3.11) \quad \Phi = f_n \left( f' \right)^{n_1} \cdots \left( f^{(k)} \right)^{n_k} - \frac{p}{g_n \left( g' \right)^{n_1} \cdots \left( g^{(k)} \right)^{n_k} - p}.\]

We deduce from (3.11) that

\[(3.12) \quad \Phi \equiv e^\beta,\]

where \(\beta\) is an entire function. Let \(f_1 = F_1, f_2 = -e^\beta G_1\) and \(f_3 = e^\beta\). Here \(f_1\) is transcendental. Now from (3.12) we have \(f_1 + f_2 + f_3 \equiv 1\). Hence by Lemma 3.7 we get

\[
\sum_{j=1}^{3} N(r, 0; f_j) + 2 \sum_{j=1}^{3} N(r, \infty; f_j) \leq N(r, 0; F_1) + N(r, 0; e^\beta G_1) + O(\log r)
\]

\[
\leq (\lambda + o(1)) T(r),
\]

as \(r \to \infty, r \in I, \lambda < 1\) and \(T(r) = \max_{1 \leq j \leq 3} T(r, f_j)\). So by Lemma 3.6 we get either \(e^\beta G_1 \equiv -1\) or \(e^\beta \equiv 1\). But here the only possibility is that \(e^\beta G_1 \equiv -1\), i.e.,

\[
g_n \left( g' \right)^{n_1} \cdots \left( g^{(k)} \right)^{n_k} \equiv -e^{-\beta} p(z)
\]

and so from (3.8) we obtain \(F_1 \equiv e^{\gamma_1} G_1\), i.e.,

\[
f_n \left( f' \right)^{n_1} \cdots \left( f^{(k)} \right)^{n_k} \equiv e^{\gamma_1} g_n \left( g' \right)^{n_1} \cdots \left( g^{(k)} \right)^{n_k},
\]

where \(\gamma_1\) is a non-constant entire function. Then from (3.8) we get

\[(3.13) \quad f_n \left( f' \right)^{n_1} \cdots \left( f^{(k)} \right)^{n_k} \equiv c e^{\gamma_1/2} p(z), g_n \left( g' \right)^{n_1} \cdots \left( g^{(k)} \right)^{n_k} \equiv c e^{-\gamma_1/2} p(z),
\]

where \(c \pm 1\). This shows that \(f_n \left( f' \right)^{n_1} \cdots \left( f^{(k)} \right)^{n_k}\) and \(g_n \left( g' \right)^{n_1} \cdots \left( g^{(k)} \right)^{n_k}\) share 0 CM. Since \(N(r, 0; f) = O(\log r)\) and \(N(r, 0; g) = O(\log r)\), so we can take

\[(3.14) \quad f(z) = h_1(z) e^{\alpha(z)}, \quad g(z) = h_2(z) e^{\beta(z)},
\]

where \(h_1\) and \(h_2\) are nonzero polynomials and \(\alpha, \beta\) are two non-constant entire functions. We deduce from (3.8) and (3.14) that either both \(\alpha\) and \(\beta\) are transcendental entire functions or both \(\alpha\) and \(\beta\) are polynomials. We now consider the following cases.
Subcase 1.1. Let \( k \in \mathbb{N} \setminus \{1\} \). First we suppose both \( \alpha \) and \( \beta \) are transcendental entire functions. Let \( \alpha_1 = \alpha' + h_1'/h_1 \) and \( \beta_1 = \beta' + h_2'/h_2 \). Clearly both \( \alpha_1 \) and \( \beta_1 \) are transcendental. Note that

\[
S(r, \alpha_1) = S\left(r, \frac{f'}{f}\right), \quad S(r, \beta_1) = S\left(r, \frac{g'}{g}\right).
\]

Moreover, we see that

\[
N(r, 0; f^{(k)}_{\alpha}) = O(\log r), \quad N(r, 0; g^{(k)}_{\beta}) = O(\log r).
\]

From these inequalities and using (3.14) we have

\[
N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \alpha_1) = S\left(r, \frac{f'}{f}\right)
\]

and

\[
N(r, \infty; g) + N(r, 0; g) + N(r, 0; g^{(k)}) = S(r, \beta_1) = S\left(r, \frac{g'}{g}\right).
\]

