POSITIVE PERIODIC SOLUTIONS OF A NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH MULTIPLE DELAYS

Yongxiang Li, Ailan Liu, Lanzhou

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Abstract. This paper deals with the existence of positive ω-periodic solutions for the neutral functional differential equation with multiple delays

$$(u(t) - cu(t-\delta))' + a(t)u(t) = f(t, u(t-\tau_1), \dots, u(t-\tau_n)).$$

The essential inequality conditions on the existence of positive periodic solutions are obtained. These inequality conditions concern with the relations of c and the coefficient function $a(t)$, and the nonlinearity $f(t, x_1, \dots, x_n)$. Our discussion is based on the perturbation method of positive operator and fixed point index theory in cones.

Keywords: neutral delay differential equation; positive periodic solution; cone; fixed point index

MSC 2010: 34K13, 34K40, 47H11

1. Introduction

In the paper, we discuss the existence of positive ω-periodic solutions of the neutral functional differential equation with multiple delays

$$(u(t) - cu(t-\delta))' + a(t)u(t) = f(t, u(t-\tau_1), \dots, u(t-\tau_n)), \tag{1.1}$$

where $\delta > 0, |c| < 1$ are constants, $a \in C(\mathbb{R}, (0,\infty))$ is a ω-periodic function, $f\colon \mathbb{R} \times [0,\infty)^n \to [0,\infty)$ is a continuous function which is ω-periodic in t, and $\tau_1, \tau_2, \dots, \tau_n$ are positive constants.

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The delay differential equation has been proposed in many fields such as biology, physicochemistry, mechanics and economics, see [4], [7]. The existence problems of periodic solutions have attracted many authors' attention, see [2], [5], [6], [8]–[12] and references therein. In some practice models, only positive periodic solutions are significant. In [11], [12], the authors obtained the existence of positive periodic solutions for some delay first-order differential equations in the form of

$$u'(t) + a(t)u(t) = f(t, u(t - \tau(t))) \tag{1.2}$$

by employing the fixed point theorem of cone mapping, and one well-known result is that if the nonlinearity $f(t, x)$ has superlinear or sublinear growth on x, the equation (1.2) has at least one positive ω-periodic solution.

Among the previous works, there are few ones concerned with neutral differential equations. In [10], by means of the continuation theorem of coincidence degree principle, Serra discussed the existence of periodic solutions for the neutral differential equation

$$(u(t) - cu(t - \delta))' = f(t, u(t)). \tag{1.3}$$

In [8], Luo, Wang and Shen employed the Krasnoselskii fixed point theorem on the sum of a compact operator and a contractive operator to obtain the existence of positive periodic solutions for the neutral functional differential equation with delay

$$(u(t) - cu(t - \tau(t)))' + a(t)u(t) = f(t, u(t - \tau(t))). \tag{1.4}$$

Motivated by the papers mentioned above, we study the existence of positive periodic solutions of the neutral functional differential equation (1.1) with multiple delays. We aim to obtain the essential conditions on the existence of positive periodic solutions of equation (1.1) via the theory of the fixed point index in cones. Specially, we hope the well-known existence result for equation (1.2) holds for equation (1.1). We will show that if c and $a(t)$ satisfy the following restriction condition (H), the result is true for equation (1.1).

For convenience, we introduce the notations

$$\underline{a} = \min_{0\leqslant t\leqslant\omega} a(t), \quad \overline{a} = \max_{0\leqslant t\leqslant\omega} a(t), \quad \sigma = \exp\Big(-2\int_0^\omega a(r)\,\mathrm{d}r\Big) \tag{1.5}$$

and make the following assumption:

$$|c| < \frac{\sigma}{\sigma + 1}. \tag{H}$$

Our main results are as follows:

Theorem 1.1. *Let $a \in C(\mathbb{R}, (0, \infty))$ be a ω-periodic function, c satisfy assumption* (H), *$f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$ and $f(t, x_1, \dots, x_n)$ be ω-periodic in t. If f satisfies the conditions*

(F1) *there exist positive constants $c_1, \dots, c_n$ satisfying $c_1 + \dots + c_n < \underline{a}$ and $\eta > 0$ such that*

$$f(t, x_1, \dots, x_n) \leqslant c_1 x_1 + \dots + c_n x_n$$

for $t \in \mathbb{R}$ and $x_1, \dots, x_n \in [0, \eta]$;

