SOME EQUIVALENT METRICS FOR BOUNDED NORMAL OPERATORS

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Abstract. Some stronger and equivalent metrics are defined on \( \mathcal{M} \), the set of all bounded normal operators on a Hilbert space \( H \) and then some topological properties of \( \mathcal{M} \) are investigated.

Keywords: Hilbert space; normal operator; equivalent metrics; composition operator

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1. INTRODUCTION AND PRELIMINARIES

Let \( H \) be a separable, infinite dimensional, complex Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and let \( \mathcal{B}(H) \) denote the algebra of all bounded linear operators on \( H \). The problem of the topological structure of \( \mathcal{C}(H) \), the subsets of closed and densely defined linear operators on \( H \) has been considered starting with the paper by Cordes and Labrousse [2]; see also [7]. They prove that the metric distance between two densely defined unbounded operators \( A \) and \( B \) may be taken as \( \| (I+AA^*)^{-1}-(I+BB^*)^{-1} \| \).

As the authors show, this metric defines the same topology for bounded operators as the ordinary metric \( \| A - B \| \). For \( A \in \mathcal{C}(H) \), let \( \alpha \) denote the contraction defined as \( \alpha(T) = A(1+A^*A)^{-1/2} \). Kaufman [5] studies a metric \( \delta \) on \( \mathcal{C}(H) \) defined as \( \delta(A,B) = \| \alpha(A) - \alpha(B) \| \) and then the author discusses connections between \( \delta \)-convergence and strong-operator-topology convergence. Also, he shows that this metric is stronger than the gap metric \( d \) (see [4], page 197) and not equivalent to it. In [6], Kittaneh presents quantitative improvements of the result of Kaufman [5] concerning equivalence of three metrics on the space of bounded linear operators on a Hilbert space. In [1], Benharrat and Messirdi defined some new strictly stronger metrics than the gap metric \( d \) and characterized the closure with respect to these metrics of the subset \( \mathcal{B}(H) \) of bounded elements of \( \mathcal{C}(H) \).

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Let \( \mathcal{M} \) be the subset of bounded normal operators in \( \mathcal{B}(H) \), \( A \in \mathcal{M} \) and let \( 0 < a < \| A \|^{-1} \). In this paper, by motivation of the above mentioned results, we shall replace \( 1 + A^*A \) with \( I + a^2A^*A + a^4(A^*)^2A^2 + \ldots \), and then we obtain some analogous results on topological properties of \( \mathcal{M} \).

In Section 2, we show that \( K_a(A) := \sum_{n=0}^{\infty} a^{2n} A^{*n} A^n \) is positive, invertible and then we obtain the relation between the operators \( K_a(A) \), \( K_a^{-1}(A) \) and \( (K_a(A))^{-1/2} \) in the case when \( A \) is normal. Moreover, we introduce some special types of metrics on normal operators in \( \mathcal{B}(H) \) and then we compare the topologies induced by these metrics.

In Section 3, inspired by definition of bisecting for \( A \in C(H) \) in [8], we define \( \tilde{A}_a \) for \( A \in \mathcal{M} \). Then using \( \tilde{A}_a \) and the metrics defined in Section 2, we introduce the \( F_1, \ldots, F_4 \) maps on \( \mathcal{M} \) with different metrics into \( \mathcal{M} \) with the aid of usual operator norm. Then we will proceed on investigating the continuity of these maps. At the end, as an example we determine \( K_a(C_\varphi), R_a(C_\varphi), S_a(C_\varphi), (\tilde{C}_\varphi)_a \) for \( C_\varphi \in \mathcal{M} \), where \( C_\varphi(f) = f \circ \varphi \) is the composition operator on \( L^2(\Sigma) \).

2. Stronger and equivalent metrics on \( \mathcal{M} \)

For \( A \in \mathcal{B}(H) \), let \( A^* \), \( \mathcal{N}(A) \), \( \mathcal{R}(A) \), \( r(A) \) and \( \| A \| \) denote the adjoint, the null space, the range, the spectral radius and the usual operator norm of \( A \), respectively. Note that \( r(A) = \lim_{n \to \infty} \|A^n\|^{1/n} \leq \| A \| \) and that the equality holds if \( A \) is normal. \( A \) is called positive if \( \langle Ax, x \rangle \geq 0 \) holds for every \( x \in H \) in which case we write \( A \succeq 0 \).

For an operator \( A \in \mathcal{B}(H) \) let \( 0 < a < (r(A))^{-1} \) be an arbitrary but fixed number. Define \( K_a(A) = \sum_{n=0}^{\infty} a^{2n} A^{*n} A^n \). The definition of \( K_a(A) \) is due to Gilfeather [3], Lambert and Petrovic [9].