Then from (3.15), (3.16) and Lemma 3.3 we have

\[
f(z) = e^{az+b}, \quad g(z) = e^{cz+d},
\]

where \( a, c \in \mathbb{C} \setminus \{0\} \), \( b, d \in \mathbb{C} \). But these types of \( f \) and \( g \) do not agree with the relation (3.8). Next we suppose both \( \alpha \) and \( \beta \) are polynomials. Also from (3.8) we get \( \alpha + \beta \equiv C \) i.e., \( \alpha' \equiv -\beta' \). Therefore \( \deg(\alpha) = \deg(\beta) \). We deduce from (3.14) that

\[
f^{n_1}(f')^{n_1} \cdots (f^{(k)})^{n_k} = \prod_{i=1}^{k}(h_1(\alpha')^i + P_{i-1}(\alpha', h_1'))^{n_i}e^{(n+s)\alpha}
\]

\[
\equiv p(z)e^{(n+s)\alpha},
\]

and

\[
g^{n_k}(g')^{n_k} \cdots (g^{(k)})^{n_k} = \prod_{i=1}^{k}(h_2(\beta')^i + Q_{i-1}(\beta', h_2'))^{n_i}e^{(n+s)\beta}
\]

\[
\equiv p(z)e^{(n+s)\beta},
\]

where \( A, B \in \mathbb{C}\setminus\{0\} \), and \( P_{i-1}(\alpha', h_1') \) and \( Q_{i-1}(\beta', h_2') \), \( i = 1, 2, \ldots, k \) are differential polynomials in \( \alpha', h_1' \) and \( \beta', h_2' \), respectively.
Since \( p(z) \) is a polynomial, from (3.18) and (3.19) we conclude that both \( h_1, h_2 \in \mathbb{C} \setminus \{0\} \). So we can rewrite \( f \) and \( g \) as

\[
(3.20) \quad f = e^{\gamma_2}, \quad g = e^{\delta_2},
\]

where \( \gamma_2 + \delta_2 \equiv C \in \mathbb{C} \setminus \{0\} \) and \( \deg(\gamma_2) = \deg(\delta_2) \). Clearly \( \gamma'_2 \equiv -\delta'_2 \). If \( \deg(\gamma_2) = \deg(\delta_2) = 1 \), we then again get a contradiction from (3.8). Next we suppose \( \deg(\gamma_2) = \deg(\delta_2) \geq 2 \). We deduce from (3.20) that

\[
\begin{align*}
f' &= \gamma'_2 e^{\gamma_2}, \\
f'' &= ((\gamma'_2)^2 + \gamma''_2)e^{\gamma_2}, \\
f''' &= ((\gamma'_2)^3 + 3\gamma'_2 \gamma''_2 + \gamma'''_2)e^{\gamma_2}, \\
f^{(iv)} &= ((\gamma'_2)^4 + 6(\gamma'_2)^2 \gamma''_2 + 3(\gamma''_2)^2 + 4\gamma'_2 \gamma'''_2 + \gamma^{(iv)}_2)e^{\gamma_2}, \\
f^{(v)} &= ((\gamma'_2)^5 + 10(\gamma'_2)^3 \gamma''_2 + 15\gamma'_2 (\gamma''_2)^2 + 10(\gamma'_2)^2 \gamma'''_2 + 10\gamma''_2 \gamma''''_2 + 5\gamma'_2 \gamma^{(iv)}_2 + \gamma^{(v)}_2)e^{\gamma_2}, \\
& \vdots \\
f^{(k)} &= ((\gamma'_2)^k + \frac{k(k-1)}{2} (\gamma'_2)^{k-2} \gamma''_2 + P_{k-2}(\gamma'_2))e^{\gamma_2}.
\end{align*}
\]

