ON NON-OSCILLATION ON SEMI-AXIS OF SOLUTIONS OF SECOND ORDER DEVIATING DIFFERENTIAL EQUATIONS

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Abstract. We obtain conditions for existence and (almost) non-oscillation of solutions of a second order linear homogeneous functional differential equations

$$u''(x) + \sum_i p_i(x)u'(h_i(x)) + \sum_i q_i(x)u(g_i(x)) = 0$$

without the delay conditions $h_i(x), g_i(x) \leqslant x$, $i = 1, 2, \ldots$, and

$$u''(x) + \int_0^\infty u'(s)\mathrm{d}_s r_1(x,s) + \int_0^\infty u(s)\mathrm{d}_s r_0(x,s) = 0.$$

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1. Introduction

1.1. Non-oscillation and delay equations. Any solution of the Euler equation

$$u'' + \frac{1}{4x^2}u = 0, \quad x \in (0, \infty)$$

has the form $u(x) = c_1\sqrt{x}\ln x + c_2\sqrt{x} = c_1u_1 + c_2u_2$. This equation may serve as a model for generalizations. It is non-oscillating on $[0, \infty)$ according to the following definition.

Definition 1.1. The homogeneous equation

$$u''(x) + p(x)u' + q(x)u = 0 \tag{1.1}$$

is called *non-oscillating* on a finite interval $I = [0, l]$, or infinite interval $I = [0, \infty)$ or $(0, \infty)$ if any its nonzero solution can have at most one simple zero in I.

Taking into consideration the Sturm separation theorem (see [6], page 335) the non-oscillation property of equation (1.1) is equivalent to the existence of a positive solution. The non-oscillation property along with other issues of the distribution of zeros of solutions are intensively studied because of its relation to spectral properties of differential operators in quantum mechanics.

The non-oscillating property (as well as non-oscillatory property[1]) of the equation $u'' + q(x)u = 0$ is well studied. See, for example, [8], [7] and the references therein. The non-oscillatory condition

$$\int_0^\infty xq(x)\,\mathrm{d}x < \infty \tag{1.2}$$

(see [7]) is not fulfilled for the mentioned Euler equation. We will see that condition (1.2) must be regarded as too strict. In this paper, devoted to the following equation (1.5), we obtain, in particular, the non-oscillation condition

$$\int_x^\infty q(s)\sqrt{s}\,\mathrm{d}s \leqslant \frac{1}{2\sqrt{x}}, \quad x > 0. \tag{1.3}$$

This inequality becomes an identity in the case $u'' + \frac{1}{4}x^{-2}u = 0$. For a deviating equation see Corollary 3.2.

In [4], [1], the question about the existence of a finally positive solution of the initial problem on semi-axis (t_0, ∞) for delay equation

$$x''(t) + \sum_i p_i(t)x'(h_i(t)) + \sum_i q_i(t)x(g_i(t)) = 0, \quad h_i(t),\, g_i(t) \leqslant t, \tag{1.4}$$

under conditions $\lim\limits_{t\to\infty} h_i(t) = \infty$, $\lim\limits_{t\to\infty} g_i(t) = \infty$, is considered.

For the delay equation Definition 1.1 has some problems in its interpretation because of the difficulties in determining the concept of the solution. There are two ways to define the solutions to delay equation (1.4). The first is considering the equation on all axis $(-\infty, \infty)$, and the second is to use an initial function on the left of a point x_0. The solution itself is considered on the semi-axis $[x_0, \infty)$. But the first

[1] A nonzero solution is oscillatory if it has arbitrarily large zeros.

case is not simple and has to be considered as a singular boundary value problem. In the second case the initial problem is not homogeneous even at zero right side as an equation $\mathcal{L}u = 0$ with linear operator $\mathcal{L}$. However, the homogeneity of the equation is essential for the non-oscillation problem.

1.2. Deviating homogeneous equation. We use Definition 1.1 for the semi-axis $[0, \infty)$ and for the homogeneous equation

$$u''(x) + Su(x) = 0, \quad x \in [0, \infty), \tag{1.5}$$

where the operator[2] S is defined by

$$Su(x) := \int_0^\infty u'(s)\,\mathrm{d}_s r_1(x, s) + \int_0^\infty u(s)\,\mathrm{d}_s r_0(x, s) \tag{1.6}$$

and satisfies the positivity condition

$$\{u \geqslant 0,\, u' \geqslant 0\} \to Su \geqslant 0. \tag{1.7}$$

Assumptions are in Section 3. Note that deviating equation (1.4) (no delay condition $h_i(x), g_i(x) \leqslant x$) can be represented in the form

$$u''(x) + \int_{-\infty}^\infty u'(s)\,\mathrm{d}_s r_1(x, s) + \int_{-\infty}^\infty u(s)\,\mathrm{d}_s r_0(x, s) = 0.$$

The initial problem ($u(x) = \varphi(x)$ if $x < 0$) can be represented as non-homogeneous equation on $[0, \infty)$ as

$$u'' + \int_0^\infty u'(s)\,\mathrm{d}_s r_1(x, s) + \int_0^\infty u(s)\,\mathrm{d}_s r_0(x, s) = f(x).$$

The idea of such separation was first used by Azbelev (see [3]).