**Lemma 2.1.** Let \( A \in \mathcal{B}(H) \). Then \( 0 \leq K_a(A) \in \mathcal{B}(H) \) and \( K_a(A) \) is invertible with \( \| K_a^{-1}(A) \| \leq 1 \).

**Proof.** Since \( \lim_{n \to \infty} \|a^{2n} A^{*n} A^n\|^{1/n} < (r(A))^{-2} \lim_{n \to \infty} \|A^n\|^{2/n} = 1 \), so the infinite series \( K_a(A) \) converges absolutely. Also, for all \( x \in H \) we have

\[
\langle K_a(A)(x), x \rangle = \sum_{n=0}^{\infty} a^{2n} \|A^n(x)\|^2 \geq 0.
\]

Thus,

\[
\| \sqrt{K_a(A)}(x) \|^2 = \langle K_a(A)(x), x \rangle = \| x \|^2 + \sum_{n=1}^{\infty} a^{2n} \|A^n(x)\|^2 \geq \| x \|^2,
\]

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and so
\[ R(\sqrt{K_a(A)}) = R(\sqrt{K_a(A)}) = N(\sqrt{K_a(A)})^{-1} = H. \]

It follows that \( \sqrt{K_a(A)} \) and hence \( K_a(A) \) is invertible. Now, replacing \( x \) by \( (K_a(A))^{-1/2}(x) \) we obtain \( \|(K_a(A))^{-1/2}(x)\| \leq \|x\| \). This implies that
\[ \frac{1}{\|K_a(A)\|} \leq \frac{1}{\|\sqrt{K_a(A)}\|^2} \leq 1. \]

\[ \square \]

For \( A \in \mathcal{B}(H) \) set \( R_a(A) = (K_a(A))^{-1} \) and \( S_a(A) = \sqrt{R_a(A)} \). Then by Lemma 2.1, \( R_a(A) \) and \( S_a(A) = (K_a(A))^{-1/2} \) are positive and \( S_a(A) \) is a contraction.

Moreover, when \( A \) is a normal operator, i.e. \( AA^* = A^*A \), then \( R_a(A) = R_a(A^*) \), \( AR_a(A) = R_a(A)A \) and \( A^*R_a(A) = R_a(A^*) \).

Recall that for \( A \in \mathcal{C}(H) \), the fundamental properties of \( R_A = (I + A^*A)^{-1} \) and \( S_A = (I + A^*A)^{-1/2} \) have been investigated by many authors, e.g. [2], [1]. In the following lemma we obtain a relationship between the concepts of \( R_a(A) \) and \( S_a(A) \) when \( A \in \mathcal{B}(H) \) is a normal operator.

**Lemma 2.2.** Let \( A \in \mathcal{B}(H) \) be a normal operator and let \( n \in \mathbb{N} \cup \{0\} \). Then the following assertions hold.

(a) \( A^n R_a(A) = R_a(A)A^n \);
(b) \( A^n S_a(A) = S_a(A)A^n \);
(c) \( S_a(A)(K_a(A) - I)S_a(A) = I - R_a(A) \);
(d) \( \sqrt{K_a(A)} - I = a|A|(S_a(A))^{-1} \);
(e) \( R_a(A) = I - a^2|A|^2 \);
(f) \( \mathcal{N}(S_a(A)) \cap \mathcal{N}(A) = \{0\} \).

**Proof.** (a) Since \( A \) is normal, from direct computations we obtain that
\[ A^n K_a(A) = A^n (I + a^2A^*A + a^4(A^*)^2A^2 + \ldots) \]
\[ = A^n + a^2A^n A^*A + a^4 A^n (A^*)^2A^2 + \ldots \]
\[ = (I + a^2AA^* + a^4 A^2 (A^*)^2 + \ldots)A^n = K_a(A^*)A^n = K_a(A)A^n. \]

Therefore, the inverse of \( K_a(A) \) is also commute with all \( A^n \).