Similarly we get

\[
\begin{align*}
g^{(k)} &= (\delta'_2)^k + \frac{k(k-1)}{2} (\delta'_2)^{k-2} \delta''_2 + P_{k-2}(\delta'_2)\right)e^{\delta_2} \\
&= \left((-1)^k(\gamma'_2)^k + \frac{k(k-1)}{2} (-1)^{k-1} (\gamma'_2)^{k-2} \gamma''_2 + P_{k-2}(-\gamma'_2)\right)e^{\delta_2},
\end{align*}
\]

where \( P_{k-2}(\gamma'_2) \) is a differential polynomial in \( \gamma'_2 \). Since \( \deg(\gamma_2) \geq 2 \), we observe that \( \deg((\gamma'_2)^k) \geq k \deg(\gamma'_2) \) and so \( (\gamma'_2)^{k-2} \gamma''_2 \) is either a nonzero constant or \( \deg((\gamma'_2)^{k-2} \gamma''_2) \geq (k-1) \deg(\gamma'_2) - 1 \). Also we see that

\[
\deg((\gamma'_2)^k) > \deg((\gamma'_2)^{k-2} \gamma''_2) > \deg(P_{k-2}(\gamma'_2)) \quad (\text{or } \deg(P_{k-2}(-\gamma'_2))).
\]

Since \( f \) and \( g \) have no zeros, from (3.13) it follows that \((f')^{n_1} \cdots (f^{(k)})^{n_k} \) and \((g')^{n_1} \cdots (g^{(k)})^{n_k} \) share 0 CM and so

\[
(3.21) \quad ((\gamma'_2)^i)^{n_1} \prod_{i=2}^{k} \left((\gamma'_2)^i + \frac{i(i-1)}{2} (\gamma'_2)^{i-2} \gamma''_2 + P_{i-2}(\gamma'_2)\right)^{n_i} \equiv d^{n_1} ((\gamma'_2)^i)^{n_1} \\
\times \prod_{i=2}^{k} \left((-1)^i(\gamma'_2)^i + \frac{i(i-1)}{2} (-1)^{i-1} (\gamma'_2)^{i-2} \gamma''_2 + P_{i-2}(-\gamma'_2)\right)^{n_i},
\]

where \( d \in \mathbb{C} \setminus \{0\} \).
Now from (3.21) we arrive at a contradiction since \( k \geq 2 \).

**Subcase 1.2.** Let \( k = 1 \). Suppose that \( \alpha \) and \( \beta \) are transcendental. Then from (3.8) and (3.14) we get

\[
(3.22) \quad (h_1 h_2)^n (h_1 \alpha' + h_1')^{n_1} (h_2 \beta' + h_2')^{n_1} e^{(n+n_1)(\alpha+\beta)} \equiv p^2(z).
\]

Let \( \alpha + \beta = \gamma \) and \( s_1 = n + n_1 \). From (3.22) we know that \( \gamma \) is not a constant since in that case we get a contradiction. Now from (3.22) we get

\[
(3.23) \quad (h_1 h_2)^n (h_1 \alpha' + h_1')^{n_1} (h_2 (\gamma' - \alpha') + h_2')^{n_1} e^{s_1 \gamma} \equiv p^2(z).
\]

We have \( T(r, \gamma') = m(r, s_1 \gamma') + O(1) = m(r, (e^{s_1 \gamma})'/e^{s_1 \gamma}) = S(r, e^{s_1 \gamma}) \). Thus from (3.23) we get

\[
T(r, e^{s_1 \gamma}) \leq T\left(r, \frac{p^2}{(h_1 h_2)^n (h_1 \alpha' + h_1')^{n_1} (h_2 (\gamma' - \alpha') + h_2')^{n_1}}\right) + O(1)
\]

\[
\leq n_1 T(r, \alpha') + n_1 T(r, \gamma' - \alpha') + O(\log r) + O(1)
\]

\[
\leq 2n_1 T(r, \alpha') + S(r, \alpha') + S(r, e^{s_1 \gamma}),
\]

which implies that \( T(r, e^{s_1 \gamma}) = O(T(r, \alpha')) \) and so \( S(r, e^{s_1 \gamma}) \) can be replaced by \( S(r, \alpha') \). Thus we get \( T(r, \gamma') = S(r, \alpha') \) and so \( \gamma' \) is a small function with respect to \( \alpha' \). In view of (3.23) and by Lemma 3.9 we get

\[
T(r, \alpha') \leq \mathcal{N}(r, \infty; \alpha') + \mathcal{N}(r, 0; h_1 \alpha' + h_1') + \mathcal{N}(r, 0; h_2 (\gamma' - \alpha') + h_2') + S(r, \alpha')
\]

