ENTROPY SOLUTIONS TO PARABOLIC EQUATIONS IN MUSIELAK FRAMEWORK INVOLVING NON COERCIVITY TERM IN DIVERGENCE FORM

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Abstract. We prove the existence of solutions to nonlinear parabolic problems of the following type:

\[
\begin{align*}
\frac{\partial b(u)}{\partial t} + A(u) &= f + \text{div}(\Theta(x; t; u)) \quad \text{in } Q, \\
u(x; t) &= 0 \quad \text{on } \partial \Omega \times [0; T], \\
b(u)(t = 0) &= b(u_0) \quad \text{on } \Omega,
\end{align*}
\]

where \( b: \mathbb{R} \to \mathbb{R} \) is a strictly increasing function of class \( C^1 \), the term

\[ A(u) = -\text{div}(a(x, t, u, \nabla u)) \]

is an operator of Leray-Lions type which satisfies the classical Leray-Lions assumptions of Musielak type, \( \Theta: \Omega \times [0; T] \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory, noncoercive function which satisfies the following condition: \( \sup_{|s| \leq k} |\Theta(\cdot, \cdot; s)| \in E_{\psi}(Q) \) for all \( k > 0 \), where \( \psi \) is the Musielak complementary function of \( \Theta \), and the second term \( f \) belongs to \( L^1(Q) \).

Keywords: inhomogeneous Musielak-Orlicz-Sobolev space; parabolic problems; Galerkin method

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1. Introduction

Our aim is to prove the existence of solutions $u$ to the following nonlinear parabolic problem:

$$
\begin{aligned}
& \frac{\partial b(u)}{\partial t} + A(u) = f + \text{div}(\Theta(x, t, u)) \quad \text{in } Q,
& u(x, t) = 0 \quad \text{on } \partial \Omega \times [0, T],
& b(u)(t = 0) = b(u_0) \quad \text{on } \Omega,
\end{aligned}
$$

where $\Omega$ is an open subset $\mathbb{R}^N$ which satisfies the segment property and $Q = \Omega \times [0, T]$, $T > 0$, $b: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function of class $C^1$ with $b(0) = 0$ and $\lim_{t \to \pm \infty} b'(t) = l < \infty$, $A(u) = -\text{div}(a(x, t, u, \nabla u))$ is a Leray-Lions operator defined on $D(A) \subset W^{1,x}_{0,L} \phi(Q)$ into its dual satisfying some conditions in Section 3, $\phi$ is Musielak function and $W^{1,x}_{0,L} \phi(Q)$ is the Musielak space defined in Section 2, $f \in L^1(Q)$ and $\Theta: \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a noncoercive function which satisfies the following condition: $\sup_{|s| \leq k} |\Theta(\cdot, \cdot, s)| \in E_\psi(Q)$ for all $k > 0$, where $\psi$ is the complementary function of $\phi$ and $E_\psi(Q)$ is a Musielak space defined in Section 2.

Under our assumptions, the above problem does not admit, in general, a weak solution since the field $a(x, t, u, \nabla u)$ does not belong to $(L^1_{\text{loc}}(Q))^N$ in general. To overcome this difficulty we use in this paper the framework of entropy solutions. This notion was introduced by Benilan et al. [9] for the study of nonlinear elliptic problems.

In the classical Sobolev spaces, Aberqi et al. in [1] have proved the existence of renormalized solutions (1.1) in the case where $b(u) \equiv b(x, u)$ and $\Theta$ satisfies a growth condition (for the definition of this notion of solution see [1], [20]), Redwane in [19] has proved the existence of renormalized solutions of (1.1), where $\Theta(x, t, u) = \Theta(u)$.

In the Sobolev variable exponent setting, Azroul, Benboubker, Redwane, and Yazough [6] have proved the existence result of renormalized solutions to a class of nonlinear parabolic equations without sign condition involving nonstandard growth in the particular case, where $\text{div}(\Theta(x, t, u)) = H(x, t, u, \nabla u)$ and in the elliptic case (see [8]).

In Orlicz framework, Redwane in [20] has proved the existence of renormalized solutions of (1.1), where $b(u) \equiv b(x, u)$ and $\Theta(x, t, u) = \Theta(u)$, Hadj Nassar, Moussa and Rhoudaf in [16] have studied the existence of renormalized solutions of (1.1) in $W^{1,x}M(Q)$, where $b(u) \equiv b(x, u)$ and $\Theta$ satisfies $|\Theta(x, u)| \leq P^{-1}P(|u|)$, where $P$ and $P$ are two complementary Orlicz functions with $P \ll M$. See also [7], [13], and [14] for related topics. For some existing results for strongly nonlinear elliptic and parabolic equations in Musielak-Orlicz-Sobolev spaces see [2], [3], [4], [5], [21].
This research is divided into several parts. In Section 2 we recall some important definitions and results of Musielak-Orlicz-Sobolev spaces. We introduce the assumptions that allow us to demonstrate our result in Section 3. Section 4 contains some important and useful lemmas to prove our main result. In Section 5 we prove the main result of this paper (Theorem 5.1) concerning the existence of solutions.

2. Preliminary

2.1. Musielak-Orlicz-Sobolev spaces. Let \( \Omega \) be an open set in \( \mathbb{R}^N \) and let \( \varphi \) be a real-valued function defined in \( \Omega \times \mathbb{R}_+ \), and satisfying the following conditions:

(a) \( \varphi(x, \cdot) \) is an N-function (convex, increasing, continuous, \( \varphi(x, 0) = 0 \), \( \varphi(x, t) > 0 \) for all \( t > 0 \)), \( \lim_{t \to 0} \sup_{x \in \Omega} \varphi(x, t)t^{-1} = 0 \), \( \lim_{t \to \infty} \inf_{x \in \Omega} \varphi(x, t)t^{-1} = \infty \).

(b) \( \varphi(\cdot, t) \) is a measurable function.

A function \( \varphi \), which satisfies conditions (a) and (b) is called Musielak-Orlicz function.

For a Musielak-Orlicz function \( \varphi \) we put \( \varphi_x(t) = \varphi(x, t) \) and we associate its nonnegative reciprocal function \( \varphi_x^{-1} \) with respect to \( t \), that is

\[
\varphi_x^{-1}(\varphi(x, t)) = \varphi(x, \varphi_x^{-1}(t)) = t.
\]

The Musielak-Orlicz function \( \varphi \) is said to satisfy the \( \Delta_2 \)-condition if for some \( k > 0 \) and a nonnegative function \( h \) integrable in \( \Omega \) we have

\[
\varphi(x, 2t) \leq k \varphi(x, t) + h(x) \quad \forall x \in \Omega \text{ and } t \geq 0.
\]

If (2.1) holds only for \( t \geq t_0 > 0 \), then \( \varphi \) is said to satisfy \( \Delta_2 \) near infinity.

