A NOTE ON STAR LINDELÖF, FIRST COUNTABLE AND NORMAL SPACES

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Abstract. A topological space $X$ is said to be star Lindelöf if for any open cover $\mathcal{U}$ of $X$ there is a Lindelöf subspace $A \subseteq X$ such that $\text{St}(A, \mathcal{U}) = X$. The “extent” $e(X)$ of $X$ is the supremum of the cardinalities of closed discrete subsets of $X$. We prove that under $V = L$ every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under $\text{MA} + \neg \text{CH}$, which shows that a star Lindelöf, first countable and normal space may not have countable extent.

Keywords: star Lindelöf space; first countable space; normal space; countable extent

MSC 2010: 54D20, 54E35

1. Introduction

Recently, the authors in [9], Theorem 2.9 proved that if $X$ is a first countable star Lindelöf normal space and has a $G_\delta$-diagonal, then the cardinality of $X$ does not exceed $\mathfrak{c}$. This result suggests the following question.

Question 1.1 (see Question 2.10 of [9]). Let $X$ be a first countable star Lindelöf normal space. Does $X$ have to have countable extent?

It is well known that the cardinality of a space which has countable extent and a $G_\delta$-diagonal is at most $\mathfrak{c}$ (see [4]). Therefore, a positive answer to Question 1.1 would imply a trivial proof of the above result.

In this paper, we prove that under $V = L$ every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under

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MA + \neg\text{CH}, which shows that a star Lindelöf, first countable and normal space may not have countable extent. This gives a complete answer to Question 1.1.

2. Notation and terminology

All the spaces are assumed to be Hausdorff if not stated otherwise. We write \(\omega\) for the first infinite cardinal and \(c\) for the cardinality of the continuum.

If \(A\) is a subset of \(X\) and \(U\) is a family of subsets of \(X\), then \(\text{St}(A, U) = \bigcup\{U \in U : U \cap A \neq \emptyset\}\). If \(A = \{x\}\) for some \(x \in X\), then we write, for simplicity, \(\text{St}(x, U)\) instead of \(\text{St}(\{x\}, U)\).

**Definition 2.1 ([8]).** Let \(P\) be a topological property. A topological space \(X\) is said to be star \(P\), if for any open cover \(U\) of \(X\) there is a subset \(A \subset X\) with property \(P\) such that \(\text{St}(A, U) = X\). The set \(A\) will be called a star kernel of the cover \(U\).

Therefore, a topological space \(X\) is said to be star Lindelöf if for any open cover \(U\) of \(X\) there is a Lindelöf subspace \(A \subset X\) such that \(\text{St}(A, U) = X\).

**Definition 2.2 ([5]).** The extent \(e(X)\) of \(X\) is the supremum of the cardinalities of closed discrete subsets of \(X\).

**Definition 2.3 ([5]).** The character of \(X\) is defined as:

\[
\chi(X) = \sup\{\chi(p, X) : p \in X\} + \omega,
\]

where \(\chi(p, X) = \min\{|B| : B\text{ is a local base for } p\}\).

**Definition 2.4 ([6]).** An uncountable subset \(X\) of real line \(\mathbb{R}\) is called a \(Q\)-set if every subset of \(X\) is a \(G_\delta\)-set in \(X\).

It should be pointed out that \(Q\)-set exists under certain set-theoretic assumption such as Martin Axiom and the negation of the Continuum Hypothesis (see [6], Theorem 4.2).

**Definition 2.5 ([1]).** A topological space \(X\) is collectionwise Hausdorff if any closed discrete set \(S \subset X\) has a disjoint open expansion.

All notation and terminology not explained here is given in [2].
3. Results

We begin with an easy lemma, which will be useful later.

**Lemma 3.1.** If $S$ is a closed discrete set in a normal space $X$ and $\mathcal{U} = \{U(x) : x \in S\}$ is a disjoint open expansion of $S$, then there is a discrete open expansion $\mathcal{V} = \{V(x) : x \in S\}$ of $S$ with $\bigcup \mathcal{V} \subset \bigcup \mathcal{U}$.