(b) Since \( A^n R_a(A) = R_a(A)A^n \), it follows that \( A^n P(R_a(A)) = P(R_a(A))A^n \), where \( P \) is any polynomial. Now let \( \{P_m\} \) be a sequence of polynomials converging uniformly to a continuous function \( g \). Then for each \( x, y \in H \) we have
\[ \langle A^n g(R_a(A))(x), y \rangle = \lim_{m \to \infty} \langle P_m(R_a(A))(x), (A^n)^*y \rangle \]
\[ = \lim_{m \to \infty} \langle P_m(R_a(A))A^n(x), y \rangle \quad \text{(by part (a))} \]
\[ = \langle g(R_a(A))A^n(x), y \rangle. \]
Thus, $A^n g(R_a(A)) = g(R_a(A)) A^n$. Let $g$ be a square root function. Consequently, $A^n \sqrt{R_a(A)} = \sqrt{R_a(A)} A^n$, and so $A^n S_a(A) = S_a(A) A^n$.

(c) Since $R_a(A) = S_a^2(A)$, then

$$I - R_a(A) = (R_a^{-1}(A) - I)R_a(A) = a^2 A^* A R_a(A) + a^4 (A^*)^2 A^2 R_a(A) + \ldots$$

$$= a^2 A^* S_a(A) S_a(A) A + a^4 (A^*)^2 S_a(A) S_a(A) A^2 + \ldots$$

$$= a^2 A^* S_a(A) A S_a(A) + a^4 (A^*)^2 S_a(A) A^2 S_a(A) + \ldots$$

$$= \sum_{n=1}^{\infty} a^{2n} (A^*)^n S_a(A) A^n S_a(A) = S_a(A) (K_a(A) - I) S_a(A).$$

(d) Normality of $A$ implies that

$$K_a(A) - I = a^2 A^* A (I + a^2 A^* A + a^4 (A^*)^2 A^2 + \ldots) = a^2 |A|^2 K_a(A).$$

Thus, $\sqrt{K_a(A)} - I = a |A| \sqrt{K_a(A)} = a |A| (S_a(A))^{-1}$.

(e) It follows from (c) and (d).

(f) It suffices to show that $\|S_a(A) u\|^2 + \|a |A| u\|^2 = \|u\|^2$ for all $u \in H$. For this, let $u \in H$. Then by (e) we have

$$\|S_a(A) u\|^2 + \|a |A| u\|^2 = \langle S_a(A) u, S_a(A) u \rangle + \langle a |A| u, a |A| u \rangle$$

$$= \langle u, R_a(A) u \rangle + \langle u, a^2 |A|^2 u \rangle$$

$$= \langle u, R_a(A) u \rangle + \langle u, (I - R_a(A)) u \rangle = \langle u, u \rangle = \|u\|^2.$$

Lemma 2.3 ([2]). Let $A$ be closed. Then

$$\Pi_{G(A)} = \begin{bmatrix} R_A & A^* R_A^* \\ AR_A & I - R_A^* \end{bmatrix},$$

where $\Pi_{G(A)}$ denotes the orthogonal projection onto $G(A) = \{(x, Ax) : x \in D(A)\}$.

Now inspired by matrix $\Pi_{G(A)}$, we define $\Pi_a(A) \in B(H \otimes H)$ for $A \in \mathcal{M}$:

$$\Pi_a(A) = \begin{bmatrix} R_a(A) & a |A| S_a(A) \\ a |A| S_a(A) & I - R_a(A) \end{bmatrix}.$$

In [1], Benharrat and Messirdi introduced metrics $g_G(T, S)$, $p_G(T, S)$, $q_G(T, S)$ and $\Sigma_G(T, S)$ for $S, T \in C(H_1, H_2)$ and a positive bijection $G \in \mathcal{L}^+(H_1)$.

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Now, inspired by these metrics we define special types of metrics on $\mathcal{M}$:

\[
\begin{align*}
    d_{(a,b)}^1(A,B) &= \|\Pi_a(A) - \Pi_b(B)\|; \\
    d_{(a,b)}^2(A,B) &= \sqrt{\|R_a(A) - R_b(B)\|^2 + \|a|A|S_a(A) - b|B|S_b(B)\|^2}; \\
    d_{(a,b)}^3(A,B) &= \|a|A| - b|B\|; \\
    d_{(a,b)}^4(A,B) &= \sqrt{2\|a|A| - b|B\|^2 + 2\|S_a(A) - S_b(B)\|^2},
\end{align*}
\]

where $0 < a < \|A\|^{-1}$ and $0 < b < \|B\|^{-1}$ are arbitrary but fixed numbers, whenever $A$ and $B$ are nonzero elements of $\mathcal{M}$. Note that $d_{(a,b)}^3 \leq d_{(a,b)}^4$. Hence, the topology induced from the metric $d_{(a,b)}^4$ on $\mathcal{M}$ is stronger than that induced from $d_{(a,b)}^3$.

Lemma 2.4 ([6]).