(F2) *there exist positive constants $d_1, \dots, d_n$ satisfying $d_1 + \dots + d_n > \overline{a}$ and $H > 0$ such that*

$$f(t, x_1, \dots, x_n) \geqslant d_1 x_1 + \dots + d_n x_n$$

for $t \in \mathbb{R}$ and $x_1, \dots, x_n \geqslant H$,

then equation (1.1) *has at least one positive ω-periodic solution.*

Theorem 1.2. *Let $a \in C(\mathbb{R}, (0, \infty))$ be a ω-periodic function, c satisfy assumption* (H), *$f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$ and $f(t, x_1, \dots, x_n)$ be ω-periodic in t. If f satisfies the conditions*

(F3) *there exist positive constants $d_1, \dots, d_n$ satisfying $d_1 + \dots + d_n > \overline{a}$ and $\eta > 0$ such that*

$$f(t, x_1, \dots, x_n) \geqslant d_1 x_1 + \dots + d_n x_n$$

for $t \in \mathbb{R}$ and $x_1, \dots, x_n \in [0, \eta]$;

(F4) *there exist positive constants $c_1, \dots, c_n$ satisfying $c_1 + \dots + c_n < \underline{a}$ and $H > 0$ such that*

$$f(t, x_1, \dots, x_n) \leqslant c_1 x_1 + \dots + c_n x_n$$

for $t \in \mathbb{R}$ and $x_1, \dots, x_n \geqslant H$,

then equation (1.1) *has at least one positive ω-periodic solution.*

In Theorem 1.1, conditions (F1) and (F2) allow $f(t, x_1, \dots, x_n)$ to have superlinear growth on $x_1, \dots, x_n$. For example,

$$f(t, x_1, \dots, x_n) = a_1(t) x_1^2 + \dots + a_n(t) x_n^2$$

satisfies (F1) and (F2), where $a_1(t), \dots, a_n(t)$ are positive and continuous ω-periodic functions.

In Theorem 1.2, conditions (F3) and (F4) allow $f(t, x_1, \ldots, x_n)$ to have sublinear growth on $x_1, \ldots, x_n$. For example,

$$f(t, x_1, \ldots, x_n) = b_1(t)\sqrt{|x_1|} + \ldots + b_n(t)\sqrt{|x_n|}$$

satisfies (F3) and (F4), where $b_1(t), \ldots, b_n(t)$ are positive and continuous ω-periodic functions.

Conditions (F1) and (F2) in Theorem 1.1 and conditions (F3) and (F4) in Theorem 1.2 are optimal for the existence of positive periodic solutions of equation (1.1). This fact can be shown from the neutral differential equation with linear delays

$$(u(t) - cu(t-\delta))' + a_0 u(t) = a_1 u(t-\tau_1) + \ldots + a_n u(t-\tau_n) + h(t), \tag{1.6}$$

where $a_0, a_1, \ldots, a_n$ are positive constants, $h \in C(\mathbb{R})$ is a positive ω-periodic function. If $a_1, \ldots, a_n$ satisfy

$$a_1 + a_2 + \ldots + a_n = a_0, \tag{1.7}$$

equation (1.6) has no positive ω-periodic solutions. In fact, if equation (1.6) has a positive ω-periodic solution, integrating the equation on $[0, \omega]$ and using the periodicity of $u(t)$, we can obtain that $\int_0^\omega h(t)\,\mathrm{d}t = 0$, which contradicts the positivity of $h(t)$. Hence, equation (1.6) has no positive ω-periodic solution. For $a(t) \equiv a_0$ and $f(t, x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n + h(t)$, if condition (1.7) holds, conditions (F1) and (F2) in Theorem 1.1 and conditions (F3) and (F4) in Theorem 1.2 are not satisfied. From this we see that the conditions in Theorems 1.1–1.2 are optimal.

The proofs of Theorems 1.1 and 1.2 are based on the fixed point index theory in cones, which will be given in Section 3. Some preliminaries to discuss equation (1.1) are presented in Section 2.

2. Preliminaries

Let $C_\omega(\mathbb{R})$ denote the Banach space of all continuous ω-periodic functions $u(t)$ with norm $\|u\|_C = \max\limits_{0 \leqslant t \leqslant \omega} |u(t)|$. Let $C^1_\omega(\mathbb{R})$ be the continuous differentiable ω-periodic function space and $C^+_\omega(\mathbb{R})$ be the cone of all nonnegative functions in $C_\omega(\mathbb{R})$.