\[
\leq O(\log r) + S(r, \alpha'),
\]

which shows that \( \alpha' \) is a polynomial and so \( \alpha \) is a polynomial. Similarly we can prove that \( \beta \) is also a polynomial. This contradicts the fact that \( \alpha \) and \( \beta \) are transcendental. Next suppose without loss of generality that \( \alpha \) is a polynomial and \( \beta \) is a transcendental entire function. Then \( \gamma \) is transcendental. So in view of (3.23) we obtain

\[
s_1 T(r, e^{\gamma}) \leq T\left(r, \frac{p^2}{(h_1 h_2)^n (h_1 \alpha' + h_1')^{n_1} (h_2 (\gamma' - \alpha') + h_2')^{n_1}}\right) + O(1)
\]

\[
\leq n_1 T(r, \alpha') + n_1 T(r, \gamma' - \alpha') + S(r, e^{\gamma})
\]

\[
\leq n_1 T(r, \gamma') + S(r, e^{\gamma}) = S(r, e^{\gamma}),
\]

which leads to a contradiction. Thus both \( \alpha \) and \( \beta \) are polynomials. From (3.8) we conclude that \( \alpha(z) + \beta(z) \equiv C \in \mathbb{C} \) and so \( \alpha'(z) + \beta'(z) \equiv 0 \). We deduce from (3.8) that

\[
(3.24) \quad f^n (f')^{n_1} \equiv h_1^n (h_1 \alpha' + h_1')^{n_1} e^{(n+n_1)\alpha} \equiv p(z) e^{(n+n_1)\alpha},
\]

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and

\[(3.25) \quad g^n(g')^{n_1} = h_2^\prime (h_2^{\prime \prime} + h_2')e^{(n+n_1)\beta} \equiv p(z)e^{(n+n_1)\beta}.
\]

Since \(p(z)\) is a polynomial, from (3.24) and (3.25) we conclude that both \(h_1\) and \(h_2\) are nonzero constant. So we can rewrite \(f\) and \(g\) as

\[(3.26) \quad f = e^{\gamma_3}, \quad g = e^{\delta_3}.
\]

Now from (3.8) we get

\[(3.27) \quad (\gamma'_3)^{n_1}(\delta'_3)^{n_1}e^{(n+n_1)(\gamma_3+\delta_3)} \equiv p^2.
\]

From (3.27) we can conclude that \(\gamma_3(z) + \delta_3(z) \equiv C \in \mathbb{C}\) and so \(\gamma'_3(z) + \delta'_3(z) \equiv 0\).

Thus from (3.27) we get \(e^{(n+n_1)C}(\gamma'_3)^{n_1}(\delta'_3)^{n_1} \equiv p^2(z)\), i.e.,

\[(3.28) \quad (-1)^n_1e^{(n+n_1)C(\gamma'_3)^{2n_1}} \equiv p^2(z).
\]

We now consider the following two subcases.

**Subcase 1.2.1.** Suppose at least one of \(l_i, i = 1, 2, \ldots, t\) is not a multiple of \(n_1\).

As \(\gamma'_3\) is a polynomial, from (3.28) we arrive at a contradiction.

**Subcase 1.2.2.** Suppose \(l_i\) is a multiple of \(n_1\) for all \(i = 1, 2, \ldots, t\). By computation, from (3.28) we get

\[(3.29) \quad \gamma'_3 = cp^{1/n_1}(z), \quad \delta'_3 = -cp^{1/n_1}(z).
\]

Hence

\[(3.30) \quad \gamma_3(z) = cQ(z) + b_1, \quad \delta_3(z) = -cQ(z) + b_2,
\]

where \(Q(z) = \int_0^z p^{1/n_1}(t) \, dt\) and \(b_1, b_2 \in \mathbb{C}\). Finally, we take \(f\) and \(g\) as

\[f(z) = c_1e^{cQ(z)}, \quad g(z) = c_2e^{-cQ(z)},
\]

where \(c_1, c_2 \in \mathbb{C}\) and \(c \in \mathbb{C} \setminus \{0\}\) such that \(c^{2n_1}(c_1c_2)^{n+n_1} = (-1)^n_1\).