Let \( \varphi \) and \( \gamma \) be two Musielak-Orlicz functions. We say that \( \varphi \) dominates \( \gamma \), and we write \( \gamma \prec \varphi \), near infinity (or globally) if there exist two positive constants \( c \) and \( t_0 \) such that for almost all \( x \in \Omega \)

\[
\gamma(x, t) \leq \varphi(x, ct) \quad \forall t \geq t_0, \quad \text{(or } \forall t \geq 0, \text{ i.e. } t_0 = 0\).
\]

We say that \( \gamma \) grows essentially less rapidly than \( \varphi \) at 0 (or near infinity), and we write \( \gamma \ll \varphi \), if for every positive constant \( c \) we have

\[
\lim_{t \to 0} \left( \sup_{x \in \Omega} \frac{\gamma(x, ct)}{\varphi(x, t)} \right) = 0 \quad \text{and } \lim_{t \to \infty} \left( \sup_{x \in \Omega} \frac{\gamma(x, ct)}{\varphi(x, t)} \right) = 0.
\]

Remark 2.1 ([11]). If \( \gamma \ll \varphi \) near infinity, then for all \( \varepsilon > 0 \) there exists \( k(\varepsilon) > 0 \) such that for almost all \( x \in \Omega \) we have

\[
\gamma(x, t) \leq k(\varepsilon) \varphi(x, \varepsilon t) \quad \forall t \geq 0.
\]
We define the functional
\[ \varphi_{\varphi, \Omega}(u) = \int_{\Omega} \varphi(x, |u(x)|) \, dx, \]
where \( u: \Omega \to \mathbb{R} \) is a Lebesgue measurable function. In the following, the measurability of function \( u: \Omega \to \mathbb{R} \) means the Lebesgue measurability. The set
\[ K_\varphi(\Omega) = \{ u: \Omega \to \mathbb{R} \text{ measurable: } \varphi_{\varphi, \Omega}(u) < \infty \}, \]
is called the generalized Orlicz class.

The Musielak-Orlicz space (or the generalized Orlicz space) \( L_\varphi(\Omega) \) is the vector space generated by \( K_\varphi(\Omega) \), that is, \( L_\varphi(\Omega) \) is the smallest linear space containing the set \( K_\varphi(\Omega) \). Equivalently,
\[ L_\varphi(\Omega) = \{ u: \Omega \to \mathbb{R} \text{ measurable: } \varphi_{\varphi, \Omega}\left(\frac{|u(x)|}{\lambda}\right) < \infty \text{ for some } \lambda > 0 \}. \]

We define the Musielak-Orlicz function complementary to \( \varphi \) in the sense of Young with respect to the variable \( s \) as
\[ \psi(x, s) = \sup_{t \geq 0} \{ st - \varphi(x, t) \}. \]

We define in the space \( L_\varphi(\Omega) \) the two norms:
\[ \| u \|_{\varphi, \Omega} = \inf \left\{ \lambda > 0: \int_{\Omega} \varphi(x, \frac{|u(x)|}{\lambda}) \, dx \leq 1 \right\}, \]
which is called the Luxemburg norm and the so called Orlicz norm defined as
\[ \| u \|_{\varphi, \Omega} = \sup_{\| v \|_{\psi, \Omega} \leq 1} \int_{\Omega} |u(x)v(x)| \, dx, \]
where \( \psi \) is the Musielak-Orlicz function complementary to \( \varphi \) and \( \| v \|_{\psi, \Omega} \) is the Luxemburg norm of \( v \) associate to the Musielak function \( \psi \). These two norms are equivalent (see [18]).

The closure in \( L_\varphi(\Omega) \) of the bounded measurable functions with compact support in \( \Omega \) is denoted by \( E_\varphi(\Omega) \). It is a separable space.

We say that a sequence of functions \( u_n \in L_\varphi(\Omega) \) is modular convergent to \( u \in L_\varphi(\Omega) \) if there exists a constant \( \lambda > 0 \) such that
\[ \lim_{n \to \infty} \varphi_{\varphi, \Omega}\left(\frac{u_n - u}{\lambda}\right) = 0. \]
For any fixed nonnegative integer $m$ we define

$$W^m L^\varphi(\Omega) = \left\{ u \in L^\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in L^\varphi(\Omega) \right\}$$

and

$$W^m E^\varphi(\Omega) = \left\{ u \in E^\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in E^\varphi(\Omega) \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers $\alpha_i$, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives. The space $W^m L^\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\mathcal{G}_{\varphi, \Omega}(u) = \sum_{|\alpha| \leq m} \varphi_{\varphi}(D^\alpha u) \text{ and } \|u\|_{m, \varphi, \Omega} = \inf \left\{ \lambda > 0 : \mathcal{G}_{\varphi, \Omega}\left( \frac{u}{\lambda} \right) \leq 1 \right\}.$$

For $u \in W^m L^\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^m L^\varphi(\Omega)$, respectively, and the pair $(W^m L^\varphi(\Omega), \|\cdot\|_{m, \varphi, \Omega})$ is a Banach space if $\varphi$ satisfies the following condition (see [18]):

$$\exists c > 0 : \inf_{x \in \Omega} \varphi(x, 1) \geq c.$$

The space $W^m L^\varphi(\Omega)$ will always be identified to a subspace of the product

$$\prod_{|\alpha| \leq m} L^\varphi(\Omega) = \Pi L^\varphi;$$

this subspace is $\sigma(\Pi L^\varphi, \Pi E^\psi)$ closed.

We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions with compact support in $\Omega$ and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(\mathbb{R}^N)$ on $\Omega$.

Let $W^0_L^\varphi(\Omega)$ be the $\sigma(\Pi L^\varphi, \Pi E^\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L^\varphi(\Omega)$.

Let $W^m E^\varphi(\Omega)$ be the space of functions $u$ such that $u$ and its distributional derivatives up to order $m$ lie in $E^\varphi(\Omega)$, and $W^0_m E^\varphi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L^\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L^\psi(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L^\psi(\Omega) \right\}$$

and

$$W^{-m} E^\psi(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E^\psi(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^m L^\varphi(\Omega)$ is modular convergent to $u \in W^m L^\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n \to \infty} \mathcal{G}_{\varphi, \Omega}\left( \frac{u_n - u}{k} \right) = 0.$$
For \( \varphi \) and its complementary function \( \psi \) the following inequality is called the Young inequality (see [18]):

\[
(2.4) \quad ts \leq \varphi(x,t) + \psi(x,s) \quad \forall t, s \geq 0, \ x \in \Omega.
\]

This inequality implies that

\[
(2.5) \quad \|u\|_{\varphi,\Omega} \leq g_{\varphi,\Omega}(u) + 1.
\]

In \( L_\varphi(\Omega) \) we have the relation between the norm and the modular:

\[
(2.6) \quad \|u\|_{\varphi,\Omega} \leq g_{\varphi,\Omega}(u) \quad \text{if} \quad \|u\|_{\varphi,\Omega} > 1,
\]

\[
(2.7) \quad \|u\|_{\varphi,\Omega} \geq g_{\varphi,\Omega}(u) \quad \text{if} \quad \|u\|_{\varphi,\Omega} \leq 1.
\]

For two complementary Musielak-Orlicz functions \( \varphi \) and \( \psi \) let \( u \in L_\varphi(\Omega) \) and \( v \in L_\psi(\Omega) \). Then we have the Hölder inequality (see [18])

\[
(2.8) \quad \left| \int_\Omega u(x)v(x) \, dx \right| \leq \|u\|_{\varphi,\Omega} \|v\|_{\psi,\Omega}.
\]

**Definition 2.1.** We say that \( \Omega \subset \mathbb{R}^N \) satisfies the segment propriety if there exists a locally finite open covering \( \{O\} \) of \( \partial\Omega \) and corresponding vectors \( \{y_i\} \) such that for \( x \in \Omega \cap O \) and \( 0 < t < 1 \) one has \( x + ty_i \in \Omega \).

### 2.2. Inhomogeneous Musielak-Orlicz-Sobolev spaces.

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \), \( T > 0 \) and set \( Q = \Omega \times [0,T] \). Let \( m \geq 1 \) be an integer and let \( \varphi \) and \( \psi \) be two complementary Musielak-Orlicz functions. For each \( \alpha \in \mathbb{N}^N \) denote by \( D^\alpha_x \) the distributional derivative on \( Q \) of order \( \alpha \) with respect to \( x \in \mathbb{R}^N \). The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as

\[
W^{m,x}_\varphi(Q) = \{ u \in L_\varphi(Q) : D^\alpha_x u \in L_\varphi(Q) \ \forall |\alpha| \leq m \}
\]

and

\[
W^{m,x}_\varphi(Q) = \{ u \in E_\varphi(Q) : D^\alpha_x u \in E_\varphi(Q) \ \forall |\alpha| \leq m \}.
\]

This second space is a subspace of the first one, and both are Banach spaces with the norm

\[
\|u\|_{m,x} = \sum_{|\alpha| \leq m} \|D^\alpha_x u\|_{\varphi,Q}.
\]

These spaces constitute a complementary system since \( \Omega \) satisfies the segment property. These spaces are considered subspaces of the product space \( \Pi L_\varphi(Q) \), which
have as many copies as there is \( \alpha \) order derivatives, \( |\alpha| \leq m \). We shall also consider the weak topologies \( \sigma(\Pi L_\varphi, \Pi E_\psi) \) and \( \sigma(\Pi L_\varphi, \Pi L_\psi) \).