In order to discuss the existence of positive ω-periodic solutions of equation (1.1), we need to build the existence and uniqueness results of ω-periodic solutions for the corresponding linear neutral differential equation

$$(u(t) - cu(t-\delta))' + a(t)u(t) = h(t), \quad t \in \mathbb{R}, \tag{2.1}$$

where $h \in C_\omega(\mathbb{R})$. For this we consider the linear differential equation

$$u'(t) + 2a(t)u(t) = h(t), \quad t \in \mathbb{R}. \tag{2.2}$$

By a direct calculation, we easily prove that for every $h \in C_\omega(\mathbb{R})$ equation (2.2) has a unique ω-periodic solution given by

$$u(t) = \int_t^{t+\omega} G(t,s)h(s)\,\mathrm{d}s := Th(t), \tag{2.3}$$

where

$$G(t,s) = \frac{\exp\bigl(2\int_t^s a(r)\,\mathrm{d}r\bigr)}{\exp\bigl(2\int_0^\omega a(r)\,\mathrm{d}r\bigr) - 1}. \tag{2.4}$$

Clearly, the operator $T\colon C(\mathbb{R}) \to C(\mathbb{R})$ defined by (2.3) is a completely continuous linear operator.

Define a subcone K_0 of $C_\omega^+(\mathbb{R})$ in $C_\omega(\mathbb{R})$ by

$$K_0 = \{u \in C_\omega(\mathbb{R})\colon\ u(t) \geqslant \sigma\|u\|_C,\ t \in \mathbb{R}\}. \tag{2.5}$$

Lemma 2.1. *For every $h \in C_\omega^+(\mathbb{R})$, the ω-periodic solution of equation* (2.2), *$u = Th \in K_0$. Namely, $T(C_\omega^+(\mathbb{R})) \subset K_0$.*

Proof. By the expression (2.4) of the Green function $G(t,s)$,

$$\begin{aligned} \overline{G} &:= \max\{G(t,s)\colon\ t \in \mathbb{R},\ t \leqslant s \leqslant t+\omega\} = \frac{\exp\bigl(2\int_0^\omega a(r)\,\mathrm{d}r\bigr)}{\exp\bigl(2\int_0^\omega a(r)\,\mathrm{d}r\bigr) - 1}, \\ \underline{G} &:= \min\{G(t,s)\colon\ t \in \mathbb{R},\ t \leqslant s \leqslant t+\omega\} = \frac{1}{\exp\bigl(2\int_0^\omega a(r)\,\mathrm{d}r\bigr) - 1}. \end{aligned} \tag{2.6}$$

Let $h \in C_\omega^+(\mathbb{R})$ and $u = Th$. For every $t \in \mathbb{R}$, from (2.3) it follows that

$$u(t) = \int_t^{t+\omega} G(t,s)h(s)\,\mathrm{d}s \leqslant \overline{G}\int_t^{t+\omega} h(s)\,\mathrm{d}s = \overline{G}\int_0^\omega h(s)\,\mathrm{d}s,$$

and therefore,

$$\|u\|_C \leqslant \overline{G}\int_0^\omega h(s)\,\mathrm{d}s.$$

Noting that $\underline{G}/\overline{G} = \sigma$, by (2.3) we obtain that

$$u(t) = \int_t^{t+\omega} G(t,s)h(s)\,\mathrm{d}s \geqslant \underline{G}\int_t^{t+\omega} h(s)\,\mathrm{d}s = \underline{G}\int_0^\omega h(s)\,\mathrm{d}s \geqslant \sigma\|u\|_C.$$

Hence, $Th = u \in K_0$. $\square$

Let $A\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ be the linear bounded operator defined by

$$Au(t) = u(t) - cu(t-\delta), \quad t \in \mathbb{R},\ u \in C_\omega(\mathbb{R}). \tag{2.7}$$

We easily verify the following lemma:

Lemma 2.2. *If $|c| < 1$, then A has a bounded inverse operator $A^{-1}\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$, which is given by*

$$A^{-1}v(t) = \sum_{j=0}^{\infty} c^j v(t - j\delta), \quad v \in C_\omega(\mathbb{R}), \tag{2.8}$$

and its norm satisfies $\|A^{-1}\| \leqslant 1/(1-|c|)$.

Let $v = Au$. Then equation (2.1) becomes

$$v'(t) + a(t)A^{-1}v(t) = h(t), \quad t \in \mathbb{R}. \tag{2.9}$$

From (2.8) we easily verify that if $v \in C^1_\omega(\mathbb{R})$, then $A^{-1}v \in C^1_\omega(\mathbb{R})$ and $(A^{-1}v)' = A^{-1}v'$. Hence, $u \in C^1_\omega(\mathbb{R})$ is a ω-periodic solution of equation (2.1) if and only if $v = Au \in C^1_\omega(\mathbb{R})$ is a ω-periodic solution of equation (2.9).