(a) If $A, B \in \mathcal{B}(H)$ are positive, then

\[\|A - B\| \leq \sqrt{\|A^2 - B^2\|}.\]

(b) If $T \in \mathcal{B}(H \oplus H)$ and

\[T = \begin{bmatrix} A & B \\ C & D \end{bmatrix},\]

then $\|T\|^2 \leq \|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2$.

It was proved that in [1] the topology induced from the metric $g_G(T, S)$ on $\mathcal{C}(H_1, H_2)$ is strictly stronger than that induced from $p_G(T, S)$. But the following proposition proves that the metrics $d_{(a,b)}^1$ and $d_{(a,b)}^2$ on $\mathcal{M}$ generate the same topology.

Proposition 2.5. The topology induced from the metric $d_{(a,b)}^1$ on $\mathcal{M}$ is equivalent to the topology induced from $d_{(a,b)}^2$ on $\mathcal{M}$.

Proof. Let $A, B \in \mathcal{M}$. Evidently, $d_{(a,b)}^2(A, B) \leq d_{(a,b)}^1(A, B)$. On the other hand, by Lemma 2.4 (b) we have

\[\|\Pi_a(A) - \Pi_b(B)\|^2 \leq 2\|R_a(A) - R_b(B)\|^2 + 2\|a|A|S_a(A) - b|B|S_b(B)\|^2.\]

Thus, $d_{(a,b)}^1(A, B) \leq \sqrt{2}d_{(a,b)}^2(A, B)$.

Lemma 2.6. Let $A$ and $B$ be two nonzero elements of $\mathcal{B}(H)$. Then

\[\frac{A}{\|A\|} - \frac{B}{\|B\|} \leq \frac{2\|A - B\|}{\|A\|}.\]

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In the following theorem, we show that

\[ \frac{A}{\|A\|} - \frac{B}{\|B\|} \leq \|B\|\|A - B\| + \|B\|(\|B\| - \|A\|) \leq 2\|B\|\|A - B\|. \]

The result follows. \(\square\)

Now, let \(A\) and \(B\) be two nonzero normal elements of \(B(H)\). Then \(r(A) = \|A\|\) and \(r(B) = \|B\|\). For \(0 < \alpha < 1\) put \(a_\alpha = \alpha\|A\|^{-1}\) and \(b_\alpha = \alpha\|B\|^{-1}\). By Lemma 2.6 we obtain

\[ \|a_\alpha A - b_\alpha B\| = \left\| \frac{\alpha A}{\|A\|} - \frac{\alpha B}{\|B\|} \right\| \leq \frac{2\alpha\|A - B\|}{\|A\|}. \]

In the following theorem, we show that \(d_{(a_\alpha, b_\alpha)}^{[i]}(A, 0) < \|\cdot\|\) for \(i = 3, 4\) on \(M\). This is why, in the study carried out by Benharrat and Messirdi, it was found that the restriction of the metric \(g_G(T, S)\) to \(L(H_1, H_2)\) is equivalent to the operator norm.

**Theorem 2.7.** The topology induced from the operator norm on \(M\) is strictly stronger than that induced from \(d_{(a_\alpha, b_\alpha)}^{[i]}(A, 0)\) for \(i = 3, 4\) on \(M\).

**Proof.** Let \(A, B \in M\). Let \(A \neq 0\) and \(B = 0\). Then by Lemma 2.4 (a) we have

\[ \|S_{a_\alpha}(A) - I\| = \|\sqrt{I - a_\alpha^2|A|^2} - I\| \leq \sqrt{\|a_\alpha^2|A|^2\|} \leq a_\alpha\|A\| \]

and \(\|a_\alpha|A|\| = a_\alpha\|A\|\). It follows that \(d_{(a_\alpha, b_\alpha)}^{[3]}(A, 0) = a_\alpha\|A\|\) and

\[ d_{(a_\alpha, b_\alpha)}^{[4]}(A, 0) = \sqrt{2(\|a_\alpha|A|\|)^2 + 2\|S_{a_\alpha}(A) - I\|^2} \leq 2a_\alpha\|A\|. \]

Now, let \(A\) and \(B\) be two nonzero elements of \(M\). Then by Lemma 2.4 (a) and Lemma 2.6 we have

\[ d_{(a_\alpha, b_\alpha)}^{[3]}(A, B) = \|a_\alpha|A| - b_\alpha|B|\| \leq \sqrt{\|a_\alpha^2A^*A - b_\alpha^2B^*B\|} \]

\[ \leq \sqrt{\|a_\alpha A^* - b_\alpha B^*\|\|a_\alpha A\| + \|b_\alpha B^*\|\|a_\alpha A - b_\alpha B\|} \]

\[ = \sqrt{(\|a_\alpha A\| + \|b_\alpha B\|)\|a_\alpha A - b_\alpha B\|} \]

\[ \leq \sqrt{\|a_\alpha A\| + \|b_\alpha B\|} \sqrt{2\alpha\|A - B\|}. \]