If \( u \in W^{m,x}L_\varphi(Q) \), then the function \( t \to u(t) = u(\cdot, t) \) is defined on \([0, T]\) with values in \( W^mL_\varphi(\Omega) \). If \( u \in W^{m,x}E_\varphi(Q) \), then \( u \in W^mE_\varphi(\Omega) \) and it is strongly measurable.

Furthermore, the imbedding \( W^{m,x}E_\varphi(Q) \subset L^1(0, T, W^mE_\varphi(\Omega)) \) holds. The space \( W^{m,x}L_\varphi(Q) \) is not in general separable, for \( u \in W^{m,x}L_\varphi(Q) \) we cannot conclude that the function \( u(t) \) is measurable on \([0, T]\).

However, the scalar function \( t \to \|u(t)\|_{\varphi, \Omega} \in L^1(0, T) \). The space \( W^{m,x}_0E_\varphi(Q) \) is defined as the norm closure of \( D(Q) \) in \( W^{m,x}E_\varphi(Q) \). We can easily show as in \([15]\) that when \( \Omega \) has the segment property, then each element \( u \) of the closure of \( D(Q) \) with respect to the weak* topology \( \sigma(\Pi L_\varphi, \Pi E_\psi) \) is a limit in \( W^{m,x}L_\varphi(Q) \) of some subsequence \( (v_j) \in D(Q) \) for the modular convergence, i.e. there exists \( \lambda > 0 \) such that for all \( |\alpha| \leq m \)

\[
\int_Q \varphi \left( x, \frac{D^\alpha_x v_j - D^\alpha_x u}{\lambda} \right) \, dx \, dt \to 0, \quad \text{as } j \to \infty,
\]

which gives that \( (v_j) \) converges to \( u \) in \( W^{m,x}L_\varphi(Q) \) for the weak topology \( \sigma(\Pi L_\varphi, \Pi L_\psi) \).

Consequently,

\[
D(Q)^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = D(Q)^{\sigma(\Pi L_\varphi, \Pi L_\psi)}.
\]

The space of functions satisfying such a property will be denoted by \( W^{m,x}_0L_\varphi(Q) \). Furthermore, \( W^{m,x}_0E_\varphi(Q) = W^{m,x}_0L_\varphi(Q) \cap \Pi E_\varphi(Q) \). Thus, both sides of the last inequality are equivalent norms on \( W^{m,x}_0L_\varphi(Q) \). We then have the following complementary system:

\[
\begin{pmatrix}
W^{m,x}_0L_\varphi(Q) \\
W^{m,x}_0E_\varphi(Q)
\end{pmatrix}
F =
\begin{pmatrix}
F_0
\end{pmatrix},
\]

where \( F \) states for the dual space of \( W^{m,x}_0E_\varphi(Q) \) and can be defined, except for an isomorphism, as the quotient of \( \Pi L_\psi \) by the polar set \( W^{m,x}_0E_\varphi(Q)^\perp \). It will be denoted by \( F = W^{-m,x}L_\psi(Q) \), where

\[
W^{-m,x}L_\psi(Q) = \left\{ f = \sum_{|\alpha| \leq m} D^\alpha_x f_\alpha \quad \text{with} \quad f_\alpha \in L_\psi(Q) \right\}.
\]

This space will be equipped with the usual quotient norm

\[
\|u\|_F = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{\psi, \Omega},
\]

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where the infimum is taken over all possible decompositions

\[ f = \sum_{|\alpha| \leq m} D_\alpha^x f_\alpha, \quad f_\alpha \in L_\psi(Q). \]

The space \( F_0 \) is then given by

\[ F_0 = \left\{ f : f = \sum_{|\alpha| \leq m} D_\alpha^x f_\alpha, \quad f_\alpha \in E_\psi(Q) \right\}, \]

and is denoted by \( W^{-m,x}E_\psi(Q) \), see [4].

3. Essential assumptions

Let \( \varphi \) be a Musielak-Orlicz function which decreases with respect to one of the coordinates of \( x \). We denote by \( \psi \) the Musielak complementary function of \( \varphi \). Throughout this paper, we assume that the following assumptions hold true:

\[ b : \mathbb{R} \to \mathbb{R} \text{ is strictly increasing } C^1 \text{ function} \]

with \( b(0) = 0 \) and \( \lim_{t \to \pm\infty} b'(t) = l < \infty \),

\[ a : \Omega \times ]0, T[ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Carathéodory function satisfying the following conditions:} \]

for almost every \((x, t) \in \Omega \times ]0, T[\) and all \(s \in \mathbb{R}, \xi \neq \xi^* \in \mathbb{R}^N;\)

\[ \left| a(x, t, s, \xi) \right| \leq \beta (h_1(x, t) + \psi^{-1}_x \gamma(x, \nu |s|) + \psi^{-1}_x \varphi(x, \nu |\xi|)), \]

\[ (a(x, t, s, \xi) - a(x, t, s, \xi^*)) (\xi - \xi^*) > 0, \]

\[ a(x, t, s, \xi) \xi \geq \alpha \varphi \left( x, \frac{|\xi|}{\lambda} \right) \]

with \( h_1(x, t) \in E_\psi(Q), h_1 \geq 0, \alpha, \beta \) and \( \nu > 0 \).

Furthermore, let \( \Theta : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}^N \) be a Carathéodory function such that

\[ \sup_{|s| \leq k} |\Theta(\cdot, \cdot, s)| \in E_\psi(Q) \quad \forall \ k > 0 \]

and

\[ f \in L^1(Q). \]
We consider the following parabolic initial-boundary problem:

\[
\begin{align*}
\frac{\partial b(u)}{\partial t} + A(u) &= f + \text{div}(\Theta(x,t,u)) & \text{in } Q, \\
u(x,t) &= 0 & \text{on } \partial \Omega \times [0,T], \\
u(x,0) &= u_0(x) & \text{on } \Omega,
\end{align*}
\]

where \(u_0\) is a given function in \(L^1(\Omega)\).

4. SOME TECHNICAL LEMMAS

**Lemma 4.1 ([10]).** Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^N\) and let \(\varphi\) and \(\psi\) be two complementary Musielak-Orlicz functions which satisfy the following conditions:

(i) There exists a constant \(c > 0\) such that \(\inf_{x \in \Omega} \varphi(x,1) \geq c\).

(ii) There exists a constant \(A > 0\) such that for all \(x,y \in \Omega\) with \(|x - y| \leq \frac{1}{2}\) we have

\[
\frac{\varphi(x,t)}{\varphi(y,t)} \leq \frac{t^A}{-\log |x-y|} \quad \forall t \geq 1.
\]

(iii)

\[
(4.1)
\]

(iv) If \(D \subset \Omega\) is a bounded measurable set, then \(\int_D \varphi(x,1) \, dx < \infty\).

Consequence, the action of a distribution \(S\) in \(W^{-1}L_\psi(\Omega)\) on an element \(u\) of \(W^1_0L_\psi(\Omega)\) is well defined. It will be denoted by \(\langle S, u \rangle\).