Lemma 2.3. *If $|c| < \frac{1}{2}$, then for every $h \in C_\omega(\mathbb{R})$, equation (2.9) has a unique ω-periodic solution $v \in C^1_\omega(\mathbb{R})$. Furthermore, for $h \in C^+_\omega(\mathbb{R})$, the solution $v \in K_0$ when $|c| < \sigma/(\sigma+1)$.*

Proof. Rewrite equation (2.9) to the form of

$$v'(t) + 2a(t)v(t) = Bv(t) + h(t), \quad t \in \mathbb{R}, \tag{2.10}$$

where $B\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a linear bounded operator defined by

$$\begin{aligned} Bv(t) &= 2a(t)v(t) - a(t)A^{-1}v(t) \\ &= a(t)v(t) + a(t)(I - A^{-1})v(t) \\ &= a(t)v(t) - ca(t)A^{-1}v(t-\delta), \quad t \in \mathbb{R}. \end{aligned} \tag{2.11}$$

From (2.10) it is easy to see that the ω-periodic solution problem of equation (2.9) is equivalent to the operator equation in Banach space $C_\omega(\mathbb{R})$

$$(I - TB)v = Th, \tag{2.12}$$

where $T\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is the ω-periodic solution operator of equation (2.2) given by (2.3), I is the identity operator in $C_\omega(\mathbb{R})$. We prove that the norm of TB in $\mathcal{L}(C_\omega(\mathbb{R}), C_\omega(\mathbb{R}))$ satisfies $\|TB\| < 1$.

For every $v \in C_\omega(\mathbb{R})$, by the definition of B we have

$$\begin{aligned}|Bv(t)| &= |a(t)v(t) - ca(t)A^{-1}v(t-\delta)| \\ &\leqslant a(t)\|v\|_C + |c|a(t)\|A^{-1}v\|_C \leqslant \Big(1 + \frac{|c|}{1-|c|}\Big)\|v\|_C a(t).\end{aligned}$$

By this and the definition (2.3) of T and the positivity of $G(t,s)$, we have

$$\begin{aligned}|TBv(t)| &\leqslant \int_t^{t+\omega} G(t,s)|Bv(s)|\,\mathrm{d}s \\ &\leqslant \Big(1 + \frac{|c|}{1-|c|}\Big)\|v\|_C \int_{t-\omega}^{t} G(t,s)a(s)\,\mathrm{d}s = \frac{1}{2}\Big(1 + \frac{|c|}{1-|c|}\Big)\|v\|_C,\end{aligned}$$

from which it follows that $\|TBv\|_C \leqslant \frac{1}{2}(1 + |c|/(1-|c|))\|v\|_C$. Therefore,

$$\|TB\| \leqslant \frac{1}{2}\Big(1 + \frac{|c|}{1-|c|}\Big). \tag{2.13}$$

Since $|c| < \frac{1}{2}$, it follows that

$$\frac{1}{2}\Big(1 + \frac{|c|}{1-|c|}\Big) < 1.$$

By this inequality and (2.13), we obtain that $\|TB\| < 1$.

Thus, $I - TB$ has a bounded inverse operator given by the series

$$(I - TB)^{-1} = \sum_{n=0}^{\infty} (TB)^n.$$

Consequently, equation (2.12), equivalently equation (2.9), has a unique ω-periodic solution

$$v = (I - TB)^{-1}Th = \sum_{n=0}^{\infty} (TB)^n Th. \tag{2.14}$$

For $h \in C_\omega^+(\mathbb{R})$, let $w = Th$. By Lemma 2.1, $w \in K_0$. Hence we have

$$\begin{aligned}Bw(t) &= a(t)w(t) - ca(t)A^{-1}w(t-\delta) \\ &\geqslant a(t)w(t) - |c|a(t)\|A^{-1}w\|_C \\ &\geqslant a(t)\sigma\|w\|_C - a(t)|c|\|A^{-1}w\|_C \\ &\geqslant a(t)\Big(\sigma - \frac{|c|}{1-|c|}\Big)\|w\|_C.\end{aligned}$$

Hence, when $|c| < \sigma/(\sigma+1)$, $Bw(t) \geqslant 0$ for every $t \in \mathbb{R}$. Namely, $Bw \in C_\omega^+(\mathbb{R})$. By Lemma 2.1, $(TB)w = T(Bw) \in K_0$. Using the inductive method we easily prove that $(TB)^n w \in K_0$ for every $n \in \mathbb{N}$. Thus according to (2.14), the unique ω-periodic solution of equation (2.9) is

$$v = \sum_{n=0}^{\infty} (TB)^n Th = \sum_{n=0}^{\infty} (TB)^n w \in K_0.$$

This completes the proof of Lemma 2.3. $\square$

Let c satisfy assumption (H). Choose a positive constant α as

$$\alpha = \begin{cases} \sigma(1-|c|), & \text{if } c \geqslant 0, \\ \dfrac{\sigma - |c|}{1+|c|}, & \text{if } c < 0, \end{cases} \tag{2.15}$$

and define another cone K in $C_\omega(\mathbb{R})$ by

$$K = \{u \in C_\omega(\mathbb{R})\colon\ u(t) \geqslant \alpha \|u\|_C,\ t \in \mathbb{R}\}. \tag{2.16}$$