Let us discuss the form of equation (1.5). The notation $x(t)$ is more popular and used to underline that variable t denotes physical time. For this reason, the majority of research of non-oscillation has been devoted to the delay equation; that may describe an evolutionary processes. On the other hand, non-oscillation property has its origin in the problem of positivity of a quadratic functional and has a mechanical interpretation (see [11]). In this case the delay condition does not make sense, and the form (1.5) looks more natural.

[2] It will be considered in a space defined in Section 3.

2. Non-oscillation and positive solvability

2.1. Ordinary and delay equation. For the ordinary equation it is sufficient to establish the existence of a positive solution on $(0,\infty)$. In the case of the ordinary equation it follows from the Sturm theorem based on non-vanishing of the Wronskian. But for the delay and deviating equations the Wronskian may have zeros. Moreover, for the deviating equation we do not know the structure of solutions. For delay equations of the form

$$u''(x) + \int_0^x u(s)\,\mathrm{d}_s r(x,s) = 0, \quad x \geqslant 0, \tag{2.1}$$

conditions for non-vanishing of the Wronskian were considered in articles [2], [14]. In this regard, see also [13], [5]. Using the ideas from [2] one can conclude, that the Wronskian will be different from zero for equation (2.1) in the case of almost non-oscillation.

Theorem 2.1. *Suppose $r(x,s)$ is non-decreasing with respect to s. Suppose also that equation* (2.1) *has a positive solution on* $(0,\infty)$. *Then equation* (2.1) *is non-oscillating on* $(0,\infty)$.

Proof. Let $u(x) > 0$ for $x \in (0,\infty)$. Let x_0 be the first zero of the Wronskian. It can be shown (as in [2] or directly) that the Wronskian is not decreasing on $[0,x_0)$. This contradiction shows that the Wronskian is different from zero on all semi-axis $[0,\infty)$. □

2.2. Almost non-oscillation. Suppose equation (1.1) is non-oscillating on $[0,\infty)$ and it has two positive solutions u_1 and u_2,

$$u_1(x) > 0, \quad u_2(x) > 0, \quad x \in (0,\infty), \tag{2.2}$$

satisfying conditions

$$u_1(0) = 0, \quad u_2(0) > 0. \tag{2.3}$$

Then we can assume that

$$\lim_{x\to\infty} \frac{u_2(x)}{u_1(x)} = 0. \tag{2.4}$$

Indeed, u_2/u_1 is strictly decreasing because this is equivalent to non-vanishing of the Wronskian:

$$\left(\frac{u_2}{u_1}\right)' = \frac{1}{u_1^2}\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} < 0. \tag{2.5}$$

If $\lim_{x\to\infty} u_2/u_1 = c > 0$, we can put $\widetilde{u}_2 = u_2 - cu_1$.

Based on properties (2.2), (2.3) and (2.4) we can formulate the following definition.

Definition 2.1 (almost non-oscillation)**.** A second order linear equation (1.5) is said to be *almost non-oscillating* on $[0,\infty)$ if there exist two solutions u_1 and u_2 satisfying (2.2), (2.3), (2.4).

Remark 2.1. The almost non-oscillation coincides with non-oscillation if and only if u_2/u_1 is strictly decreasing, because of (2.5).

2.3. Non-oscillation and positive solvability. How can we establish the existence of a positive solution on all semi-axis $[0,\infty)$? Let us analyse this using the equation

$$u''(x) + q(x)u = 0$$

with non-negative $q(x)$ satisfying the condition $q(x) \not\equiv 0$ on $[l,\infty)$ for any $l > 0$.

Proposition 2.1. *The equation $u''(x) + q(x)u = 0$ is non-oscillating on $[0,\infty)$ if and only if the problem*

$$u''(x) + q(x)u = 0, \quad u(0) = 0, \quad u'(l) = 0 \tag{2.6}$$

is uniquely solvable for any $l > 0$.

Proof. Let $u(x)$ be the solution satisfying initial conditions $u(0) = 0$, $u'(0) = 1$. If it is increasing on all semi-axis, then problem (2.6) is uniquely solvable for each $l > 0$.

If $u(x)$ is not increasing, its derivative has a zero in a point $x = l$, and $u(x)$ is a nonzero solution to problem (2.6). The function $u(x)$ must have a zero for $x > l$. This is the oscillating case. □

The Fredholm property of problem (2.6) and the solvability of Cauchy problem were used essentially. So, this approach is useful for the delay equation but does not serve for the arbitrary deviating equation.