**Truncation operator.** For \(k > 0\) we define the truncation at height \(k\) as

\[
T_k(s) = \begin{cases} 
  s & \text{if } |s| \leq k, \\
  k \left| \frac{s}{|s|} \right| & \text{if } |s| > k.
\end{cases}
\]

In the following lemma we give the modular Poincaré’s inequality in Musielak-Orlicz spaces.
Lemma 4.2 ([12]). Under the assumptions of Lemma 4.1 and by assuming that \( \varphi(x,t) \) decreases with respect to one of the coordinates of \( x \), there exists a constant \( c > 0 \), which depends only on \( \Omega \), such that

\[
(4.4) \quad \int_{\Omega} \varphi(x,|u(x)|) \, dx \leq \int_{\Omega} \varphi(x,c|\nabla u(x)|) \, dx \quad \forall u \in W^{1}_{0}L_{\varphi}(\Omega).
\]

Remark 4.1. The following function is an example of a function that satisfies the previous lemma:

\[
\varphi(x,t) = t^{|x|^2_2 - |x|^2_1} \log(1 + t).
\]

Lemma 4.3 (The Nemytskii operator [5]). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with finite measure and let \( \varphi \) and \( \psi \) be two Musielak-Orlicz functions. Let \( f : \Omega \times \mathbb{R}^p \to \mathbb{R}^q \) be a Carathéodory function such that for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R}^p \)

\[
(4.5) \quad |f(x,s)| \leq c(x) + k_1 \psi^{-1}_x \varphi(x,k_2|s|),
\]

where \( k_1 \) and \( k_2 \) are real positive constants and \( c(\cdot) \in E(\Omega) \). Then the Nemytskii operator \( N_f \) defined by \( N_f(u)(x) = f(x,u(x)) \) is continuous from

\[
\left( \mathcal{P}(E_{\varphi}(\Omega),\frac{1}{k_2}) \right)^p = \prod \left\{ u \in L_{\varphi}(\Omega) : d(u,E_{\varphi}(\Omega)) < \frac{1}{k_2} \right\}
\]

into \( (L_{\psi}(\Omega))^q \) for the modular convergence.

Furthermore, if \( c(\cdot) \in E_{\gamma}(\Omega) \) and \( \gamma \ll \psi \), then \( N_f \) is strongly continuous from \( (\mathcal{P}(E_{\varphi}(\Omega),k_2^{-1}))^p \) to \( (E_{\gamma}(\Omega))^q \).

Lemma 4.4 ([12]). Assume that (3.2)–(3.4) are satisfied and let \( (z_n)_n \) be a sequence in \( W^{1,x}_{0}L_{\varphi}(\Omega) \) such that

(i) \( z_n \rightharpoonup z \) in \( W^{1,x}_{0}L_{\varphi}(\Omega) \) for \( \sigma(\Pi L_{\varphi},\Pi E_{\psi}) \),

(ii) \( (a(\cdot,t,z_n,\nabla z_n))_n \) is bounded in \( (L_{\psi}(\Omega))^N \),

(iii) \( \int_{\Omega}(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s))(|\nabla z_n - \nabla z\chi_s|) \, dx \to 0 \) as \( n,s \to \infty \), where \( \chi_s \) is the characteristic function of \( \Omega_s = \{ x \in \Omega : |\nabla z| \leq s \} \).

Then we have

\[
z_n \to z \text{ for the modular convergence in } W^{1}_{0}L_{\varphi}(\Omega).
\]

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5. Main result

We shall prove the following existence theorem.

**Theorem 5.1.** Let $\varphi$ and $\psi$ be two complementary Musielak-Orlicz functions satisfying the assumptions of Lemma 4.2, we assume that (3.1)-(3.6) hold true. Then problem (P) has at least one entropy solution $u \in D(A) \cap W^{1,x}_0 L_{\varphi}(Q) \cap \mathcal{C}([0,T],L^2(\Omega))$ in the following sense:

\[
\begin{aligned}
T_k(u) & \in W^{1,x}_0 L_{\varphi}(Q) \quad \forall k > 0, \\
\left\langle \frac{\partial b(u)}{\partial t}, T_k(u-v) \right\rangle & + \int_Q a(x,t,u,\nabla u) \nabla T_k(u-v) \, dx \, dt \\
\leq & \int_Q fT_k(u-v) \, dx \, dt + \int_Q \Theta(x,t,u)\nabla T_k(u-v) \, dx \, dt \\
\forall v & \in W^{1,x}_0 L_{\varphi}(Q) \cap L^\infty(Q) \text{ such that } \frac{\partial v}{\partial t} \in W^{-1,x} L_{\psi}(Q) + L^1(Q).
\end{aligned}
\]

**Proof.** We will use the Galerkin method due to Landes and Mustonen (see [17]), we choose a sequence $\{w_1, w_2, \ldots \}$ in $D(\Omega)$ such that $\bigcup_{p=0}^{\infty} V_p$ with $V_p = \{w_1, \ldots, w_p\}$ is dense in $H^m_0(\Omega)$ with $m$ large enough so that $H^m_0(\Omega)$ is continuously embedded in $C^1(\bar{\Omega})$. For every $v \in H^m_0(\Omega)$ there exists a sequence $(v_j) \subset \bigcup_{p=0}^{\infty} V_p$ such that $v_n \to v$ in $H^m_0(\Omega)$ and in $C^1(\bar{\Omega})$.

We denote further $V_p = \mathcal{C}([0,T], V_p)$. It is easy to see that the closure of $\bigcup_{p=0}^{\infty} V_p$ with respect to the norm

$$
\|v\|_{C^{1,0}(Q)} = \sup_{|\alpha| \leq 1} \{ |D^\alpha_x v(x,t)| : (x,t) \in Q \}
$$

contains $D(Q)$. This implies that for any $f \in W^{-1,x} E_{\psi}(Q)$ there exists a sequence $(f_n) \subset \bigcup_{p=0}^{\infty} V_p$ such that $f_n \to f$ strongly in $W^{-1,x} E_{\psi}(Q)$.

Indeed, let $\varepsilon > 0$ be given. Write $f = \sum_{|\alpha| \leq 1} D^\alpha_x f_\alpha$. There exists $g_\alpha \in \mathcal{D}(Q)$ such that $\|f_\alpha - g_\alpha\|_{\psi, Q} \leq \varepsilon(2N+2)^{-1}$. Moreover, by setting $g = \sum_{|\alpha| \leq 1} D^\alpha_x g_\alpha$, we see that $g \in \mathcal{D}(Q)$, and so there exists $v \in \bigcup_{p=0}^{\infty} V_p$ such that $\|g - v\|_{\infty, Q} \leq \varepsilon(2\text{meas}(Q))^{-1}$. We deduce that

$$
\|f - v\|_{W^{-1,x} L_{\psi}(Q)} \leq \sum_{|\alpha| \leq 1} \|f_\alpha - g_\alpha\|_{\psi, Q} + \|g - v\|_{\psi, Q} \leq \varepsilon.
$$
We devide the proof into six steps.