Also, since

\[ \|S_{a_\alpha}(A) - S_{b_\alpha}(B)\| = \|\sqrt{I - a_\alpha^2|A|^2} - \sqrt{I - b_\alpha^2|B|^2}\| \]

\[ \leq \sqrt{(I - a_\alpha^2|A|^2) - (I - b_\alpha^2|B|^2)} \]

\[ = \sqrt{a_\alpha^2A^*A - b_\alpha^2B^*B} \leq \sqrt{\|a_\alpha A\| + \|b_\alpha B\|} \sqrt{2\alpha\|A - B\|}, \]

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we get that
\[ d_{[a, b]}^{[4]}(A, B) \leq \sqrt{4(\|a_A A\| + \|b_B B\|) + 2\alpha\|A - B\|}. \]
This completes the proof. \hfill \Box

Recall that in the study carried out by Benharrat and Messirdi in [1], it was proved that the topology induced from the metric \(q_G(T, S)\) on \(C(H_1, H_2)\) is strictly stronger than that induced from \(g_G(T, S)\). However, in the following theorem we show that \(d^{[1]} \cong d^{[3]}\).

**Theorem 2.8.** The topology induced from the metric \(d^{[1]}\) on \(\mathcal{M}\) is equivalent to the topology induced from to the metric \(d^{[3]}\) on \(\mathcal{M}\).

**Proof.** Let \(A, B \in \mathcal{M}\). Then by Lemma 2.4 (a) and the definition of \(d^{[i]}\) for \(i = 1, 3\) we have
\[
d^{[3]}_{(a, b)}(A, B) = \|aA - bB\| = \|aA|S_a(A)S_a^{-1}(A) - bB|S_b(B)S_b^{-1}(B)\| \\
\leq \|aA|S_a(A) - bB|S_b(B)\|\|S_a^{-1}(A)\| + \|bB|S_b(B)\|\|S_a^{-1}(A) - S_b^{-1}(B)\| \\
\leq d^{[1]}_{(a, b)}(A, B)\|S_a^{-1}(A)\| + \|bB|S_b(B)\|\sqrt{\|S_a^{-2}(A) - S_b^{-2}(B)\|} \\
= d^{[1]}_{(a, b)}(A, B)\|S_a^{-1}(A)\| + \|bB|S_b(B)\|\sqrt{\|R_a^{-1}(A) - R_b^{-1}(B)\|} \\
= d^{[1]}_{(a, b)}(A, B)\|S_a^{-1}(A)\| \\
+ \|bB|S_b(B)\|\sqrt{\|R_a^{-1}(A)\|}\sqrt{\|R_b^{-1}(B)\|}d^{[1]}_{(a, b)}(A, B).
\]

Conversely, by Lemma 2.2 (e) and Lemma 2.4 (a) we obtain
\[
\|R_a(A) - R_b(B)\| = \|(I - R_a(A)) - (I - R_b(B))\| = \|a^2A^2 - b^2B^2\| \\
\leq \|aA - bB\|\|(aA\| + \|bB\|\| = d^{[3]}_{(a, b)}(A, B)(\|aA\| + \|bB\|)
\]
and
\[
\|aA|S_a(A) - bB|S_b(B)\| \leq \|aA - bB\|\|S_a(A)\| + \|bB\|\|S_a(A) - S_b(B)\| \\
\leq \|aA - bB\| + \|bB\|\sqrt{\|R_a(A) - R_b(B)\|} \\
\leq d^{[3]}_{(a, b)}(A, B) + \|bB\|\sqrt{\|d^{[3]}_{(a, b)}(A, B)(\|aA\| + \|bB\|).}
\]
But
\[ (d_{(a,b)}^1(A, B))^2 \leq 2\|R_a(A) - R_b(B)\|^2 + 2\|a|S_a(A) - b|S_b(B)\|^2. \]

This completes the proof. \qed

3. Some operator transformations

The following lemma will be used in this section to obtain a new operator transform.

