# A NOTE ON STAR LINDELÖF, FIRST COUNTABLE AND NORMAL SPACES 

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#### Abstract

A topological space $X$ is said to be star Lindelöf if for any open cover $\mathcal{U}$ of $X$ there is a Lindelöf subspace $A \subset X$ such that $\operatorname{St}(A, \mathcal{U})=X$. The "extent" $e(X)$ of $X$ is the supremum of the cardinalities of closed discrete subsets of $X$. We prove that under $V=L$ every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under MA $+\neg \mathrm{CH}$, which shows that a star Lindelöf, first countable and normal space may not have countable extent.


Keywords: star Lindelöf space; first countable space; normal space; countable extent
MSC 2010: 54D20, 54E35

## 1. Introduction

Recently, the authors in [9], Theorem 2.9 proved that if $X$ is a first countable star Lindelöf normal space and has a $G_{\delta}$-diagonal, then the cardinality of $X$ does not exceed $\boldsymbol{c}$. This result suggests the following question.

Question 1.1 (see Question 2.10 of [9]). Let $X$ be a first countable star Lindelöf normal space. Does $X$ have to have countable extent?

It is well known that the cardinality of a space which has countable extent and a $G_{\delta}$-diagonal is at most $\mathfrak{c}$ (see [4]). Therefore, a positive answer to Question 1.1 would imply a trivial proof of the above result.

In this paper, we prove that under $V=L$ every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under

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MA $+\neg \mathrm{CH}$, which shows that a star Lindelöf, first countable and normal space may not have countable extent. This gives a complete answer to Question 1.1.

## 2. Notation and terminology

All the spaces are assumed to be Hausdorff if not stated otherwise. We write $\omega$ for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

If $A$ is a subset of $X$ and $\mathcal{U}$ is a family of subsets of $X$, then $\operatorname{St}(A, \mathcal{U})=\bigcup\{U \in \mathcal{U}$ : $U \cap A \neq \emptyset\}$. If $A=\{x\}$ for some $x \in X$, then we write, for simplicity, $\operatorname{St}(x, \mathcal{U})$ instead of $\operatorname{St}(\{x\}, \mathcal{U})$.

Definition 2.1 ([8]). Let $\mathcal{P}$ be a topological property. A topological space $X$ is said to be star $\mathcal{P}$, if for any open cover $\mathcal{U}$ of $X$ there is a subset $A \subset X$ with property $\mathcal{P}$ such that $\operatorname{St}(A, \mathcal{U})=X$. The set $A$ will be called a star kernel of the cover $\mathcal{U}$.

Therefore, a topological space $X$ is said to be star Lindelöf if for any open cover $\mathcal{U}$ of $X$ there is a Lindelöf subspace $A \subset X$ such that $\operatorname{St}(A, \mathcal{U})=X$.

Definition 2.2 ([5]). The extent $e(X)$ of $X$ is the supremum of the cardinalities of closed discrete subsets of $X$.

Definition 2.3 ([5]). The character of $X$ is defined as:

$$
\chi(X)=\sup \{\chi(p, X): p \in X\}+\omega
$$

where $\chi(p, X)=\min \{|\mathcal{B}|: \mathcal{B}$ is a local base for $p\}$.
Definition 2.4 ([6]). An uncountable subset $X$ of real line $\mathbb{R}$ is called a $Q$-set if every subset of $X$ is a $G_{\delta}$-set in $X$.

It should be pointed out that $Q$-set exists under certain set-theoretic assumption such as Martin Axiom and the negation of the Continuum Hypothesis (see [6], Theorem 4.2).

Definition 2.5 ([1]). A topological space $X$ is collectionwise Hausdorff if any closed discrete set $S \subset X$ has a disjoint open expansion.

All notation and terminology not explained here is given in [2].