**Step 1: Approximate problem.** For \( n \in \mathbb{N} \) we define the following approximations:

\[
\begin{align*}
 b_n(r) &= T_n(b(r)) + \frac{r}{n} \quad \forall \, r \in \mathbb{R}, \\
 \Theta_n(x, t, s) &= \Theta(x, t, T_n(s)).
\end{align*}
\]

\((f_n)_n\) is a sequence in \( W^{-1} E_\psi(Q) \cap L^1(Q) \) such that

\[
 f_n \to f \quad \text{in} \quad L^1(Q) \quad \text{with} \quad \| f_n \|_{L^1(Q)} \leq \| f \|_{L^1(Q)},
\]

and \( u_{0n} \) is a sequence of \( D(\Omega) \) such that

\[
 b_n(u_{0n}) \to b(u_0) \quad \text{strongly in} \quad L^1(\Omega) \quad \text{with} \quad \| b_n(u_{0n}) \|_{L^1(\Omega)} \leq \| b(u_0) \|_{L^1(\Omega)}.
\]

We consider the approximate problem

\[
 (P_n) \quad \begin{cases} 
 u_n \in V_n, & \frac{\partial b_n(u_n)}{\partial t} \in L^1(0, T, V_n), \quad u_n(\cdot, 0) = u_{0n} \quad \text{a.e. in} \quad \Omega, \\
 \frac{\partial b_n(u_n)}{\partial t} - \text{div}(a(x, t, u_n, \nabla u_n)) = f_n + \text{div}(\Theta_n(x, t, u_n)).
\end{cases}
\]

There exists at least one solution \( u_n \) of \((P_n)\) (this solution \( u_n \) can be obtained from Galerkin solution (see [17]).

**Step 2: A priori estimates.** In this section we denote by \( c_i, i = 1, 2, \ldots \) constants not depending on \( k \) and \( n \).

For \( \tau \in [0, T] \), taking \( T_k(u_n) \chi_{[0, \tau]} \) as test function in \((P_n)\), we obtain

\[
 \int_{Q_\tau} \frac{\partial b_n(u_n)}{\partial t} T_k(u_n) \, dx \, dt + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n) \, dx \, dt \\
= \int_{Q_\tau} f_n T_k(u_n) \, dx \, dt + \int_{Q_\tau} \Theta_n(x, t, u_n) \nabla T_k(u_n) \, dx \, dt.
\]

We set

\[
 S^k_n(\sigma) = \int_0^\sigma b_n'(r) T_k(r) \, dr.
\]

Then we have

\[
 \int_{Q_\tau} \frac{\partial b_n(u_n)}{\partial t} T_k(u_n) \, dx \, dt = \int_{Q_\tau} \frac{\partial u_n}{\partial t} b_n'(u_n) T_k(u_n) \, dx \, dt \\
= \int_\Omega S^k_n(u_n(\tau)) \, dx - \int_\Omega S^k_n(u_{0n}) \, dx.
\]

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Hence, we have
\[
\int_\Omega S_n^k(u_n(\tau)) \, dx - \int_\Omega S_n^k(u_{0n}) \, dx + \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n) \, dx \, dt \\
= \int_Q f_n T_k(u_n) \, dx \, dt + \int_{Q_r} \Theta_n(x, t, u_n) \nabla T_k(u_n) \, dx \, dt.
\]

Due to the definition of $S_n^k$, (3.1) and (5.5), one has
\[
\int_\Omega S_n^k(u_{0n}) \, dx \leq k \int_\Omega |b_n(u_{0n})| \, dx \leq \|b(u_0)\|_{L^1(\Omega)}.
\]

Using (5.4) and (5.6), we obtain
\[
\int_\Omega S_n^k(u_n(\tau)) \, dx + \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n) \, dx \, dt \\
\leq k(\|f\|_{L^1(\Omega)} + \|b(u_0)\|_{L^1(\Omega)}) + \int_{Q_r} \Theta_n(x, t, u_n) \nabla T_k(u_n) \, dx \, dt \\
\leq c_1 k + \int_{Q_r} \Theta_n(x, t, u_n) \nabla T_k(u_n) \, dx \, dt.
\]

For $n \geq k$, condition (3.5) and Young’s inequality gives
\[
\int_{Q_r} \Theta_n(x, t, u_n) \nabla T_k(u_n) \, dx \, dt \leq \int_{Q_r} |\Theta(x, t, u_n)| \nabla T_k(u_n) \, dx \, dt \\
= \int_{Q_r} |\Theta_n(x, t, T_k(u_n))| \nabla T_k(u_n) \, dx \, dt \\
= \int_{Q_r} \sup_{\|s\| \leq k} |\Theta(x, t, s)| \nabla T_k(u_n) \, dx \, dt \\
\leq \int_{Q_r} \psi\left(x, c_\alpha \sup_{\|s\| \leq k} |\Theta(x, t, s)|\right) \, dx \, dt \\
+ \frac{\alpha}{2(\alpha + 1)} \int_{Q_r} \varphi(x, |\nabla T_k(u_n)|) \, dx \, dt \\
\leq r(k) + \frac{\alpha}{2(\alpha + 1)} \int_{Q_r} \varphi(x, |\nabla T_k(u_n)|) \, dx \, dt
\]
where $r(k) = \int_{Q_r} \psi\left(x, c_\alpha \sup_{\|s\| \leq k} |\Theta(x, t, s)|\right) \, dx \, dt$. Then by condition (3.4) and by combining (5.7) and (5.8), we get
\[
\int_\Omega S_n^k(u_n(\tau)) \, dx + \frac{2\alpha + 1}{2(\alpha + 1)} \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n) \, dx \, dt \leq c_1 k + r(k).
\]
Now, using the fact that \( S_n^k(u_n(\tau)) \geq 0 \), one has

\[
\int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt \leq \frac{2(\alpha + 1)}{2\alpha + 1}(c_1 k + r(k)).
\]

Then using (3.4), we have

\[
\int_Q \varphi(x, \frac{\nabla T_k(u_n)}{\lambda}) \, dx \, dt \leq \frac{2(\alpha + 1)(c_1 k + r(k))}{\alpha(2\alpha + 1)}.
\]

Using Lemma 4.2, we have that \((T_k(u_n))\) is bounded in \( W_0^{1, x}L_\varphi(Q) \), then there exists \( v_k \) such that

\[
\begin{cases}
T_k(u_n) \rightarrow v_k \quad \text{in} \quad W_0^{1, x}L_\varphi(Q) \quad \text{for} \quad \sigma(\Pi L_\varphi, \Pi E_\psi), \\
T_k(u_n) \rightarrow v_k \quad \text{strongly in} \quad E_\varphi(Q).
\end{cases}
\]

Therefore, we can assume that \((T_k(u_n))_n\) is a Cauchy sequence in measure in \( \Omega \). Then for all \( k > 0 \) and \( \delta, \varepsilon > 0 \) there exists \( n_0 = n_0(k, \delta, \varepsilon) \) such that

\[
\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \quad \forall \ m, n \geq n_0.
\]

It is easy to show that

\[
\inf_{x \in \Omega} \varphi(x, \frac{k}{\lambda c}) \text{meas}\{|u_n| > k\} = \int_{\{|u_n| > k\} \subset \Omega} \inf_{x \in \Omega} \varphi(x, \frac{k}{\lambda c}) \, dx \, dt
\]

\[
\leq \int_Q \varphi(x, \frac{|T_k(u_n)|}{\lambda c}) \, dx \, dt
\]

\[
\leq \int_Q \varphi(x, \frac{|\nabla T_k(u_n)|}{\lambda}) \, dx \, dt \quad \text{(using Lemma 4.2)}
\]

\[
\leq \frac{2(\alpha + 1)(c_1 k + r(k))}{\alpha(2\alpha + 1)} \quad \text{(using (5.11))},
\]

where this \( c \) is the constant of Lemma 4.2. Then, by using the definition of \( \varphi \),

\[
\text{meas}\{|u_n| > k\} \leq \frac{2(\alpha + 1)(c_1 k + r(k))}{\alpha(2\alpha + 1)} \inf_{x \in \Omega} \varphi(x, k/\lambda c) \rightarrow 0, \quad \text{as} \quad k \rightarrow \infty.
\]

Since for all \( \delta > 0 \),

\[
\text{meas}\{|u_n - u_m| > \delta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\}
\]

\[
+ \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.
\]
Using (5.14), we get for all $\varepsilon > 0$ there exists $k_0 > 0$ such that

\begin{equation}
\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3}, \quad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3} \quad \forall k \geq k_0(\varepsilon).
\end{equation}

