A NOTE ON STAR LINDELÖF, FIRST COUNTABLE AND NORMAL SPACES

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Abstract. A topological space X is said to be star Lindelöf if for any open cover \mathcal{U} of X there is a Lindelöf subspace $A \subset X$ such that $\operatorname{St}(A,\mathcal{U}) = X$. The "extent" e(X) of X is the supremum of the cardinalities of closed discrete subsets of X. We prove that under V = L every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under MA + \neg CH, which shows that a star Lindelöf, first countable and normal space may not have countable extent.

Keywords: star Lindelöf space; first countable space; normal space; countable extent

MSC 2010: 54D20, 54E35

1. INTRODUCTION

Recently, the authors in [9], Theorem 2.9 proved that if X is a first countable star Lindelöf normal space and has a G_{δ} -diagonal, then the cardinality of X does not exceed \mathfrak{c} . This result suggests the following question.

Question 1.1 (see Question 2.10 of [9]). Let X be a first countable star Lindelöf normal space. Does X have to have countable extent?

It is well known that the cardinality of a space which has countable extent and a G_{δ} -diagonal is at most \mathfrak{c} (see [4]). Therefore, a positive answer to Question 1.1 would imply a trivial proof of the above result.

In this paper, we prove that under V = L every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under

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 $MA + \neg CH$, which shows that a star Lindelöf, first countable and normal space may not have countable extent. This gives a complete answer to Question 1.1.

2. NOTATION AND TERMINOLOGY

All the spaces are assumed to be Hausdorff if not stated otherwise. We write ω for the first infinite cardinal and \mathfrak{c} for the cardinality of the continuum.

If A is a subset of X and \mathcal{U} is a family of subsets of X, then $\operatorname{St}(A, \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : U \cap A \neq \emptyset \}$. If $A = \{x\}$ for some $x \in X$, then we write, for simplicity, $\operatorname{St}(x, \mathcal{U})$ instead of $\operatorname{St}(\{x\}, \mathcal{U})$.

Definition 2.1 ([8]). Let \mathcal{P} be a topological property. A topological space X is said to be star \mathcal{P} , if for any open cover \mathcal{U} of X there is a subset $A \subset X$ with property \mathcal{P} such that $St(A, \mathcal{U}) = X$. The set A will be called a star kernel of the cover \mathcal{U} .

Therefore, a topological space X is said to be star Lindelöf if for any open cover \mathcal{U} of X there is a Lindelöf subspace $A \subset X$ such that $\operatorname{St}(A, \mathcal{U}) = X$.

Definition 2.2 ([5]). The extent e(X) of X is the supremum of the cardinalities of closed discrete subsets of X.

Definition 2.3 ([5]). The character of X is defined as:

$$\chi(X) = \sup\{\chi(p, X) \colon p \in X\} + \omega,\$$

where $\chi(p, X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a local base for } p\}.$

Definition 2.4 ([6]). An uncountable subset X of real line \mathbb{R} is called a Q-set if every subset of X is a G_{δ} -set in X.

It should be pointed out that Q-set exists under certain set-theoretic assumption such as Martin Axiom and the negation of the Continuum Hypothesis (see [6], Theorem 4.2).

Definition 2.5 ([1]). A topological space X is collectionwise Hausdorff if any closed discrete set $S \subset X$ has a disjoint open expansion.

All notation and terminology not explained here is given in [2].

3. Results

We begin with an easy lemma, which will be useful later.

Lemma 3.1. If S is a closed discrete set in a normal space X and $\mathcal{U} = \{U(x): x \in S\}$ is a disjoint open expansion of S, then there is a discrete open expansion $\mathcal{V} = \{V(x): x \in S\}$ of S with $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.

Proof. By normality there exists an open set W in X such that $S \subset W \subset \overline{W} \subset \bigcup \mathcal{U}$. $\bigcup \mathcal{U}$. For all $x \in S$ let $V(x) = U(x) \cap W$. It is easily verified that $\mathcal{V} = \{V(x): x \in S\}$ is a discrete open collection of cardinality |S| and satisfies $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.

Theorem 3.2. Assuming V = L, if X is a star Lindelöf and normal space with $\chi(X) \leq \mathfrak{c}$, then X has countable extent.

Proof. Assume the contrary. It follows that there exists an uncountable closed and discrete subset S of X. Fleissner in [3] proved that under V = L, all normal spaces with character $\leq \mathfrak{c}$ are collectionwise Hausdorff, so S has a disjoint open expansion $\mathcal{U} = \{U(x): x \in S\}$. We apply Lemma 3.1 to conclude that there is a discrete open expansion $\mathcal{V} = \{V(x): x \in S\}$ of S satisfying $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.

Let $\mathcal{W} = \mathcal{V} \cup \{X \setminus S\}$. Obviously, \mathcal{W} is an open cover of X. Since X is star Lindelöf, it follows that there exists a Lindelöf subset Y of X such that $\operatorname{St}(Y, \mathcal{W}) = X$. For each $x \in S$, clearly $Y \cap V(x) \neq \emptyset$; pick $\xi(x) \in Y \cap V(x)$ and let $A = \{\xi(x) \colon x \in S\}$. Since Y is Lindelöf, there exists a limit point ξ for A. Therefore

$$\xi \in \overline{A} \subset \overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}.$$

It follows that there is U(x) for some $x \in S$ which contains ξ . This implies that infinitely many points of A are in U(x), which is a contradiction. This completes the proof.

Clearly, every first countable space X satisfies $\chi(X) = \omega < \mathfrak{c}$. So the following result is an immediate consequence of Theorem 3.2.

Corollary 3.3. Assuming V = L, if X is a star Lindelöf, first countable and normal space, then X has countable extent.

We present an example below, in which the Q-set is used.

E x a m p l e 3.4 ([7], Example F). Assume MA + \neg CH. There exists a star Lindelöf, first countable and normal space, which has an uncountable closed and discrete subset.

Proof. Take an uncountable Q-set A in \mathbb{R} . Let L be the closed upper halfplane, $L_1 = \{(x,0): x \in R\}$. Let $X = \{(x,0): x \in A\} \cup (L \setminus L_1)$. Define a basis for a topology on X as follows. For every $p \in X$ and $\varepsilon > 0$, let $B(p,\varepsilon)$ be the set of all points of X inside the circle of radius ε and the center at p, and define

$$U(p,\varepsilon) = B((x,\varepsilon),\varepsilon) \cup \{p\} \text{ for } p \in A$$

and

$$U(p,\varepsilon) = B(p,\varepsilon) \text{ for } p \in X \setminus A.$$

By a quick observation, we conclude that X is first countable. Moreover, $\{(x, y): x, y \in \mathbb{Q}^+\} \subset X$ (where \mathbb{Q} is the set of all rational numbers) witnesses that X is separable, and it follows that X is star countable, and hence star Lindelöf. It has been proved in [7], Example F, that X is normal. Finally, it is not difficult to see that $\{\{(x, 0)\}: x \in A\}$ is an uncountable closed and discrete subset of X. This completes the proof.

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