**Lemma 3.1.** Let \( A \in \mathcal{B}(H) \) be a normal operator. Then
\[
\|(I + S_a(A))^{-1}\| \leq 1.
\]

**Proof.** For all \( x \in H \) we have
\[
\|\sqrt{(I + S_a(A))(x)}\|^2 = \langle \sqrt{I + S_a(A)}(x), \sqrt{I + S_a(A)}x \rangle = \langle (I + S_a(A))(x), x \rangle = \langle x, x \rangle + \langle (S_a(A))x, x \rangle \geq \|x\|^2,
\]
and \( R(\sqrt{I + S_a(A)}) = N(\sqrt{I + S_a(A)})^\perp = H. \) Thus, \( \sqrt{I + S_a(A)} \) and hence \( I + S_a(A) \) is invertible. Now, replacing \( x \) by \( \sqrt{I + S_a(A)}(x) \) we obtain
\[
\|\sqrt{I + S_a(A)}(x)\| \leq \|x\|.
\]
It follows that
\[
\|(I + S_a(A))^{-1}\| \leq \|\sqrt{I + S_a(A)}\|^2 \leq 1.
\]
\qed

**Definition 3.2.** For \( A \in \mathcal{M} \) and \( 0 < a < \|A\|^{-1} \) the bisecting of \( A \), in the sense of Lambert and Petrovic, is the operator \( \tilde{A}_a \) defined as
\[
\tilde{A}_a = a|A|(I + S_a(A))^{-1}.
\]

The bisecting of \( A \) was originally introduced in [8] by Labrousse in order to study closed operators. By Lemma 3.1, \( I + S_a(A) \) is invertible and so \( \tilde{A}_a \) as a positive operator is well defined. Moreover, \( \|\tilde{A}_a\| \leq \|a|A||\|(I + S_a(A))^{-1}\| \leq 1. \)
Now we consider the maps

\[ F_1: (\mathcal{M}, \|\cdot\|) \to (\mathcal{M}, \|\cdot\|), \quad A \to (I + S_{a_\alpha}(A))^{-1}; \]
\[ F_2: (\mathcal{M}, \|\cdot\|) \to (\mathcal{M}, \|\cdot\|), \quad A \to \widetilde{A}_a; \]
\[ F_3: (\mathcal{M}, d^{[3]}) \to (\mathcal{M}, \|\cdot\|), \quad A \to \widetilde{A}_a; \]
\[ F_4: (\mathcal{M}, d^{[4]}) \to (\mathcal{M}, \|\cdot\|), \quad A \to \widetilde{A}_a. \]

We note that in \((\mathcal{M}, \|\cdot\|), \|\cdot\|\) is the norm of \(H\). We pose the following question:

For which operators \(A \in \mathcal{M}\) is the map \(F_i\) continuous?

**Theorem 3.3.** The maps \(F_1, F_2, F_3\) and \(F_4\) are continuous.

**Proof.** Let \(A \in \mathcal{M}\) and \(\|A\| \to 0\). By Theorem 2.7 and Lemma 3.1 we obtain

\[
\|F_1(A) - F_1(0)\| = \|(I + S_{a_\alpha}(A))^{-1} - (I + I)^{-1}\|
\leq \|(I + S_{a_\alpha}(A))^{-1} - \|I + S_{a_\alpha}(A) - 2I\|\|(2I)^{-1}\|
\leq \|S_{a_\alpha}(A) - I\| \leq a_\alpha \|A\| \to 0.
\]

Now, let \(A\) and \(B\) be two nonzero elements of \(\mathcal{M}\) and \(\|A - B\| \to 0\). We show that \(\|F_1(A) - F_1(B)\| \to 0\). Again by Theorem 2.7 and Lemma 3.1, if \(\|A - B\| \to 0\), we have

\[
\|F_1(A) - F_1(B)\| = \|(I + S_{a_\alpha}(A))^{-1} - (I + S_{b_\alpha}(B))^{-1}\|
\leq \|(I + S_{a_\alpha}(A))^{-1}\|\|S_{a_\alpha}(A) - S_{b_\alpha}(B)\|\|(I + S_{b_\alpha}(B))^{-1}\|
\leq \sqrt{\|a_\alpha A\| + \|b_\alpha B\|} \sqrt{\frac{2\alpha\|A - B\|}{\|A\|}} \to 0.
\]

Thus, \(F_1\) is continuous.