Combining (5.13), (5.15) and (5.16), we obtain that for all $\delta, \varepsilon > 0$ there exists $n_0 = n_0(\delta, \varepsilon)$ such that

\begin{equation}
\text{meas}\{|u_m - u_n| > \delta\} \leq \varepsilon \quad \forall n, m \geq n_0.
\end{equation}

It follows that $(u_n)_n$ is a Cauchy sequence in measure. Then the there exists a function $u$ such that

\begin{equation}
\begin{cases}
T_k(u_n) \to T_k(u) \quad \text{in } W^1_0 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \\
T_k(u_n) \to T_k(u) \quad \text{strongly in } E_\varphi(\Omega).
\end{cases}
\end{equation}

**Step 3: Boundness of $(a(x, t, T_k(u_n)), \nabla T_k(u_n))_n$ in $(L_\psi(Q))^N$.** Let $w \in (E_\varphi(Q))^N$ be arbitrary such that $\|w\|_{\varphi, Q} = 1$. By (3.3) we have

\begin{equation}
(a(x, t, T_k(u_n)), \nabla T_k(u_n)) - a\left(x, t, T_k(u_n), \frac{w}{\nu}\right) \left(\nabla T_k(u_n) - \frac{w}{\nu}\right) > 0.
\end{equation}

Hence,

\begin{equation}
\int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \frac{w}{\nu} \, dx \, dt 
\leq \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt 
- \int_Q a\left(x, t, T_k(u_n), \frac{w}{\nu}\right) \left(\nabla T_k(u_n) - \frac{w}{\nu}\right) \, dx \, dt,
\end{equation}

and hence, using (5.10),

\begin{equation}
\int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt 
\leq \frac{2(\alpha + 1)(c_1 k + r(k))}{\alpha(2\alpha + 1)}.
\end{equation}

For $\mu$ large enough ($\mu > \beta$), using (3.2) we have

\begin{equation}
\begin{aligned}
\int_Q \psi_x \left(\frac{a(x, t, T_k(u_n), w\nu^{-1})}{3\mu}\right) \, dx \, dt 
\leq \int_Q \psi_x \left(\frac{\beta(h_1(x, t) + \psi^{-1}_x(\gamma(x, \nu|T_k(u_n)|))) + \psi^{-1}_x(\varphi(x, |w|))}{3\mu}\right) \, dx \, dt 
\leq \frac{\beta}{3\mu} \int_Q \psi_x \left(h_1(x, t) + \psi^{-1}_x(\gamma(x, \nu|T_k(u_n)|)) + \psi^{-1}_x(\varphi(x, |w|))\right) \, dx \, dt 
\leq \frac{\beta}{3\mu} \left(\int_Q \psi_x(h_1(x, t)) \, dx \, dt + \int_Q \gamma(x, \nu|T_k(u_n)|) \, dx \, dt + \int_Q \varphi(x, |w|) \, dx \, dt\right) 
\leq c_2(k).
\end{aligned}
\end{equation}
Now, since $\gamma$ grows essentially less rapidly than $\varphi$ near infinity and by using Remark 2.1, there exists $r'(k) > 0$ such that $\gamma(x, v_k) \leq r'(k) \varphi(x, 1)$ and so we have
\[
\int_Q \psi_x \left( \frac{a(x, t, T_k(u_n), w v^{-1})}{3\mu} \right) dx dt \\
\leq \frac{\beta}{3\mu} \left( \int_Q \psi_x(h_1(x, t)) dx dt + r'(k) \int_Q \varphi(x, 1) dx dt + \int_Q \varphi(x, |w|) dx dt \right).
\]
Hence $a(x, t, T_k(u_n), w v^{-1})$ is bounded in $(L_\psi(Q))^N$. This implies that the second term of the right-hand side of (5.18) is bounded, consequently, we obtain
\[
\int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) w dx dt \leq c_2(k) \quad \forall w \in (L^\sigma(Q))^N \text{ with } \|w\|_{\psi, Q} \leq 1.
\]
Hence, by the theorem of Banach Steinhaus, the sequence $(a(x, t, T_k(u_n), \nabla T_k(u_n)))_n$ remains bounded in $(L_\psi(Q))^N$, which implies that for all $k > 0$ there exists a function $l_k \in (L_\psi(Q))^N$ such that
\[
(5.20) \quad a(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup l_k \text{ weak star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi, \Pi E \varphi).
\]

Step 4: Modular convergence of the truncations. Since $T_k(u) \in W^{1,*}L_\varphi(Q)$, there exists a sequence $(v_j^k) \subset D(\Omega)$ such that $v_j^k \to T_k(u)$. For the sake of simplicity, we denote by $\varepsilon(n, j, \mu, s)$ any quantity (possible different) such that
\[
\lim_{s \to \infty} \lim_{\mu \to \infty} \lim_{j \to \infty} \lim_{n \to \infty} \varepsilon(n, j, \mu, s) = 0.
\]
If the quantity we consider does not depend on one of the parameters $n, j, \mu$ and $s$, we will omit the dependence on the corresponding parameter: as an example, $\varepsilon(n, j)$ is any quantity such that
\[
\lim_{j \to \infty} \lim_{n \to \infty} \varepsilon(n, j) = 0.
\]
We denote also by $\chi_{j, s}$ (or $\chi_s$) the characteristic functions of the set
\[
Q_{j, s} = \{(x, t) \in Q : |\nabla T_k(v_j^k)| \leq s\} \quad \text{or} \quad Q_s = \{(x, t) \in Q : |\nabla T_k(u)| \leq s\}.
\]
For $k > 0$, taking $T_k(u_n) - T_k(v_j^k)_\mu$ as a test function in $(P_n)$, we get
\[
(5.21) \quad \int_Q \frac{\partial b_{n}(u_n)}{\partial t}(T_k(u_n) - T_k(v_j^k)_\mu) dx dt \\
\quad + \int_Q a(x, t, u_n, \nabla u_n) \nabla(T_k(u_n) - T_k(v_j^k)_\mu) dx dt \\
= \int_Q f_n(T_k(u_n) - T_k(v_j^k)_\mu) dx dt \\
\quad + \int_Q \Theta_n(x, t, u_n) \nabla(T_k(u_n) - T_k(v_j^k)_\mu) dx dt.
\]
Firstly, for the first term of the left-hand side of (5.21) we get
\[
\int_Q \frac{\partial b_n(u_n)}{\partial t}(T_k(u_n) - T_k(v^k_j))_\mu \, dx \, dt
= \int_Q \frac{\partial b_n(u_n)}{\partial t}T_k(u_n) \, dx \, dt - \int_Q \frac{\partial b_n(u_n)}{\partial t}T_k(v^k_j)_\mu \, dx \, dt = I_1 + I_2.
\]
For $I_1$ we have
\[
I_1 = \int_\Omega B^k_n(u_n(T)) \, dx - \int_\Omega B^k_n(u_0) \, dx,
\]
where $B^k_n(s) = \int_0^s b'(r)T_k(r) \, dr$. Then, by passing to the limit as $n \to \infty$, we get
\[
(5.22)
I_1 = \int_\Omega B^k(u(T)) \, dx - \int_\Omega B^k(u_0) \, dx + \varepsilon(n),
\]
where $B^k(s) = \int_0^s b'(r)T_k(r) \, dr$. For $I_2$, by integration by parts with respect to $t$, we find
\[
I_2 = \int \Omega b_n(u_0n)T_k(v^k_j)_\mu(0) \, dx - \int \Omega b_n(u_n(T))T_k(v^k_j)_\mu(T) \, dx
+ \mu \int_Q (T_k(v^k_j) - T_k(v^k_j)_\mu)b_n(u_n) \, dx \, dt.
\]
Passing to the limit as $n, j \to \infty$ and since $u_n \to u$ a.e. in $Q$ and by Lebesgue dominated convergence theorem, we get
\[
(5.23)
I_2 = \int \Omega b(u_0)T_k(u)_\mu(0) \, dx - \int \Omega b(u(T))T_k(u)_\mu(T) \, dx
+ \mu \int Q (T_k(u) - T_k(u)_\mu)b(u) \, dx \, dt + \varepsilon(n, j)
= J_1 + J_2 + \varepsilon(n, j).
\]
For $J_2$ we have
\[
J_2 = \mu \int_Q (T_k(u) - T_k(u)_\mu)b(u) \, dx \, dt
= \mu \int_Q (T_k(u) - T_k(u)_\mu)(b(u) - b(T_k(u))) \, dx \, dt
+ \mu \int_Q (T_k(u) - T_k(u)_\mu)(b(T_k(u)) - b(T_k(u)_\mu)) \, dx \, dt
+ \mu \int_Q (T_k(u) - T_k(u)_\mu)b(T_k(u)_\mu) \, dx \, dt.
\]
Since $b$ is increasing, we get
\[
J_2 \geq \mu \int_Q (T_k(u) - T_k(u)_\mu)(b(u) - b(T_k(u))) \, dx \, dt \\
+ \mu \int_Q (T_k(u) - T_k(u)_\mu)b(T_k(u)_\mu) \, dx \, dt \\
\geq \mu \int_{u > k} (k - T_k(u)_\mu)(b(u) - b(k)) \, dx \, dt \\
+ \mu \int_{u < -k} (-k - T_k(u)_\mu)(b(u) - b(-k)) \, dx \, dt \\
+ \int_Q \frac{\partial T_k(u)_\mu}{\partial t} b(T_k(u)_\mu) \, dx \, dt.
\]