**Case 2.** Let \(p(z) = b \in \mathbb{C} \setminus \{0\}\). Then from (3.8) we get

\[(3.31) \quad f^n(f')^{n_1} \ldots (f^{(k)})^{n_k}g^n(g')^{n_1} \ldots (g^{(k)})^{n_k} \equiv b^2,
\]

where \(f\) and \(g\) are transcendental entire functions. Clearly \(f\) and \(g\) have no zeros and so we can take \(f\) and \(g\) as

\[(3.32) \quad f = e^\alpha, \quad g = e^\beta,
\]

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where $\alpha(z)$, $\beta(z)$ are two non-constant entire functions. We now consider the following two subcases.

**Subcase 2.1.** Let $k \geq 2$. From (3.31) it is clear that $f f^{(k)} \neq 0$ and $g g^{(k)} \neq 0$. Then by Lemma 3.8 we have

\[(3.33)\]
\[f(z) = e^{az+b}, \quad g(z) = e^{cz+d},\]

where $a, c \in \mathbb{C} \setminus \{0\}$, $b, d \in \mathbb{C}$. But from (3.31) we see that $a + c = 0$.

**Subcase 2.2.** Let $k = 1$. Considering Subcase 1.2 one can easily get

\[(3.34)\]
\[f(z) = e^{az+b}, \quad g(z) = e^{cz+d},\]

where $a, c \in \mathbb{C} \setminus \{0\}$, $b, d \in \mathbb{C}$. Finally, we can take $f$ and $g$ as

\[f(z) = c_3 e^{dz}, \quad g(z) = c_4 e^{-dz},\]

where $c_3, c_4, d \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^{m_1} (c_3 c_4)^{n+s} d^{2m_1} = b^2$. This completes the proof. \(\square\)

**Lemma 3.17.** Let $f$ and $g$ be two transcendental meromorphic functions and let $F = f^{n}(f')^{n_1} \cdots (f^{(k)})^{n_k}/p$ and $G = g^{n}(g')^{n_1} \cdots (g^{(k)})^{n_k}/p$, where $p(z)$ is defined as in (2.1) and $n, n_k, k \in \mathbb{N}, n_i \in \mathbb{N} \cup \{0\}, i = 1, 2, \ldots, k-1$ are such that $n > s+t+m+2$. If $f, g$ share $(\infty, 0)$ and $H \equiv 0$ then either

\[f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} g^n(g')^{n_1} \cdots (g^{(k)})^{n_k} \equiv p^2(z),\]

where $f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} - p(z)$ and $g^n(g')^{n_1} \cdots (g^{(k)})^{n_k} - p(z)$ share 0 CM or

\[f^n(f')^{n_1} \cdots (f^{(k)})^{n_k} \equiv g^n(g')^{n_1} \cdots (g^{(k)})^{n_k}.\]

**Proof.** Since $H \equiv 0$, by integration we get

\[(3.35)\]
\[\frac{1}{F-1} = \frac{bG + a - b}{G - 1},\]

where $a, b \in \mathbb{C}$ and $a \in \mathbb{C} \setminus \{0\}$. From (3.35) it is clear that $F$ and $G$ share $(1, \infty)$. We now consider the following cases.

**Case 1.** Let $b \in \mathbb{C} \setminus \{0\}$ and $a \neq b$. If $b = -1$, then from (3.35) we have

\[F = \frac{-a}{G - a - 1},\]
Therefore $\overline{N}(r, a + 1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p)$. So in view of Lemma 3.12 and the second fundamental theorem we get

\[
(n - s)T(r, g) \leq T(r, G) - sN(r, \infty; g) - N(r, 0; (g')^{n_1} \ldots (g^{(k)})^{n_k}) + S(r, g)
\]
\[
\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, a + 1; G) - dN(r, \infty; g)
\]
\[
- N(r, 0; (g')^{n_1} \ldots (g^{(k)})^{n_k}) + S(r, g)
\]
\[
\leq \overline{N}(r, 0; g) + \overline{N}(r, 0; (g')^{n_1} \ldots (g^{(k)})^{n_k}) + \overline{N}(r, \infty; f)
\]
\[
- N(r, 0; (g')^{n_1} \ldots (g^{(k)})^{n_k}) + S(r, g)
\]
\[
\leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + S(r, g)
\]
\[
\leq 2T(r, g) + S(r, g),
\]

which is a contradiction since $n > s + 2$. If $b \neq -1$, from (3.35) we obtain that

\[
F - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2(G + (a - b)/b)}.
\]

So $\overline{N}(r, (b - a)/b; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p)$. Using Lemma 3.12 and the same argument as the one used in the case when $b = -1$ we get a contradiction.