Let \(A \in \mathcal{M}\) and \(\|A\| \to 0\). By Lemma 3.1 we have

\[
\|F_2(A) - F_2(0)\| = \|\widetilde{A}_a - \widetilde{0}\| = \|a_\alpha A\|\|(I + S_{a_\alpha}(A))^{-1}\| \leq \|a_\alpha A\| = a_\alpha \|A\| \to 0.
\]

Now, let \(A\) and \(B\) be two nonzero elements of \(\mathcal{M}\) and \(\|A - B\| \to 0\). Then from Theorem 2.7 we obtain

\[
\|F_2(A) - F_2(B)\| = \|\widetilde{A}_a - \widetilde{B}_b\| = \|a_\alpha A\|\|(I + S_{a_\alpha}(A))^{-1} - b_\alpha |D|\|(I + S_{b_\alpha}(B))^{-1}\|
\leq \sqrt{\|a_\alpha^2 A^* A - b_\alpha^2 B^* B\|\|I + S_{a_\alpha}(A))^{-1}\|
\quad + \sqrt{\|b_\alpha B^* B\|\|I + S_{a_\alpha}(A)\|^{-1} - (I + S_{b_\alpha}(B))^{-1}\|
\leq \sqrt{\|a_\alpha A\| + \|b_\alpha B\|} \sqrt{2\alpha\|A - B\|\|A\|} \sqrt{\frac{2\alpha\|A - B\|\|A\|}{\|A\|}} \to 0.
\]

This implies that \(F_2\) is continuous.
Let $A \in M$ such that $d^{[3]}_{(a,0)}(A,0) \to 0$. Then $\|a|A|| \to 0$. Then we have

$$\|F_3(A) - F_3(0)\| = \|\tilde{A}_a - \tilde{0}\| = \|a|A| (I + S_a(A))^{-1} - 0\| \leq \|a|A|| \to 0.$$ 

Let $A$ and $B$ be two nonzero elements of $M$ and $d^{[3]}_{(a,b)}(A,B) \to 0$. Then

$$\|a|A| - b|B|| \to 0.$$ 

Again by Theorem 2.7 and definition of $d^{[3]}$ we have

$$\|F_3(A) - F_3(B)\| = \|\tilde{A}_a - \tilde{B}_b\| = \|a|A| (I + S_a(A))^{-1} - b|B| (I + S_b(B))^{-1}\|
\leq \|a|A| - b|B|| \| (I + S_a(A))^{-1}\|
+ \|b|B|| \| (I + S_a(A))^{-1}\| \|S_a(A) - S_b(B)\| \| (I + S_b(B))^{-1}\|
\leq \|a|A| - b|B|| \|b|B|| \sqrt{\|a^2|A|^2 - b^2|B|^2\|}
\leq \|a|A| - b|B|| \|b|B|| \sqrt{\|a^2|A|^2 - b^2|B|^2\|} \to 0.$$ 

Thus, $F_3$ is also continuous.

Let $A \in M$ and $d^{[3]}_{(a,0)}(A,0) \to 0$. Then $\|a|A|| \to 0$. Then

$$\|F_4(A) - F_4(0)\| = \|\tilde{A}_a - \tilde{0}\| = \|a|A| (I + S_a(A))^{-1} - 0\| \leq \|a|A|| \| (I + S_a(A))^{-1}\| \leq \|a|A|| \to 0.$$ 

Let $A, B \in M$ such that $d^{[4]}_{(a,b)}(A,B) \to 0$. Then $\|a|A| - b|B|| \to 0$ and $\|S_a(A) - S_b(B)|| \to 0$. Then we have

$$\|F_4(A) - F_4(B)\| = \|\tilde{A}_a - \tilde{B}_b\| = \|a|A| (I + S_a(A))^{-1} - b|B| (I + S_b(B))^{-1}\|
\leq \|a|A| - b|B|| \| (I + S_a(A))^{-1}\|
+ \|b|B|| \| (I + S_a(A))^{-1}\| \|S_a(A) - S_b(B)\| \| (I + S_b(B))^{-1}\|
\leq \|a|A| - b|B|| \|b|B|| \|S_a(A) - S_b(B)\| \to 0.$$ 

Consequently, $\|F_4(A) - F_4(B)\| \to 0$ as $d^{[4]}_{(a,b)}(A,B) \to 0$. \hfill \Box

**Definition 3.4.** If $A, B \in M$, $0 < a < \|A\|^{-1}$ and $0 < b < \|B\|^{-1}$. The Cordes-Labrousse transform with respect to the pair $(A,B)$ is the operator $V_{A,B}^{(a,b)}$ given by

$$V_{A,B}^{(a,b)} = S_a(A)S_b(B) + (a|A|)(b|B|).$$ 

We will write $V_{A,B}^{(a,b)}$ simply as $V_{A,B}$ for fixed elements $A$ and $B$ when no confusion can arise. Since $A$ and $B$ are normal operators then $V_{A,B}^* = V_{B,A}$. Also, $V_{A,A} = R_a(A) + a^2|A|^2 = R_a(A) + I - R_a(A) = I$. 