Since $b$ is increasing and $-k \leq T_k(u)_\mu \leq k$, we get
\[
(5.24) \quad J_2 \geq \int_\Omega \overline{B}(T_k(u(T)))_\mu \, dx - \int_\Omega \overline{B}(T_k(u_0)_\mu) \, dx,
\]
where $\overline{B}(s) = \int_0^s b(\tau) \, d\tau$.

Combining (5.22), (5.23) and (5.24), we get
\[
(5.25) \quad \int_Q \frac{\partial b_n(u_n)}{\partial t} (T_k(u_n) - T_k(v^k_{j})_{\mu}) \, dx \, dt \\
\geq \int_\Omega B^k(u(T)) \, dx - \int_\Omega B^k(u_0) \, dx + \int_\Omega b(u_0)T_k(u)_\mu(0) \, dx \\
- \int_\Omega b(u(T))T_k(u)_\mu(T) \, dx + \int_\Omega \overline{B}(T_k(u(T))_\mu) \, dx \\
- \int_\Omega \overline{B}(T_k(u_0)_\mu) \, dx + \varepsilon(n, j).
\]

Passing now to the limit for $\mu \to \infty$, we obtain
\[
(5.26) \quad \int_Q \frac{\partial b_n(u_n)}{\partial t} (T_k(u_n) - T_k(v^k_{j})_{\mu}) \, dx \, dt \\
\geq \int_\Omega B^k(u(T)) \, dx - \int_\Omega B^k(u_0) \, dx + \int_\Omega b(u_0)T_k(u_0) \, dx \\
- \int_\Omega b(u(T))T_k(u(T)) \, dx + \int_\Omega \overline{B}(T_k(u(T))) \, dx \\
- \int_\Omega \overline{B}(T_k(u_0)) \, dx + \varepsilon(n, j, \mu).
\]

Observe that for all $z \in \mathbb{R}$ we have
\[
\overline{B}(T_k(z)) = b(z)T_k(z) - B^k(z).
\]
Then, we deduce that
\[(5.27) \quad \int_Q \frac{\partial b_n(u_n)}{\partial t} (T_k(u_n) - T_k(v_j^k)_{\mu}) \, dx \, dt \geq \varepsilon(n,j,\mu).\]

Secondly, since \(f_n \to f\) strongly in \(L^1(Q)\) and \(T_k(u_n) - T_k(v_j^k)_{\mu}\) converges to \(T_k(u) - T_k(v_j^k)_{\mu}\) weakly star in \(L^\infty(Q)\), the first term of the right-hand side can be written as
\[
\int_Q f_n(T_k(u_n) - T_k(v_j^k)_{\mu}) \, dx \, dt = \int_Q f(T_k(u) - T_k(v_j^k)_{\mu}) \, dx \, dt + \varepsilon(n).
\]

Hence, by letting \(j\) and \(\mu\) to infinity, one has
\[(5.28) \quad \int_Q f_n(T_k(u_n) - T_k(v_j^k)_{\mu}) \, dx \, dt = \varepsilon(n,j,\mu).\]

Thirdly, for the last term of the right-hand side, one has for \(n \geq 2k\)
\[
\int_Q \Theta_n(x,t,u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_{\mu}) \, dx \, dt
\]
\[
= \int_Q \Theta_n(x,t,T_{2k}(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j^k)_{\mu}) \, dx \, dt
\]
\[
= \int_Q \Theta(x,t,T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j^k)_{\mu}) \, dx \, dt,
\]
and as \(\Theta(x,t,T_{2k}(u_n))\) converges strongly to \(\Theta(x,t,T_{2k}(u))\) in \(E_{\psi}(Q)\) and \(\nabla T_k(u_n) - \nabla T_k(v_j^k)_{\mu}\) converges weakly to \(\nabla T_k(u) - \nabla T_k(v_j^k)_{\mu}\) in \((L^\infty(\varphi(Q)))^N\), we get
\[
\int_Q \Theta_n(x,t,u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_{\mu}) \, dx \, dt
\]
\[
= \int_Q \Theta(x,t,T_{2k}(u))(\nabla T_k(u) - \nabla T_k(v_j^k)_{\mu}) \, dx \, dt + \varepsilon(n).
\]

Then by letting \(j\) and \(\mu\) to infinity, we get
\[(5.29) \quad \int_Q \Theta_n(x,t,u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_{\mu}) \, dx \, dt = \varepsilon(n,j,\mu).\]

Thus, by combining (5.21), (5.27), (5.28) and (5.29), we obtain
\[(5.30) \quad \int a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j^k)_{\mu}) \, dx \, dt \leq \varepsilon(n,j,\mu).\]
Splitting the first term of the last inequality on \( \{|u_n| \leq k\} \) and \( \{|u_n| > k\} \) and observing that \( \nabla(T_k(u_n) - T_k(v^k_j)) = 0 \) on \( \{|u_n| > 2k\} \), we get

\[
(5.31) \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v^k_j)) \, dx \, dt \\
\leq \int_{\{|u_n| > k\}} a(x, t, T_2k(u_n), \nabla T_2k(u_n))\nabla T_k(v^k_j) \, dx \, dt + \varepsilon(n, j, \mu).
\]

For the first term of the right-hand side of the last inequality we have

\[
\int_{\{|u_n| > k\}} a(x, t, T_2k(u_n), \nabla T_2k(u_n))\nabla T_k(v^k_j) \, dx \, dt \\
= \int_{\{|u| > k\}} l_2k \nabla T_k(v^k_j) \, dx \, dt + \varepsilon(n).
\]

Then by letting \( j \) and \( \mu \) to infinity, we get

\[
\int_{\{|u_n| > k\}} a(x, t, T_2k(u_n), \nabla T_2k(u_n))\nabla T_k(v^k_j) \, dx \, dt = \varepsilon(n, j, \mu).
\]

Then (5.31) becomes

\[
(5.32) \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v^k_j)) \, dx \, dt \leq \varepsilon(n, j, \mu).
\]