It seems that for an deviating equation it is more natural to use a boundary value problem (BVP) on all semi-axis

$$u'' + Su = f, \quad u(0) = \alpha, \quad u'(\infty) = 0 \tag{2.7}$$

This problem may have a unique non-negative solution if $f \leqslant 0$ (and if f is in some linear manifold) and $\alpha \geqslant 0$.

The set of solutions of problem (2.7) in the mentioned manifold (denote it by D_ω as this will be done below) does not cover all solutions of the homogeneous equation. For the ordinary equation $u''+qu=0$ and for the delay equation $u''(x)+q(x)u(h(x))=0$, $h(x)\leqslant x$ (or (2.1)) it is not a problem because of Theorem 2.1. But for deviating equation (1.5) we need to establish the existence of a nonzero positive solution satisfying $u(0)=0$. This is our main difficulty.

Thus, we will use the following definition.

Definition 2.2 (positive solvability)**.** We say that the boundary value problem (2.7) is *positively solvable* if it is uniquely solvable in a space D_ω (it will be defined later) for any (f,α) (for a linear class (or space) of functions f). The solution is positive if $(-f,\alpha)\geqslant 0$, not identicaly 0, and there exists a solution u_1, positive on $(0,\infty)$, of the homogeneous problem

$$u''+Su=0,\quad u(0)=0,\quad u'(\infty)=0. \tag{2.8}$$

A solution to the problem of positive solvability is presented in Theorem 3.1.

Remark 2.2. Since problem (2.7) is uniquely solvable in D_ω, the solution $u_1\notin D_\omega$.

It seems that this property is equivalent to almost non-oscillation (Definition 2.1). Indeed, denote by $u_1(x)$ a solution of (2.8) and suppose (2.7) is uniquely solvable. The solution of (2.7) has the form $u(x)=Gf(x)+\alpha u_2(x)$. Then any solution of non-homogeneous equation

$$u''(x)+Su(x)=f(x),\quad x\in[0,\infty)$$

has the form

$$u(x)=c_1u_1(x)+\alpha u_2(x)+Gf(x).$$

But there is the difficulty to establish the property (2.4).

Remark 2.3. The boundary condition $u'(\infty)=\beta\neq 0$ can be considered if and only if condition (3.16) holds. In the case of equation $u''+q(x)u=f$, condition (3.16) has the form (1.2).

For example, condition (1.2) is not fulfilled for equation $u''+kx^{-2}u=0$, $x\in[1,\infty)$, $0<k\leqslant\frac{1}{4}$. It is non-oscillating but it does not have solutions with $u'(\infty)\neq 0$. Under condition (1.2) the problem

$$u''+q(x)u=f,\quad u(0)=\alpha,\quad u'(\infty)=\beta \tag{2.9}$$

has the Fredholm property. We call this case a *regular case* despite the fact that the equation is considered on the infinite interval.

3. Results

3.1. Notation, assumptions, definitions. Let

- ▷ $\mathbb{R} = (-\infty, \infty)$, $L(\Omega)$ be the space of Lebesgue integrable on the measurable set Ω functions ($\Omega \subset \mathbb{R}$), $L(\alpha, \beta) := L([\alpha, \beta])$,
- ▷ the symbol $\|\cdot\|_X$ denote a norm in a space X; for example, $\|z\|_L$ is the norm in the space L,
- ▷ W_{loc} be the set of locally[3] on $[0, \infty)$ absolutely continuous functions,
- ▷ $L_\infty(\alpha, \beta)$ be the space of measurable and essentially bounded on (α, β) functions.

By definition, the solution $u(x)$ of (1.5) is a continuous function which has absolutely continuous derivative $u'(x)$ on any segment in $[0, \infty)$ (in some examples an interval $[x_0, \infty)$ may be explored instead of $[0, \infty)$).

The functions r_0, r_1 are non-decreasing with respect to the second argument for almost all $x \geqslant 0$ and measurable with respect to x for all $s \geqslant 0$. Assume that $r_i(x, 0) = 0$, $i = 0, 1$. The non-decreasing condition provides positivity of the operator S defined by (1.6), that is, implication (1.7) holds.

The integrals in (1.6) are understood in the sense of Lebesgue-Stieltjes. This means that $\int_0^\infty |u'(s)| \, \mathrm{d}_s r_1(x, s)$ and $\int_0^\infty |u(s)| \, \mathrm{d}_s r_0(x, s)$ are finite for almost all $x \in [0, \infty)$. Let

- ▷ $D(S)$ be the domain of S, that is, the set of functions $u \in W_{\mathrm{loc}}$ such that $u' \in W_{\mathrm{loc}}$, and the integrals

$$\int_0^\infty u'(s) \, \mathrm{d}_s r_1(x, s) \quad \text{and} \quad \int_0^\infty u(s) \, \mathrm{d}_s r_0(x, s)$$

  exist for almost all $x \geqslant 0$.