**Proof.** By normality there exists an open set $W$ in $X$ such that $S \subset W \subset W \subset \bigcup \mathcal{U}$. For all $x \in S$ let $V(x) = U(x) \cap W$. It is easily verified that $\mathcal{V} = \{V(x) : x \in S\}$ is a discrete open collection of cardinality $|S|$ and satisfies $\bigcup \mathcal{V} \subset \bigcup \mathcal{U}$. □

**Theorem 3.2.** Assuming $V = L$, if $X$ is a star Lindelöf and normal space with $\chi(X) \leq \omega$, then $X$ has countable extent.

**Proof.** Assume the contrary. It follows that there exists an uncountable closed and discrete subset $S$ of $X$. Fleissner in [3] proved that under $V = L$, all normal spaces with character $\leq \omega$ are collectionwise Hausdorff, so $S$ has a disjoint open expansion $\mathcal{U} = \{U(x) : x \in S\}$. We apply Lemma 3.1 to conclude that there is a discrete open expansion $\mathcal{V} = \{V(x) : x \in S\}$ of $S$ satisfying $\bigcup \mathcal{V} \subset \bigcup \mathcal{U}$.

Let $\mathcal{W} = \mathcal{V} \cup \{X \setminus S\}$. Obviously, $\mathcal{W}$ is an open cover of $X$. Since $X$ is star Lindelöf, it follows that there exists a Lindelöf subset $Y$ of $X$ such that $\text{St}(Y, \mathcal{W}) = X$. For each $x \in S$, clearly $Y \cap V(x) \neq \emptyset$; pick $\xi(x) \in Y \cap V(x)$ and let $\mathcal{A} = \{\xi(x) : x \in S\}$. Since $Y$ is Lindelöf, there exists a limit point $\xi$ for $\mathcal{A}$. Therefore

$$\xi \in \overline{\mathcal{A}} \subset \overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}.$$ 

It follows that there is $U(x)$ for some $x \in S$ which contains $\xi$. This implies that infinitely many points of $\mathcal{A}$ are in $U(x)$, which is a contradiction. This completes the proof. □

Clearly, every first countable space $X$ satisfies $\chi(X) = \omega < \omega$. So the following result is an immediate consequence of Theorem 3.2.

**Corollary 3.3.** Assuming $V = L$, if $X$ is a star Lindelöf, first countable and normal space, then $X$ has countable extent.

We present an example below, in which the $Q$-set is used.

**Example 3.4 ([7], Example F).** Assume $\text{MA} + \neg \text{CH}$. There exists a star Lindelöf, first countable and normal space, which has an uncountable closed and discrete subset.
Proof. Take an uncountable $Q$-set $A$ in $\mathbb{R}$. Let $L$ be the closed upper half-plane, $L_1 = \{(x, 0): x \in R\}$. Let $X = \{(x, 0): x \in A\} \cup (L \setminus L_1)$. Define a basis for a topology on $X$ as follows. For every $p \in X$ and $\varepsilon > 0$, let $B(p, \varepsilon)$ be the set of all points of $X$ inside the circle of radius $\varepsilon$ and the center at $p$, and define

$$U(p, \varepsilon) = B((x, \varepsilon), \varepsilon) \cup \{p\} \quad \text{for } p \in A$$

and

$$U(p, \varepsilon) = B(p, \varepsilon) \quad \text{for } p \in X \setminus A.$$  

By a quick observation, we conclude that $X$ is first countable. Moreover, $\{(x, y): x, y \in \mathbb{Q}^+\} \subset X$ (where $\mathbb{Q}$ is the set of all rational numbers) witnesses that $X$ is separable, and it follows that $X$ is star countable, and hence star Lindelöf. It has been proved in [7], Example F, that $X$ is normal. Finally, it is not difficult to see that $\{(x, 0): x \in A\}$ is an uncountable closed and discrete subset of $X$. This completes the proof. □

References


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