By a simple calculus, we get

\[
\int_Q (a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)) \\
\times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx \, dt \\
= \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v^k_j)) \, dx \, dt \\
- \int_Q (a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)) \\
\times (\nabla T_k(u)\chi_s - \nabla T_k(v^k_j)) \, dx \, dt \\
- \int_Q a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(v^k_j)) \, dx \, dt \\
\leq - \int_Q (a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)) \\
\times (\nabla T_k(u)\chi_s - \nabla T_k(v^k_j)) \, dx \, dt \\
- \int_Q a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(v^k_j)) \, dx \, dt + \varepsilon(n, j, \mu) \\
= L_1 + L_2 + \varepsilon(n, j, \mu).
\]

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For $L_1$, since $a(x, t, T_k(u_n), \nabla T_k(u_n))$ weakly star converges to $l_k$ in $(L^\psi(Q))^N$ and $a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)$ strongly converges to $a(x, t, T_k(u), \nabla T_k(u)\chi_s)$ in $(L^\psi(Q))^N$, we get

$$L_1 = -\int_Q (l_k - a(x, t, T_k(u), \nabla T_k(u)\chi_s)) (\nabla T_k(u)\chi_s - \nabla T_k(v^k)\mu) \, dx \, dt + \varepsilon(n).$$

Then by letting $j$ and $\mu$ to infinity, we obtain

$$L_1 = \varepsilon(n, j, \mu, s).$$

Similarly,

$$L_2 = \varepsilon(n, j, \mu).$$

Consequently, we deduce that

$$\int_Q (a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)) \times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx \, dt \to 0, \quad \text{as } n \to \infty.$$

Using Lemma 4.4, we get

$$T_k(u_n) \to T_k(u) \text{ for the modular convergence in } W^{1, x}_0 L^\varphi(Q).$$

**Step 5: Passage to the limit.** Since the sequence $T_k(u_n)$ converges for the modular convergence in $W^{1, x}_0 L^\varphi(Q)$, there exists a subsequence, which is also denoted by $(u_n)_n$, such that

$$\nabla u_n \to \nabla u \text{ a.e. in } Q.$$

Let $v \in W^{1, x}_0 L^\varphi(\Omega) \cap L^\infty(\Omega)$ and $\lambda = k + \|v\|_\infty$ with $k > 0$. Taking $T_k(u_n - v)$ as a test function in $(\mathcal{P}_n)$, we get

$$\int_Q \frac{\partial b_n(u_n)}{\partial t} T_k(u_n - v) \, dx \, dt$$

$$+ \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - v) \, dx \, dt$$

$$= \int_Q f_n T_k(u_n - v) \, dx \, dt + \int_Q \Theta_n(x, t, u_n) \nabla T_k(u_n - v) \, dx \, dt.$$
For the first term of the left-hand side of (5.36), by using the fact that \(b_n(u_n) \rightharpoonup b(u)\) weakly in \(L^p(Q)\), we get

\[
\int_Q \frac{\partial b_n(u_n)}{\partial t} T_k(u_n - v) \, dx \, dt = \left[ \int_0^T B^k_n(u_n) \, dt \right]_0^T = \left[ \int_0^T B^k(u) \, dt \right]_0^T + \varepsilon(n) = \int_Q \frac{\partial b(u)}{\partial t} T_k(u - v) \, dx \, dt + \varepsilon(n),
\]

where \(B^k_n(s) = \int_s^\infty b'(\tau)T_k(\tau - v) \, d\tau\) and \(B^k(s) = \int_0^s b'(\tau)T_k(\tau - v) \, d\tau\).

For the second term of the left-hand side of (5.36) we have

\[
\liminf_{n \to \infty} \int_Q a(x, u_n, \nabla u_n) \nabla T_k(u_n - v) \, dx \, dt \geq \int_Q a(x, u, \nabla u) \nabla T_k(u - v) \, dx \, dt.
\]

Indeed, if \(|u_n| > \lambda\), then \(|u_n - v| \geq |u_n| - \|v\|_\infty > k\). Let \(D_n = \{|u_n - v| \leq k\}\), therefore \(D_n \subseteq \{|u_n| \leq \lambda\}\), which implies that

\[
\int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - v) = a(x, t, u_n, \nabla u_n) \nabla (u_n - v) \chi_{D_n} = a(x, t, T_\lambda(u_n), \nabla T_\lambda(u_n)) (\nabla T_\lambda(u_n) - \nabla v) \chi_{D_n}.
\]

Then

\[
\int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - v) \, dx \, dt = \int_Q a(x, t, T_\lambda(u_n), \nabla T_\lambda(u_n)) (\nabla T_\lambda(u_n) - \nabla v) \chi_{D_n} \, dx \, dt
\]

\[
= \int_Q (a(x, t, T_\lambda(u_n), \nabla T_\lambda(u_n)) - a(x, t, T_\lambda(u_n), \nabla v))
\]

\[
\times (\nabla T_\lambda(u_n) - \nabla v) \chi_{D_n} \, dx \, dt
\]

\[
+ \int_Q a(x, t, T_\lambda(u_n), \nabla v) (\nabla T_\lambda(u_n) - \nabla v) \chi_{D_n} \, dx \, dt.
\]

Let \(D = \{|u - v| \leq k\}\), then we obtain

\[
\liminf_{n \to \infty} \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - v) \, dx \, dt
\]

\[
\geq \int_Q (a(x, t, T_\lambda(u), \nabla T_\lambda(u)) - a(x, t, T_\lambda(u), \nabla v))
\]

\[
\times (\nabla T_\lambda(u) - \nabla v) \chi_D \, dx \, dt
\]

\[
+ \lim_{n \to \infty} \int_Q a(x, t, T_\lambda(u_n), \nabla v) (\nabla T_\lambda(u_n) - \nabla v) \chi_{D_n} \, dx \, dt.
\]
The second term on the right-hand side of (5.40) is equal to
\[
\int_Q a(x, T_\lambda(u), \nabla v)(\nabla T_\lambda(u) - \nabla v) \chi_D \, dx \, dt.
\]

Finally, we get
\[
(5.41) \quad \liminf_{n \to \infty} \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - v) \, dx \, dt
\geq \int_Q a(x, t, T_\lambda(u), \nabla T_\lambda(u))(\nabla T_\lambda(u) - \nabla v) \chi_D \, dx \, dt
= \int_Q a(x, t, u, \nabla u)(\nabla u - \nabla v) \chi_D \, dx \, dt
= \int_Q a(x, t, u, \nabla u) \nabla T_k(u - v) \, dx \, dt.
\]

For the first term on the right-hand side of (5.36), using the strong convergence of \((f_n)_n\), we get
\[
(5.42) \quad \int_Q f_n T_k(u_n - v) \, dx \, dt = \int_Q f T_k(u_n - v) \, dx \, dt + \varepsilon(n).
\]

For the second term on the right-hand side of (5.36), for \(n \geq \lambda = k + \|v\|_\infty\), we have
\[
(5.43) \quad \int_Q \Theta_n(x, t, u_n) \nabla T_k(u_n - v) \, dx \, dt = \int_Q \Theta(x, t, T_\lambda(u_n)) \nabla T_k(u_n - v) \, dx \, dt
= \int_Q \Theta(x, t, u) \nabla T_k(u - v) \, dx \, dt + \varepsilon(n).
\]

Combining (5.36)–(5.43), one has
\[
\int_Q \frac{\partial b(u)}{\partial t} T_k(u - v) \, dx \, dt + \int_Q a(x, t, u, \nabla u) \nabla T_k(u - v) \, dx \, dt
\leq \int_Q f T_k(u - v) \, dx \, dt + \int_Q \Theta(x, t, u) \nabla T_k(u - v) \, dx \, dt.
\]

Consequently, via all steps, the proof of Theorem 5.1 is completed. \(\square\)
References


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