For the particular case of (1.4), when

$$Su(x) = \sum_{k=1}^{\infty} p_k(x) u'(g_k(x)) + \sum_{k=1}^{\infty} q_k(x) u(h_k(x))$$

(assuming that $u(x) = 0$ if $x < 0$), the above conditions mean that $p_i(x)$, $q_i(x)$ are Lebesgue integrable on any segment in $[0, \infty)$, $h_i(x)$, $g_i(x)$ are measurable, but without delay conditions $h_i(x), g_i(x) \leqslant x$, $i = 1, 2, \ldots$ The positivity condition of S for the deviating operator means non-negativeness; $p_i(x), q_i(x) \geqslant 0$.

Definition 3.1. A function $z\colon [a, \infty) \to \mathbb{R}$ is called *non-finite* if (for every $\nu > a$) $z(x) \neq 0$ on a set of positive measure in $[\nu, \infty)$.

A function $z\colon [a, \infty) \to \mathbb{R}$ is called *finally positive* if there exists $\nu > a$ such that $z(x) > 0$ on $[\nu, \infty)$.

[3] Locally on X means: on every segment from X.

3.2. Positive solvability. We study positive solvability of the BVP (2.7).

Let $\omega(x)$ be a weight positive continuous function satisfying

$$\inf\{\omega(s)\colon\ s \in [0,\infty)\} > 0. \tag{3.1}$$

Lemma 3.1. *For any $(z,\alpha) \in L(0,\infty) \times \mathbb{R}$ the problem*

$$-\omega u'' = z, \quad u(0) = \alpha, \quad u'(\infty) = 0, \quad z \in L(0,\infty) \tag{3.2}$$

has a unique solution in W_{loc} with $u' \in W_{\mathrm{loc}}$

$$u(x) = \int_0^\infty \frac{G(x,s)}{\omega(s)} z(s)\,\mathrm{d}s + \alpha := G_\omega z(x) + \alpha, \tag{3.3}$$

where

$$G(x,s) = \min\{x,s\}. \tag{3.4}$$

If $z \geqslant 0$ and $\alpha \geqslant 0$, then $u(x) \geqslant 0$ and $u'(x) \geqslant 0$ on $[0,\infty)$.

The assertion is verified directly. Note that the integral $u'(x) = \int_x^\infty (z(s)/\omega(s))\,\mathrm{d}s$ exists since the function $u'' = -z/\omega$ is integrable.

Let D_ω be the set of all solutions of problem (3.2):

$$D_\omega = \{u = G_\omega z + \alpha\colon\ z \in L(0,\infty),\ \alpha \in \mathbb{R}\}.$$

We use relations (3.3) and (3.2) to reduce the BVP (2.7) to an equation in $L(0,\infty)$.

Lemma 3.2. *Suppose $D(S) \supset D_\omega$ and $\omega(x) r_0(x,\infty)$ is in $L(0,\infty)$. Then substitution (3.3) reduces (2.7) to the equation*

$$z - K_\omega z = -\omega f + \alpha\omega(x) r_0(x,\infty), \tag{3.5}$$

where $K_\omega = \omega S G_\omega$. Conversely, if $z - K_\omega z = f$, $z \in L(0,\infty)$, then $u = G_\omega z + \alpha$ is a solution of problem

$$u'' + Su = -\frac{f}{\omega} + \alpha r_0(x,\infty), \quad u(0) = \alpha, \quad u'(\infty) = 0.$$

Let $D := D_\omega$, $K := K_\omega$ and $K(x,s) := K_\omega(x,s)$ for $\omega = 1$. Let $\varrho(K)$ and $\varrho(K_\omega)$ be the spectral radii of the operators K and K_ω, respectively.

The following assertion is a particular case of Lemma 3.2.

Lemma 3.3. *Let $\omega = 1$. Suppose $D(S) \supset D$. If the equation*

$$z = Kz \tag{3.6}$$

has a nonzero solution $z \geqslant 0$, $z \in L(0,\infty)$, then the function

$$u(x) = \int_0^\infty G(x,s)z(s)\,\mathrm{d}s \tag{3.7}$$

is a positive solution of problem (2.8). *If z is non-finite, then $v'(x) > 0$, $x \in [0,\infty)$.*

From Lemma 3.2 it follows the simple theorem:

Theorem 3.1 (positive solvability)**.** *Suppose $K_\omega\colon L(0,\infty) \to L(0,\infty)$ is a continuous operator and $\varrho(K_\omega) < 1$. Then for any f such that $\omega f \in L(0,\infty)$ and any α, problem* (2.7) *is uniquely solvable in D_ω. From $(-f,\alpha) \geqslant 0$, not identicaly* 0 *it follows the positivity; $u(x) > 0$, $u'(x) \geqslant 0$ on $(0,\infty)$.*

Proof. If $\omega f \in L$, $f \leqslant 0$, $\alpha \geqslant 0$, then (3.5) has a unique solution $z \geqslant 0$. Thus, $u = G_\omega z + \alpha$ has the desired property. □

Lemma 3.4. *$D(S) \supset D_\omega$ if and only if*

$$K_\omega(x,\cdot) \in L_\infty(0,\infty) \quad \text{for almost all } x \in [0,\infty),$$

where $K_\omega(x,s)$ is defined by

$$K_\omega(x,s) = \frac{\omega(x)}{\omega(s)}\biggl(\int_0^\infty \mathrm{d}_\tau r_0(x,\tau)\,G(\tau,s) + r_1(x,s)\biggr). \tag{3.8}$$

If $D(S) \supset D_\omega$, then K_ω is an integral operator with the kernel $K_\omega(x,s)$.