Lemma 2.4. *If c satisfies assumption* (H), *then for every $h \in C_\omega(\mathbb{R})$, equation* (2.1) *has a unique ω-periodic solution $u := Sh \in C_\omega^1(\mathbb{R})$. Moreover, S: $C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a completely continuous linear operator, and $Sh \in K$ when $h \in C_\omega^+(\mathbb{R})$.*

P r o o f. Let $h \in C_\omega(\mathbb{R})$. By assumption (H) and Lemma 2.3, equation (2.9) has a unique ω-periodic solution $v \in C_\omega^1(\mathbb{R})$ given by (2.14). Hence

$$u = A^{-1}v = A^{-1}(I - TB)^{-1}Th := Sh \tag{2.17}$$

is a unique ω-periodic solution of equation (2.1), where

$$S = A^{-1}(I - TB)^{-1}T \tag{2.18}$$

is the corresponding ω-periodic solution operator. In (2.18), since

$$(I - TB)^{-1}T = \sum_{n=0}^{\infty} (TB)^n T = T\Bigl(I + B\Bigl(\sum_{n=0}^{\infty} (TB)^n T\Bigr)\Bigr),$$

from the complete continuity of operator $T\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ and boundedness of operator $I + B\Bigl(\sum_{n=0}^{\infty} (TB)^n T\Bigr)\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$, it follows that $(I - TB)^{-1}T$: $C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a completely continuous linear operator. Combining this with the fact that $A^{-1}\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a bounded linear operator, we see that $S = A^{-1}(I - TB)^{-1}T\colon C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is completely continuous.

Let $h \in C_\omega^+(\mathbb{R})$ and $u = Sh$. By Lemma 2.3, equation (2.9) has a unique ω-periodic solution $v \in K_0$. By (2.17), $u = A^{-1}v$.

If $c \geqslant 0$, by Lemma 2.2 and the definition of cone K_0, we have

$$\begin{aligned} u(t) = A^{-1}v(t) = \sum_{j=0}^{\infty} c^j v(t - j\delta) \geqslant v(t) \geqslant \sigma\|v\|_C = \sigma\|Au\|_C \\ \geqslant \sigma\|A^{-1}\|^{-1}\|u\|_C = \sigma(1 - |c|)\|u\|_C = \alpha\|u\|_C. \end{aligned}$$

If $c < 0$, by Lemma 2.2 and the definition of cone K_0, we have

$$\begin{aligned} u(t) &= A^{-1}v(t) = \sum_{j=0}^{\infty} c^j v(t - j\delta) \\ &= \sum_{j=0}^{\infty} |c|^{2j} v(t - 2j\delta) - \sum_{j=0}^{\infty} |c|^{2j+1} v(t - (2j+1)\delta) \\ &\geqslant \sum_{j=0}^{\infty} |c|^{2j}\sigma\|v\|_C - \sum_{j=0}^{\infty} |c|^{2j+1}\|v\|_C \\ &= \frac{\sigma - |c|}{1 - |c|^2}\|v\|_C = \frac{\sigma - |c|}{1 - |c|^2}\|Au\|_C \\ &\geqslant \frac{\sigma - |c|}{1 - |c|^2}\|A^{-1}\|^{-1}\|u\|_C = \frac{\sigma - |c|}{1 + |c|}\|u\|_C = \alpha\|u\|_c. \end{aligned}$$

Hence $Sh = u \in K$. □

Let $f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$. Now we consider the nonlinear equation (1.1). For every $u \in K$, set

$$F(u)(t) := f(t, u(t - \tau_1), \dots, u(t - \tau_n)), \quad t \in \mathbb{R}. \tag{2.19}$$

Then $F\colon K \to C_\omega^+(\mathbb{R})$ is continuous. Define a mapping $Q\colon K \to K$ by

$$Q(u) = S(F(u)), \quad u \in K. \tag{2.20}$$

By the definition of operator S, the positive ω-periodic solution of equation (1.1) is equivalent to the nontrivial fixed point of Q. By Lemma 2.4, we have the following statement:

Lemma 2.5. *$Q\colon K \to K$ is a completely continuous mapping.*

We will find the nonzero fixed point of Q by using the fixed point index theory in cones. The following two lemmas from [1], [3] are needed in our argument.