**Case 2.** Let $b \in \mathbb{C} \setminus \{0\}$ and $a = b$. If $b = -1$, then from (3.35) we have $FG \equiv 1$, i.e., $f^n(f')^{n_1} \ldots (f^{(k)})^{n_k} g^n(g')^{n_1} \ldots (g^{(k)})^{n_k} \equiv p^2$, where $f^n(f')^{n_1} \ldots (f^{(k)})^{n_k} - p(z)$ and $g^n(g')^{n_1} \ldots (g^{(k)})^{n_k} - p(z)$ share 0 CM. If $b \neq -1$, from (3.35) we have

\[
\frac{1}{F} \equiv \frac{bG}{(1 + b)G - 1}.
\]

Therefore $\overline{N}(r, 1/(1 + b); G) = \overline{N}(r, 0; F)$. So in view of Lemmas 3.2, 3.12 and the second fundamental theorem we get

\[
(n - s)T(r, g) \leq T(r, G) - sN(r, \infty; g) - N(r, 0; (g')^{n_1} \ldots (g^{(k)})^{n_k}) + S(r, g)
\]
\[
\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1 + b}; G\right) - dN(r, \infty; g)
\]
\[
- N(r, 0; (g')^{n_1} \ldots (g^{(k)})^{n_k}) + S(r, g)
\]
\[
\leq \overline{N}(r, 0; g) + \overline{N}(r, 0; F) + S(r, g)
\]
\[
\leq \overline{N}(r, 0; g) + \overline{N}(r, 0; f) + \sum_{i=1}^{k} n_i \overline{N}(r, 0; f^{(i)} \ | \ f \neq 0) + S(r, g)
\]
\[
\leq T(r, g) + tT(r, f) + mT(r, f) + S(r, f) + S(r, g).
\]
Without loss of generality, we may suppose that there exists a set $I$ with infinite measure such that $T(r, f) \leq T(r, g)$ for $r \in I$. So for $r \in I$ we have $(n - s)T(r, g) \leq (t + m + 2)T(r, g) + S(r, g)$, which is a contradiction since $n > s + t + m + 2$.

Case 3. Let $b = 0$. From (3.35) we obtain

$$(3.36) \quad F \equiv \frac{G + a - 1}{a}.$$ 

If $a \neq 1$ then from (3.36) we obtain $\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F)$. We can deduce a contradiction similarly to Case 2. Therefore $a = 1$ and from (3.36) we obtain $F \equiv G$, i.e., $f^n(f')^{n_1} \ldots (f^{(k)})^{n_k} \equiv g^n(g')^{n_1} \ldots (g^{(k)})^{n_k}$. This completes the proof. □

Lemma 3.18. Let $f$ and $g$ be non-constant meromorphic functions sharing $(1, k_1)$, where $k_1 \in \mathbb{N} \cup \{\infty\} \setminus \{1\}$. Then

$$N(r, 1; g) - \overline{N}(r, 1; g) \geq \overline{N}(r, 1; f \mid = 2) + 2\overline{N}(r, 1; f \mid = 3) + \ldots + (k_1 - 1)\overline{N}(r, 1; f \mid = k_1) + k_1\overline{N}_L(r, 1; f) + (k_1 + 1)\overline{N}_L(r, 1; g) + k_1\overline{N}_E^{k_1+1}(r, 1; g).$$

4. Proofs of the theorems

Proof of Theorem 2.1. Let

$$F = \frac{f^n(f')^{n_1} \ldots (f^{(k)})^{n_k}}{p} \quad \text{and} \quad G = \frac{g^n(g')^{n_1} \ldots (g^{(k)})^{n_k}}{p}.$$

Note that $f$ and $g$ are transcendental meromorphic functions, so $p(z)$ is a small function with respect to both $f^n(f')^{n_1} \ldots (f^{(k)})^{n_k}$ and $g^n(g')^{n_1} \ldots (g^{(k)})^{n_k}$. Also $F, G$ share $(1, k_1)$ and $f, g$ share $(\infty, 0)$.