Proof. Suppose $D(S) \supset D_\omega$. Then for all $z \in L$ there exist integrals

$$\int_0^\infty \mathrm{d}_\tau r_0(x,\tau) \int_0^\infty \frac{G(\tau,s)}{\omega(s)}|z(s)|\,\mathrm{d}s \quad \text{and} \quad \int_0^\infty \mathrm{d}_\tau r_1(x,\tau) \int_\tau^\infty \frac{|z(s)|}{\omega(s)}\,\mathrm{d}s.$$

From the Fubini's theorem we can conclude that there exists integral

$$\int_0^\infty \mathrm{d}s|z(s)|\frac{1}{\omega(s)}\biggl(\int_0^\infty \mathrm{d}_\tau r_0(x,\tau)G(\tau,s) + \int_0^s \mathrm{d}_\tau r_1(x,\tau)\biggr). \tag{3.9}$$

By virtue of arbitrariness of $z \in L(0,\infty)$ it is possible if $K(x,\cdot) \in L_\infty(0,\infty)$.

Conversely, let $K(x,\cdot) \in L_\infty(0,\infty)$ for almost all $x \in (0,\infty)$, $z \in L(0,\infty)$ and $u = G_\omega z$. Denote $u_\pm = G_\omega z_\pm$, where $z = z_+ - z_-$, $|z| = z_+ + z_-$. Apply the Fubini's theorem to (3.9), where instead of z we will have $z_\pm$. Thus, the integrals

$$\int_0^\infty u_\pm^{(i)} \,\mathrm{d}_s r_i(x,s), \quad i = 0,1$$

are finite for these $x \in (0,\infty)$. □

R e m a r k 3.1. The condition $\varrho(K_\omega) < 1$ is too tough. For this reason, we should think about the consequences of using condition $\varrho(K_\omega) \leqslant 1$. This condition is enough to establish the existence of a positive solution to the homogeneous problem (2.8).

To apply Theorem 3.1 one can use the estimation of the norm

$$\|K_\omega\| \leqslant \operatorname*{ess\,sup}_s \int_0^\infty K_\omega(x,s)\,\mathrm{d}x \tag{3.10}$$

of the operator K_ω. From (3.10) and (3.8) we get

(3.11)

$$\begin{aligned}\|K_\omega\| &\leqslant \operatorname*{ess\,sup}_s \int_0^\infty \frac{\omega(x)}{\omega(s)}\biggl(\int_0^\infty \mathrm{d}_\tau r_0(x,\tau)\,G(\tau,s) + r_1(x,s)\biggr)\,\mathrm{d}x \\ &= \operatorname*{ess\,sup}_s \int_0^\infty \frac{\omega(x)}{\omega(s)}\biggl(\int_0^s \tau\,\mathrm{d}_\tau r_0(x,\tau) + s(r_0(x,\infty) - r_0(x,s)) + r_1(x,s)\biggr)\,\mathrm{d}x.\end{aligned}$$

3.3. Example. Ordinary equation. For the ordinary equation

$$u'' + q(x)u = f, \tag{3.12}$$

$K_\omega(x,s) = (\omega(x)/\omega(s))q(x)G(x,s)$. Therefore from (3.11),

$$\|K_\omega\| \leqslant \operatorname*{ess\,sup}_s \frac{1}{\omega(s)}\biggl(\int_0^s x\omega(x)q(x)\,\mathrm{d}x + s\int_s^\infty \omega(x)q(x)\,\mathrm{d}x\biggr).$$

For example, if[4] $\omega(x) = \sqrt{x}$, $q(x) = kx^{-2}$, then the right-hand side of the inequality is equal to $4k$. Thus $\|K_\omega\| \leqslant 4k$. If $k < \frac{1}{4}$, then the unique solution (in D_ω) of problem

$$u'' + \frac{k}{x^2}u = f(x) \leqslant 0, \quad u(x_0) = \alpha > 0, \quad u'(+\infty) = 0, \quad \omega f \in L[x_0,\infty)$$

is positive on $[x_0,\infty)$.

[4] To avoid singularity in the point 0 we have to consider here $[x_0,\infty)$, $x_0 > 0$ instead of $[0,\infty)$.