Lemma 2.6. *Let E be a Banach space, Ω be a bounded open subset of E with $\theta \in \Omega$, $Q\colon K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\lambda Q(u) \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \lambda \leqslant 1$, then the fixed point index $i(Q, K \cap \Omega, K) = 1$.*

Lemma 2.7. *Let E be a Banach space, Ω be a bounded open subset of E and $Q\colon K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists an $e \in K \setminus \{\theta\}$ such that $u - Q(u) \neq \mu e$ for every $u \in K \cap \partial\Omega$ and $\mu \geqslant 0$, then the fixed point index $i(Q, K \cap \Omega, K) = 0$.*

In the next section, we will use Lemma 2.6 and Lemma 2.7 to prove Theorem 1.1 and Theorem 1.2.

3. Proofs of main results

Proof of Theorem 1.1. Choose $E = C_\omega(\mathbb{R})$. Let $K \subset E$ be the closed convex cone defined by (2.16) and $Q\colon K \to K$ be the operator defined by (2.20). Then the positive ω-periodic solution of equation (1.1) is equivalent to the nontrivial fixed point of Q. Set

$$\Omega_1 = \{u \in C_\omega(\mathbb{R})\colon \|u\|_C < r\}, \quad \Omega_2 = \{u \in C_\omega(\mathbb{R})\colon \|u\|_C < R\}, \tag{3.1}$$

where $0 < r < R < \infty$. We show that the operator Q has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Let $r \in (0, \eta)$, where η is the positive constant in condition (F1). We prove that Q satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$, namely $\lambda Qu \neq u$ for every $u \in K \cap \partial\Omega_1$ and $0 < \lambda \leqslant 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \lambda_0 \leqslant 1$ such that $\lambda_0 Q u_0 = u_0$, then by the definition of Q and Lemma 2.4, $u_0 \in C^1_\omega(\mathbb{R})$ satisfies the delay differential equation

$$(u_0(t) - cu_0(t-\delta))' + a(t)u_0(t) = \lambda_0 f(t, u_0(t-\tau_1), \ldots, u_0(t-\tau_n)), \quad t \in \mathbb{R}. \tag{3.2}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$0 \leqslant u_0(t-\tau_k) \leqslant \|u_0\|_C = r < \eta, \quad k = 1, \ldots, n, \ t \in \mathbb{R}. \tag{3.3}$$

Hence from condition (F1) it follows that

$$f(t, u_0(t-\tau_1), \ldots, u_0(t-\tau_n)) \leqslant c_1 u_0(t-\tau_1) + \ldots + c_n u_0(t-\tau_n), \quad t \in \mathbb{R}.$$

By this and (3.2), we obtain that

$$(u_0(t) - cu_0(t-\delta))' + a(t)u_0(t) \leqslant c_1 u_0(t-\tau_1) + \ldots + c_n u_0(t-\tau_n), \quad t \in \mathbb{R}.$$

Integrating both sides of this inequality from 0 to ω and using the periodicity of u_0, we have

$$\begin{aligned}\int_0^\omega a(t)u_0(t)\,\mathrm{d}t &\leqslant c_1 \int_0^\omega u_0(t-\tau_1)\,\mathrm{d}t + \ldots + c_n \int_0^\omega u_0(t-\tau_n)\,\mathrm{d}t \\ &= (c_1 + \ldots + c_n) \int_0^\omega u_0(t)\,\mathrm{d}t.\end{aligned}$$

From this it follows that

$$\underline{a} \int_0^\omega u_0(t)\,\mathrm{d}t \leqslant \int_0^\omega a(t)u_0(t)\,\mathrm{d}t \leqslant (c_1 + \ldots + c_n) \int_0^\omega u_0(t)\,\mathrm{d}t. \tag{3.4}$$

By the definition of cone K, $\int_0^\omega u_0(t)\,\mathrm{d}t \geqslant \alpha \|u_0\|_C \cdot \omega > 0$. From (3.4) it follows that $\underline{a} \leqslant c_1 + \ldots + c_n$, which contradicts the assumption in condition (F1). Hence Q satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$. By Lemma 2.6, we have

$$i(Q, K \cap \Omega_1, K) = 1. \tag{3.5}$$

Next, choose $R > \max\{H/\alpha, \eta\}$, where H is the positive constant in condition (F2), and let $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that Q satisfies the condition of Lemma 2.7 in $K \cap \partial\Omega_2$, namely $u - Qu \neq \mu e$ for every $u \in K \cap \partial\Omega_2$ and $\mu \geqslant 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\mu_1 \geqslant 0$ such that $u_1 - Qu_1 = \mu_1 e$, since $u_1 - \mu_1 e = Qu_1$, by definition of Q and Lemma 2.4, $u_1 \in C^1_\omega(\mathbb{R})$ satisfies the differential equation

$$\begin{aligned}(u_1(t) - cu_1(t-\delta))' + a(t)(u_1(t) - \mu_1) \\ = f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)), \quad t \in \mathbb{R}.\end{aligned} \tag{3.6}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_1(t-\tau_k) \geqslant \alpha \|u_1\|_C > H, \quad t \in \mathbb{R},\ k = 1, \ldots, n. \tag{3.7}$$