Case 1. Let $H \not\equiv 0$. From (3.1) it can be easily calculated that the possible poles of $H$ occur at

(i) multiple zeros of $F$ and $G$,
(ii) those 1 points of $F$ and $G$ whose multiplicities are different,
(iii) those poles of $F$ and $G$ whose multiplicities are different,
(iv) the zeros of $F'(G')$ which are not zeros of $F(F - 1)(G(G - 1))$.

Since $H$ has only simple poles we get

$$(4.1) \quad N(r, \infty; H) \leq \overline{N}_*(r, \infty; f, g) + \overline{N}_*(r, 1; F, G) + \overline{N}(r, 0; F \mid \geq 2) + \overline{N}(r, 0; G \mid \geq 2) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'),$$

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where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of $F'$ which are not zeros of $F(F - 1)$, and $\overline{N}_0(r, 0; G')$ is similarly defined.

Let $z_0$ be a simple zero of $F - 1$ but $p(z_0) \neq 0$. Then $z_0$ is a simple zero of $G - 1$ and a zero of $H$. So

\begin{equation}
(4.2) \quad N(r, 1; F | = 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, f) + S(r, g).
\end{equation}

Using (4.1) and (4.2) we get

\begin{equation}
(4.3) \quad \overline{N}(r, 1; F) \leq N(r, 1; F | = 1) + \overline{N}(r, 1; F | \geq 2)
\leq \overline{N}_*(r, \infty; f, g) + \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2)
+ \overline{N}_*(r, 0, 1; F, G) + \overline{N}(r, 1; F | \geq 2) + \overline{N}_0(r, 0; F')
+ \overline{N}_0(r, 0; G') + S(r, f) + S(r, g)
\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2)
+ \overline{N}_*(r, 1; F, G) + \overline{N}(r, 1; F | \geq 2) + \overline{N}_0(r, 0; F')
+ \overline{N}_0(r, 0; G') + S(r, f) + S(r, g).
\end{equation}

Now in view of Lemmas 3.2 and 3.18 we get

\begin{equation}
(4.4) \quad \overline{N}_0(r, 0; G') + \overline{N}(r, 1; F | \geq 2) + \overline{N}_*(r, 1; F, G)
\leq \overline{N}_0(r, 0; G') + \overline{N}(r, 1; F | = 2)
+ \overline{N}(r, 1; F | = 3) + \ldots + \overline{N}(r, 1; F | = k_1) + \overline{N}_{k_1+1}^{(k_1+1)}(r, 1; F)
+ \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_*(r, 1; F, G)
\leq \overline{N}_0(r, 0; G') - \overline{N}(r, 1; F | = 3) - \ldots - (k_1 - 2)\overline{N}(r, 1; F | = k_1)
- (k_1 - 1)\overline{N}_L(r, 1; F) - k_1\overline{N}_L(r, 1; G) - (k_1 - 1)\overline{N}_{k_1+1}^{(k_1+1)}(r, 1; F)
+ N(r, 1; G) - \overline{N}(r, 1; G) + \overline{N}_*(r, 1; F, G)
\leq \overline{N}_0(r, 0; G') + N(r, 1; G) - \overline{N}(r, 1; G)
- (k_1 - 2)\overline{N}_L(r, 1; F) - (k_1 - 1)\overline{N}_L(r, 1; G)
\leq N(r, 0; G' | G \neq 0) - (k_1 - 2)\overline{N}_L(r, 1; F) - (k_1 - 1)\overline{N}_L(r, 1; G)
\leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; g) - (k_1 - 2)\overline{N}_L(r, 1; F) - (k_1 - 1)\overline{N}_L(r, 1; G)
= \overline{N}(r, 0; G) + \overline{N}(r, \infty; g) - (k_1 - 2)\overline{N}_*(r, 1; F, G) - \overline{N}_L(r, 1; G),
\end{equation}