Let us now use Proposition 2.1. We can use the estimation of the integral operator $K_\omega^{(a,l)}$ for the segment $[a,l]$ instead of $[0,\infty)$. Then

$$K_\omega^{(a,l)}(x,s) = \frac{\omega(x)}{\omega(s)} q(x) \min\{x-a, s-a\}.$$

Corollary 3.1. *Let the inequality*

$$\int_x^\infty q(s)\sqrt{s}\,\mathrm{d}s \leqslant \frac{1}{2\sqrt{x}}, \quad x>0$$

be fulfilled. Then equation $u''+q(x)u=0$ is non-oscillating on $(0,\infty)$.

P r o o f. Show that $\|K_\omega^{(a,l)}\|<1$ for any $l>0$. This guaranties unique solvability of problem (2.6). Let $\omega(x)=\sqrt{x}$. In this case

$$\|K_\omega^{(a,l)}\| \leqslant \operatorname*{ess\,sup}_s \frac{1}{\sqrt{s}}\biggl(\int_a^s (x-a)\sqrt{x}q(x)\,\mathrm{d}x + (s-a)\int_s^l \sqrt{x}q(x)\,\mathrm{d}x\biggr)$$

or

$$\|K_\omega^{(a,l)}\| \leqslant \sup_s \widetilde{\varphi}(s), \tag{3.13}$$

where

$$\widetilde{\varphi}(s) = \varphi(s) - \frac{a}{\sqrt{s}}\int_a^l \sqrt{x}q(x)\,\mathrm{d}x,$$

$$\varphi(s) = \frac{1}{\sqrt{s}}\int_a^s x\sqrt{x}q(x)\,\mathrm{d}x + \sqrt{s}\int_s^l \sqrt{x}q(x)\,\mathrm{d}x.$$

Since $\varphi'(0)>0$, $\varphi'(l)<0$, in the maximum point

$$\int_a^s x\sqrt{x}q(x)\,\mathrm{d}x = s\int_s^l \sqrt{x}q(x)\,\mathrm{d}x,$$

and at that point

$$\varphi(s) = 2\sqrt{s}\int_s^l \sqrt{x}q(x)\,\mathrm{d}x.$$

From here, (3.13) and (1.3) it follows $\|K_\omega^{(a,l)}\|<1$. □

3.4. Singular and regular cases. Operator K is defined on $L(0,\infty)$ if $D\subset D(S)$. From (3.8) it follows

$$K(x,s) = \int_0^s \tau\,\mathrm{d}_\tau r_0(x,\tau) + s(r_0(x,\infty)-r_0(x,s)) + r_1(x,s). \tag{3.14}$$

Using Lemma 3.4 and due to $K(x,\cdot)$ being non-decreasing, we have the following condition for $D \subset D(S)$:

$$K(x,\infty) = \int_0^\infty \tau \, \mathrm{d}_\tau r_0(x,\tau) + r_1(x,\infty) \quad \text{is finite for almost all } x \geqslant 0. \tag{3.15}$$

Definition 3.2 (regular and singular)**.** We say that problem (2.7) is *regular* (or *Fredholm* one), if $D \subset D(S)$ and the operator $K = SG\colon L(0,\infty) \to L(0,\infty)$ is compact. Conversely case ($D \not\subset D(S)$ or K is not compact), the problem is *singular*.

Below we consider only the case $D \subset D(S)$. The equation

$$u''(x) + \frac{1}{8x\sqrt{x}} \int_1^\infty u(s) \, \frac{\mathrm{d}s}{s^2} = 0$$

shows the case $D \not\subset D(S)$. It has the solution $u(x) = \sqrt{x}$.

Lemma 3.5. *The condition*

$$\int_0^\infty \left(\int_0^\infty \tau \, \mathrm{d}_\tau r_0(x,\tau) \right) \mathrm{d}x + \int_0^\infty r_1(x,\infty) \, \mathrm{d}x < \infty \tag{3.16}$$

is a necessary and sufficient condition of regularity of problem (2.7).

P r o o f. Note that

$$\int_0^\infty K(x,\infty) \, \mathrm{d}x = \int_0^\infty \left(\int_0^\infty \tau \, \mathrm{d}_\tau r_0(x,\tau) \right) \mathrm{d}x + \int_0^\infty r_1(x,\infty) \, \mathrm{d}x.$$

If $\int_0^\infty K(x,\infty) \, \mathrm{d}x < \infty$, operator $K\colon L(0,\infty) \to L(0,\infty)$ is compact. To show compactness, note that $K(\cdot,s) \in L(0,\infty)$ for any s and that it is non-decreasing in s. A necessary and sufficient compactness condition of the integral operator K is compactness of the vector function $s \mapsto K(x,s)$ (see [9], Theorem 6.6, page 116). Monotonicity provides such compactness.