From this and condition (F2), it follows that

$$f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)) \geqslant d_1 u_1(t-\tau_1) + \ldots + d_n u_n(t-\tau_n), \quad t \in I.$$

By this inequality and (3.6), we have

$$(u_1(t) - cu_1(t-\delta))' + a(t)(u_1(t) - \mu_1) \geqslant d_1 u_1(t-\tau_1) + \ldots + d_n u_1(t-\tau_n), \quad t \in I.$$

Integrating this inequality on $[0,\omega]$ and using the periodicity of u_1, we have

$$\begin{aligned}\int_0^\omega a(t)(u_1(t)-\mu_1)\,\mathrm{d}t &\geqslant d_1\int_0^\omega u_1(t-\tau_1)\,\mathrm{d}t+\ldots+d_n\int_0^\omega u_1(t-\tau_n)\,\mathrm{d}t\\ &=(d_1+\ldots+d_n)\int_0^\omega u_1(t)\,\mathrm{d}t.\end{aligned}$$

Consequently,

$$\begin{aligned}\overline{a}\int_0^\omega u_1(t)\,\mathrm{d}t \geqslant \int_0^\omega a(t)u_1(t)\,\mathrm{d}t &\geqslant \int_0^\omega a(t)(u_1(t)-\mu_1)\,\mathrm{d}t\\ &\geqslant (d_1+\ldots+d_n)\int_0^\omega u_1(t)\,\mathrm{d}t.\end{aligned}\tag{3.8}$$

Since $\int_0^\omega u_1(t)\,\mathrm{d}t \geqslant \alpha\|u_1\|_C\cdot\omega > 0$, from this inequality it follows that $\overline{a}\geqslant d_1+\ldots+d_n$, which contradicts the assumption in condition (F2). This means that Q satisfies the condition of Lemma 2.7 in $K\cap\partial\Omega_2$. By Lemma 2.7,

$$i(Q,K\cap\Omega_2,K)=0.\tag{3.9}$$

Now by the additivity of fixed point index, (3.5) and (3.9) we have

$$i(Q,K\cap(\Omega_2\setminus\overline{\Omega}_1),K)=i(Q,K\cap\Omega_2,K)-i(Q,K\cap\Omega_1,K)=-1.$$

Hence Q has a fixed point in $K\cap(\Omega_2\setminus\overline{\Omega}_1)$, which is a positive ω-periodic solution of equation (1.1). □

P r o o f of Theorem 1.2. Let $\Omega_1,\Omega_2\subset C_\omega(\mathbb{R})$ be defined by (3.1). We prove that the operator Q defined by (2.20) has a fixed point in $K\cap(\Omega_2\setminus\overline{\Omega}_1)$ if r is small enough and R large enough.

Let $r\in(0,\eta)$, where η is the positive constant in condition (F3), and choose $e(t)\equiv 1$. We prove that Q satisfies the condition of Lemma 2.7 in $K\cap\partial\Omega_1$, namely $u-Qu\neq\mu e$ for every $u\in K\cap\partial\Omega_1$ and $\mu\geqslant 0$. In fact, if there exist $u_0\in K\cap\partial\Omega_1$ and $\mu_0\geqslant 0$ such that $u_0-Qu_0=\mu_0 e$, since $u_0-\mu_0 e=Qu_0$, by definition of Q and Lemma 2.4, $u_0\in C^1_\omega(\mathbb{R})$ satisfies the differential equation

$$(u_0(t)-cu_0(t-\delta))'+a(t)(u_0(t)-\mu_0)=f(t,u_0(t-\tau_1),\ldots,u_0(t-\tau_n)),\quad t\in\mathbb{R}.\tag{3.10}$$

Since $u_0\in K\cap\partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (3.3). From (3.3) and condition (F3) it follows that

$$f(t,u_0(t-\tau_1),\ldots,u_0(t-\tau_n))\geqslant d_1u_0(t-\tau_1)+\ldots+d_nu_0(t-\tau_n),\quad t\in\mathbb{R}.$$

From this and (3.10), it follows that

$$(u_0(t) - cu_0(t-\delta))' + a(t)(u_0(t) - \mu_0) \geqslant d_1 u_0(t-\tau_1) + \ldots + d_n u_0(t-\tau_n), \quad t \in \mathbb{R}.$$