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The proof of the following proposition is similar in spirit to [2], Lemma 5.3.

**Lemma 3.5.** Let $A, B \in \mathcal{M}$ and let $x \in H$. Then the following assertions hold.

(a) $||V_{A,B}(x)||^2 - ||x||^2 \leq ||x||^2 d^{[2]}_{(a,b)}(A, B)$;
(b) $||V_{A,B}(x)||^2 \geq (1 - (d^{[2]}_{(a,b)}(A, B))^2)||x||^2$;
(c) If $d^{[2]}_{(a,b)}(A, B) < 1$, then $V_{A,B}$ is invertible.

**Example 3.6.** Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. Let $\varphi: X \to X$ be a non-singular measurable point transformation, which means the measure $\mu \circ \varphi^{-1}$, defined by $\mu \circ \varphi^{-1}(B) = \mu(\varphi^{-1}(B))$ for all $B \in \Sigma$, is absolutely continuous with respect to $\mu$ (we write $\mu \circ \varphi^{-1} \ll \mu$). It follows that $\mu \circ \varphi^{-n} \ll \mu$ for every $n \in \mathbb{N}$. Then by Radon-Nikodym theorem there exists a unique non-negative $\Sigma$-measurable function $h_n$ on $X$ with $h_n = d\mu \circ \varphi^{-n}/d\mu$. Put $h_1 = h$. Now, let $C_\varphi$ defined by $C_\varphi(f) = f \circ \varphi$ be a composition operator on $L^2(\Sigma)$. Note that $C_\varphi \in B(L^2(\Sigma))$ if and only if $h \in L^\infty(\Sigma)$ and in this case $||C_\varphi|| = ||h||^{1/2}\{2\}$. Also it is a classical fact that $C_\varphi \in B(L^2(\Sigma))$ is normal if and only if $\varphi^{-1}(\Sigma) = \Sigma$ and $h \circ \varphi = h$ (see [10]). Let $\mathcal{M} = \{C_\varphi \in B(L^2(\Sigma)): \ C_\varphi \text{ is normal}\}$. Let $C_\varphi \in B(L^2(\Sigma))$ and $f \in L^2(\Sigma)$. Then we have

$$\langle C_\varphi^n C_\varphi^n f, f \rangle = \langle C_\varphi^n f, C_\varphi^n f \rangle = ||C_\varphi^n f||^2 = ||C_\varphi f||^2$$

$$= ||M_{\sqrt{h_n}} f||^2 = \langle M_{\sqrt{h_n}} f, M_{\sqrt{h_n}} f \rangle = \langle M_{h_n} f, f \rangle,$$

where $M_{h_n}$ is the multiplication operator. So, $C_\varphi^n C_\varphi^n = M_{h_n}$. In particular, if $C_\varphi \in \mathcal{M}$, then $C_\varphi^n C_\varphi^n = (C_\varphi^n C_\varphi^n)^n = (M_{h})^n = M_{h^n}$, and so $h_n = h^n$ for each $n \in \mathbb{N}$.

Let $0 < a < ||h||^{-1/2} = ||C_\varphi||^{1/2} = r(C_\varphi)^{-1}$. Then

$$K_a(C_\varphi) = \sum_{n=0}^{\infty} a^{2n} C_\varphi^n C_\varphi^n = \sum_{n=0}^{\infty} M_{a^{2n}h^n} = (I - M_{a^2h})^{-1}.$$  

Hence

$$R_a(C_\varphi) = K_a(C_\varphi)^{-1} = I - M_{a^2h}, \quad S_a(C_\varphi) = R_a\sqrt{C_\varphi} = M_{\sqrt{1 - a^2h}},$$

$$\langle \tilde{C}_\varphi \rangle_a = a|C_\varphi| (I + S_a(C_\varphi))^{-1} = M_{\sqrt{a^2h/(1 + \sqrt{1 - a^2h})}}.$$

Now, for $i = 1, 2$ let $C_{\varphi_i} \in \mathcal{M}$ and $h_i = (d\mu \circ \varphi_i^{-1})/d\mu$. Then we have

$$V_{C_{\varphi_1}, C_{\varphi_2}} = S_a(C_{\varphi_1}) S_b(C_{\varphi_2}) + (a|C_{\varphi_1}|)(b|C_{\varphi_2}|) = M_{\sqrt{(1-a^2h_1)(1-b^2h_2)+\sqrt{a^2b^2h_1h_2}}}.$$  

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