If $\int_0^\infty K(x,\infty) \, \mathrm{d}x = \infty$, then K does not act in $L(0,\infty)$. $\square$

3.5. Homogeneous problem. Singular case. In this subsection assume that the following two conditions are fulfilled:

▷ singularity:

$$\int_0^\infty \left(\int_0^\infty \tau \, \mathrm{d}_\tau r_0(x,\tau) \right) \mathrm{d}x + \int_0^\infty r_1(x,\infty) \, \mathrm{d}x = \infty, \tag{3.17}$$

▷ for any $\nu > 0$

$$\int_0^\nu \left(\int_0^\infty \tau \, \mathrm{d}_\tau r_0(x,\tau) \right) \mathrm{d}x < \infty, \quad \int_0^\nu r_1(x,\infty) \, \mathrm{d}x < \infty. \tag{3.18}$$

In this section we obtain some theorems about the existence of a positive solution to the homogeneous boundary value problem (2.8). The scheme of work will be as follows. First we use Lemma 3.3 to the connection between the boundary value problem (2.8) and the integral equation (3.6). After this we rely on the results obtained for the integral equation in a separate independent Section A.2. The existence of a nonzero non-negative solution of a homogeneous integral equation is asserted in three theorems A.1, A.2, A.3. The corresponding theorems for the boundary value problem are presented below.

Conditions (3.17) and (3.18) ensure the fulfillment of conditions (A.7) and (A.6) for the kernel (3.14).

Theorem 3.2. *Suppose $\varrho(K_\omega) \leqslant 1$ for a weight function satisfying condition* (3.1). *Then there exists a solution $u(x)$ of problem* (2.8), *$u(x) > 0$, $u'(x) \geqslant 0$ on* $(0,\infty)$.

Proof. From Theorem A.1 it follows that there exists a non-negative nontrivial solution of (3.6). The function $u(x)$ defined by (3.7) is positive on $(0,\infty)$ and $u'(x) \geqslant 0$. □

Remark 3.2. Indeed, $u'(x) > 0$ if $r_1(x,s) \equiv 0$, since the case $u' = 0$ on $[l,\infty)$ for some $l > 0$ is not possible. One can see this analysing equation (1.5).

The use of this theorem is reduced to the estimation of the norm by inequality (3.11).

Let $K^{(0,l)}$ be the integral operator with the kernel $K(x,s)$ acting in the space $L(0,l)$. Let ϱ_l be the spectral radius of $K^{(0,l)}$. Operator $K^{(0,l)}$ is compact. This fact has been used in many articles. It is recommended to look at the proof of compactness in [3].

The following two theorems are corollaries of Theorems A.2 and A.3.

Theorem 3.3. *Suppose that $\varrho_l < 1$ for any $l > 0$. Then there exists a positive strictly increasing solution of homogeneous problem* (2.8).

This theorem can be used to obtain effective conditions as it is done, for example, in [14], [15], [16].

Theorem 3.4. *Suppose there exists a solution $v(x) > 0$, $v'(x) > 0$ of the inequality $v'' + Sv \leqslant 0$ positive on $(0,\infty)$. Then there exists a positive strictly increasing solution of homogeneous problem* (2.8).

P r o o f. The function $z_1 = -v''$ satisfies the inequality $z_1 \geqslant Kz_1$. By Theorem A.3 there exists a nonzero non-finite solution $z(x)$ of (3.6). The function $u(x)$ defined by (3.7) is a necessary solution. $\square$

Corollary 3.2 (of Theorem 3.2)**.** *Let $a > 0$. Consider deviating (without condition $h(x) \leqslant x$) equation*

$$u''(x) + q(x)u(h(x)) = f(x), \quad u(x) = 0, \text{ if } x < a. \tag{3.19}$$

Assume that $h(x)$ is increasing and the inverse function $g(s)$ is differentiable. Suppose that

$$\int_a^\infty q(x)h(x)\,\mathrm{d}x = \infty \tag{3.20}$$

and for some $\alpha \in (0,1)$ and for all $s \geqslant a$,

$$\int_{g(s)}^\infty q(x)x^\alpha\,\mathrm{d}x \leqslant \frac{\alpha}{s^{1-\alpha}}. \tag{3.21}$$

Then there exists a positive strictly increasing solution of equation (3.19) *satisfying the initial condition $u(a) = 0$.*

P r o o f. Condition (3.20) is the singularity condition (3.17) for equation (3.19). In this case from (3.8) we have $K_\omega(x,s) = (\omega(x)/\omega(s))q(x)G(h(x),s)$, where $G(x,s) = \min\{x-a, s-a\}$ but $G(x,s) = 0$ if $x < a$. For the estimation of the norm we have

$$\|K_\omega\| \leqslant \operatorname*{ess\,sup}_{s\geqslant a} \widetilde{\varphi}(s), \tag{3.22}$$

where

$$\begin{aligned}
\widetilde{\varphi}(s) &:= \frac{1}{\omega(s)}\int_0^\infty q(x)\omega(x)G(h(x),s)\,\mathrm{d}x \\
&= \frac{1}{\omega(s)}\biggl(\int_{h(x)\leqslant s} q(x)\omega(x)(h(x)-a)\,\mathrm{d}x + (s-a)\int_{h(x)>s} q(x)\omega(x)\,\mathrm{d}x\biggr) \\
&= \varphi(s) - \frac{a}{\omega(s)}\int_a^\infty q(x)\omega(x)\,\mathrm{d}x.
\end{aligned}$$