Integrating this inequality on $[0, \omega]$ and using the periodicity of $u_0(t)$, we have that

$$\begin{aligned}\int_0^\omega a(t)(u_0(t) - \mu_0)\,\mathrm{d}t &\geqslant d_1 \int_0^\omega u_0(t-\tau_1)\,\mathrm{d}t + \ldots + d_n \int_0^\omega u_0(t-\tau_n)\,\mathrm{d}t \\ &= (d_1 + \ldots + d_n) \int_0^\omega u_0(t)\,\mathrm{d}t.\end{aligned}$$

From this we obtain that

$$\begin{aligned}\overline{a} \int_0^\omega u_0(t)\,\mathrm{d}t \geqslant \int_0^\omega a(t) u_0(t)\,\mathrm{d}t &\geqslant \int_0^\omega a(t)(u_0(t) - \mu_0)\,\mathrm{d}t \\ &\geqslant (d_1 + \ldots + d_n) \int_0^\omega u_0(t)\,\mathrm{d}s.\end{aligned} \tag{3.11}$$

Since $\int_0^\omega u_0(t)\,dt \geqslant \alpha \|u_0\|_C \cdot \omega > 0$, from the inequality (3.11) it follows that $\overline{a} \geqslant d_1 + \ldots + d_n$, which contradicts the assumption in (F3). Hence, Q satisfies the condition of Lemma 2.7 in $K \cap \partial\Omega_1$. By Lemma 2.7, we have

$$i(Q, K \cap \Omega_1, K) = 0. \tag{3.12}$$

Then, choosing $R > \max\{H/\alpha, \eta\}$, we show that Q satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$, namely $\lambda Q u \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \lambda \leqslant 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \lambda_1 \leqslant 1$ such that $\lambda_1 Q u_1 = u_1$, then by the definition of Q and Lemma 2.4, $u_1 \in C^1_\omega(\mathbb{R})$ satisfies the differential equation

$$(u_1(t) - cu_1(t-\delta))' + a(t) u_1(t) = \lambda_1 f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)), \quad t \in \mathbb{R}. \tag{3.13}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, u_1 satisfies (3.7). From (3.7) and condition (F4), it follows that

$$f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)) \leqslant c_1 u_1(t-\tau_1) + \ldots + c_n u_1(t-\tau_n), \quad t \in \mathbb{R}.$$

By this and (3.13), we have that

$$(u_1(t) - cu_1(t-\delta))' + a(t) u_1(t) \leqslant c_1 u_1(t-\tau_1) + \ldots + c_n u_1(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.14}$$

Integrating this inequality on $[0, \omega]$ and using the periodicity of $u_1(t)$, we have that

$$\begin{aligned}\int_0^\omega a(t) u_1(t)\,\mathrm{d}t &\leqslant c_1 \int_0^\omega u_1(t-\tau_1)\,\mathrm{d}t + \ldots + c_n \int_0^\omega u_1(t-\tau_n)\,\mathrm{d}t \\ &= (c_1 + \ldots + c_n) \int_0^\omega u_1(t)\,\mathrm{d}t.\end{aligned}$$

From this we obtain that

$$\underline{a}\int_0^\omega u_1(t)\,\mathrm{d}t \leqslant \int_0^\omega a(t)u_1(t)\,\mathrm{d}t \leqslant (c_1+\ldots+c_n)\int_0^\omega u_1(t)\,\mathrm{d}t. \tag{3.15}$$

Since $\int_0^\omega u_1(t)\,\mathrm{d}t \geqslant \alpha\|u_0\|_C\cdot\omega > 0$, from the inequality (3.15) it follows that $\underline{a} \leqslant c_1+\ldots+c_n$, which contradicts the assumption in condition (F4). Hence, Q satisfies the condition of Lemma 2.6 in $K\cap\partial\Omega_1$. By Lemma 2.6, we have

$$i(Q, K\cap\Omega_2, K) = 1. \tag{3.16}$$

Now, from (3.12) and (3.16) it follows that

$$i(Q, K\cap(\Omega_2\setminus\overline{\Omega}_1), K) = i(Q, K\cap\Omega_2, K) - i(Q, K\cap\Omega_1, K) = 1.$$

Hence Q has a fixed point in $K\cap(\Omega_2\setminus\overline{\Omega}_1)$, which is a positive ω-periodic solution of equation (1.1). □

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Authors' address: *Yongxiang Li, Ailan Liu*, Department of Mathematics, Northwest Normal University, 967 Anning East Road, Lanzhou 730070, People's Republic of China, e-mail: `liyxnwnu@163.com`, `15339860773@163.com`.