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Hence using (4.3), (4.4) and Lemma 3.1 we get from second fundamental theorem that

\[(4.5) \quad T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1; F) - N_0(r, 0; F')\]
\[\leq 2\overline{N}(r, \infty; f) + N_2(r, 0; F) + \overline{N}(r, 0; G | \geq 2) + \overline{N}(r, 1; F | \geq 2)\]
\[+ \overline{N}_*(r, 1; F, G) + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g)\]
\[\leq 3\overline{N}(r, \infty; f) + N_2(r, 0; F) + N_2(r, 0; G)\]
\[- (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g)\]
\[\leq 3\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + N(r, 0; (f')^{n_1} \cdots (f^{(k)})^{n_k})\]
\[+ 2\overline{N}(r, 0; g) + \sum_{i=1}^{k} n_{i+2}^* N_{i+2}(r, 0; g) + \sum_{i=1}^{k} p_{i+2} N(r, \infty; g)\]
\[- (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g)\]
\[\leq (3 + m_1)\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g)\]
\[+ sN(r, 0; g) + N(r, 0; (f')^{n_1} \cdots (f^{(k)})^{n_k})\]
\[- (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).\]

Now using Lemmas 3.11 and 3.12 we get from (4.5)

\[(4.6) \quad (n - s)T(r, f) \leq T(r, F) - sN(r, \infty; f) - N(r, 0; (f')^{n_1} \cdots (f^{(k)})^{n_k}) + S(r, f)\]
\[\leq (3 + m_1 - s)\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g)\]
\[+ sN(r, 0; g) - (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g)\]
\[\leq \frac{2(t + 1)(3 + m_1 - s)}{n + s + m_1 - 2m - 1} T(r) + (4 + s)T(r) + S(r)\]
\[\leq \left( \frac{4n + (6 + 2t)m_1 - 8m + 8}{n + s + m_1 - 2m - 1} + s \right) T(r) + S(r).\]

In a similar way we can obtain

\[(4.7) \quad (n - s)T(r, g) \leq \left( \frac{4n + (6 + 2t)m_1 - 8m + 8}{n + s + m_1 - 2m - 1} + s \right) T(r) + S(r).\]

Combining (4.6) and (4.7) we see that

\[(n - s)T(r) \leq \left( \frac{(s + 4)n + (6 + 2t)m_1 - 8m + sm_1 - 2sm + s^2 - s + 8}{n + s + m_1 - 2m - 1} \right) T(r) + S(r),\]

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i.e.,

\[(4.8) \quad ((n - K_1)(n - K_2))T(r) \leq S(r),\]

where

\[K_1 = \frac{2m + s + 5 - m_1 + \sqrt{L}}{2} \quad \text{and} \quad K_2 = \frac{2m + s + 5 - m_1 - \sqrt{L}}{2},\]

where

\[L = (2m + s + 5 - m_1)^2 + 8s^2 - 8s + 4(6 + 2t)m_1 + 8sm_1 - 16sm - 32m + 32.\]

Note that

\[L = (m_1 + 3s)^2 + 4(6 + 2t)m_1 + 2s - 12sm - 4mm_1 - 10m_1 + 4m^2 - 12m + 57\]
\[\leq (m_1 + 3s)^2 + 2m_1 + 8tm_1 + 2s - 4m(m_1 - m) - 12(sm - m_1) - 12m + 57\]
\[< (m_1 + 3s)^2 + 2(m_1 + 3s)(1 + 4t) + (1 + 4t)^2 = (m_1 + 3s + 4t + 1)^2.\]

Therefore

\[K_1 = \frac{2m + s + 5 - m_1 + \sqrt{L}}{2} < \frac{2m + s + 5 - m_1 + m_1 + 3s + 4t + 1}{2} = 2s + m + 2t + 3.\]

Since \(n > 2s + m + 2t + 2\), (4.8) leads to a contradiction.

\textit{Case 2.} Let \(H \equiv 0\). Then the theorem follows from Lemmas 3.17, 3.15 and 3.16. \(\square\)

\textbf{Acknowledgement.} The author wishes to thank the referee for his/her valuable comments and suggestions towards the improvement of the paper.

\textit{References}


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