## 3. Results

We begin with an easy lemma, which will be useful later.
Lemma 3.1. If $S$ is a closed discrete set in a normal space $X$ and $\mathcal{U}=\{U(x)$ : $x \in S\}$ is a disjoint open expansion of $S$, then there is a discrete open expansion $\mathcal{V}=\{V(x): x \in S\}$ of $S$ with $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.

Proof. By normality there exists an open set $W$ in $X$ such that $S \subset W \subset \bar{W} \subset$ $\cup \mathcal{U}$. For all $x \in S$ let $V(x)=U(x) \cap W$. It is easily verified that $\mathcal{V}=\{V(x): x \in S\}$ is a discrete open collection of cardinality $|S|$ and satisfies $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.

Theorem 3.2. Assuming $V=L$, if $X$ is a star Lindelöf and normal space with $\chi(X) \leqslant \mathfrak{c}$, then $X$ has countable extent.

Proof. Assume the contrary. It follows that there exists an uncountable closed and discrete subset $S$ of $X$. Fleissner in [3] proved that under $V=L$, all normal spaces with character $\leqslant \mathfrak{c}$ are collectionwise Hausdorff, so $S$ has a disjoint open expansion $\mathcal{U}=\{U(x): x \in S\}$. We apply Lemma 3.1 to conclude that there is a discrete open expansion $\mathcal{V}=\{V(x): x \in S\}$ of $S$ satisfying $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.

Let $\mathcal{W}=\mathcal{V} \cup\{X \backslash S\}$. Obviously, $\mathcal{W}$ is an open cover of $X$. Since $X$ is star Lindelöf, it follows that there exists a Lindelöf subset $Y$ of $X$ such that $\operatorname{St}(Y, \mathcal{W})=X$. For each $x \in S$, clearly $Y \cap V(x) \neq \emptyset$; pick $\xi(x) \in Y \cap V(x)$ and let $A=\{\xi(x): x \in S\}$. Since $Y$ is Lindelöf, there exists a limit point $\xi$ for $A$. Therefore

$$
\xi \in \bar{A} \subset \overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}
$$

It follows that there is $U(x)$ for some $x \in S$ which contains $\xi$. This implies that infinitely many points of $A$ are in $U(x)$, which is a contradiction. This completes the proof.

Clearly, every first countable space $X$ satisfies $\chi(X)=\omega<\mathfrak{c}$. So the following result is an immediate consequence of Theorem 3.2.

Corollary 3.3. Assuming $V=L$, if $X$ is a star Lindelöf, first countable and normal space, then $X$ has countable extent.

We present an example below, in which the $Q$-set is used.
Example 3.4 ([7], Example F). Assume MA $+\neg \mathrm{CH}$. There exists a star Lindelöf, first countable and normal space, which has an uncountable closed and discrete subset.

Proof. Take an uncountable $Q$-set $A$ in $\mathbb{R}$. Let $L$ be the closed upper halfplane, $L_{1}=\{(x, 0): x \in R\}$. Let $X=\{(x, 0): x \in A\} \cup\left(L \backslash L_{1}\right)$. Define a basis for a topology on $X$ as follows. For every $p \in X$ and $\varepsilon>0$, let $B(p, \varepsilon)$ be the set of all points of $X$ inside the circle of radius $\varepsilon$ and the center at $p$, and define

$$
U(p, \varepsilon)=B((x, \varepsilon), \varepsilon) \cup\{p\} \quad \text { for } p \in A
$$

and

$$
U(p, \varepsilon)=B(p, \varepsilon) \quad \text { for } p \in X \backslash A
$$

By a quick observation, we conclude that $X$ is first countable. Moreover, $\{(x, y)$ : $\left.x, y \in \mathbb{Q}^{+}\right\} \subset X$ (where $\mathbb{Q}$ is the set of all rational numbers) witnesses that $X$ is separable, and it follows that $X$ is star countable, and hence star Lindelöf. It has been proved in [7], Example F, that $X$ is normal. Finally, it is not difficult to see that $\{\{(x, 0)\}: x \in A\}$ is an uncountable closed and discrete subset of $X$. This completes the proof.

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