Since $\varphi(0) = 0$, it is clear that $\sup\varphi(s) = \sup\{\varphi(s)\colon\ \varphi'(s) \geqslant 0\}$. Since

$$\varphi'(s) = \Bigl(\frac{1}{\omega(s)}\Bigr)'\int_0^{g(s)} q(x)\omega(x)h(x)\,\mathrm{d}x + \Bigl(\frac{s}{\omega(s)}\Bigr)'\int_{g(s)}^\infty q(x)\omega(x)\,\mathrm{d}x,$$

from inequality $\varphi'(s) \geqslant 0$ it follows (suppose ω is increasing)

$$\int_0^{g(s)} q(x)\omega(x)h(x)\,\mathrm{d}x \leqslant -\frac{(s\omega^{-1})'}{(\omega^{-1})'}\int_{g(s)}^{\infty} q(x)\omega(x)\,\mathrm{d}x.$$

From here

$$\varphi(s) \leqslant \frac{1}{\omega}\Big(-\frac{(s\omega^{-1})'}{(\omega^{-1})'} + s\Big)\int_{g(s)}^{\infty} q(x)\omega(x)\,\mathrm{d}x.$$

Let here $\omega = s^{\alpha}$. Then

$$\varphi(s) \leqslant \frac{s^{1-\alpha}}{\alpha}\int_{g(s)}^{\infty} q(x)x^{\alpha}\,\mathrm{d}x.$$

□

Corollary 3.3 (comparison theorem)**.** *Suppose that the equation*

$$v''(x) + \sum_i \widetilde{q}_i(x)v(\widetilde{h}_i(x)) = 0$$

with non-negative coefficients has a solution positive on $(0,\infty)$*. Consider the equation with non-negative coefficients*

$$u''(x) + \sum_i q_i(x)u(h_i(x)) = 0. \tag{3.23}$$

If for all i*,* $q_i \leqslant \widetilde{q}_i$ *and* $h_i \leqslant \widetilde{h}_i$*, then equation* (3.23) *has a positive solution.*

P r o o f. It follows from Theorem 3.4. □

3.6. Positive solvability. Regular case. Let condition (3.16) be fulfilled. Then we can consider the problem

$$u'' + Su = f, \quad u(0) = \alpha, \quad u'(\infty) = \beta \tag{3.24}$$

with $\beta \neq 0$. By means of the substitution $u(x) = \int_0^{\infty} \min\{x,s\}z(s)\,\mathrm{d}s + \alpha + \beta x$, this problem is reduced to the equation (see (3.15))

$$z(x) = \int_0^{\infty} K(x,s)z(s)\,\mathrm{d}s + \alpha r_0(x,\infty) + \beta K(x,\infty).$$

Theorem 3.5. *Suppose there exists a solution, positive on* $(0,\infty)$*, of the problem*

$$u'' + Su = \psi \leqslant 0, \quad u(0) = \alpha \geqslant 0, \quad u'(\infty) = \beta \geqslant 0$$

for some non-negative nontrivial triple $(-\psi,\alpha,\beta)$*. Then* (3.24) *is uniquely solvable and for any non-negative nontrivial* $(-f,\alpha,\beta)$ *the solution of this problem is positive on* $(0,\infty)$*.*

In substance, this theorem has already been obtained in [15]. The difference is that an infinite interval is considered here. But the essence is in the compactness of the operator SG.

Appendix A. The existence of a non-finite solution of a homogeneous integral equation

The results of this section are independent of the main content of the article. These results are presented in [18]. For completeness, we give them here with the proofs. The scheme was used first in [17] without proofs and in [12].

Let $L = L(0, \infty)$,

▷ L_{loc} be the topological space with topology of convergence in $L(0, \nu)$ for any $\nu > 0$,

▷ the convergences in L and L_{loc} be $z_n \xrightarrow{L} z$ and $z_n \xrightarrow{L_{\mathrm{loc}}} z$, respectively.

In the first subsection, an abstract operator $K\colon\ L \to L_{\mathrm{loc}}$ will be considered and in the second one, the obtained result will be applied to integral equation (A.5). Note that the positivity condition $K \geqslant 0$ is used essentially.

A.1. General operator. Let $K, K_n\colon\ L \to L_{\mathrm{loc}}$, $n = 1, 2, \ldots$, be linear mappings. Consider an equation

$$z = Kz + f. \tag{A.1}$$

Suppose that there exists a sequence (z_n) of solutions of the equations

$$z_n = K_n z_n + f_n \tag{A.2}$$

such that $\|z_n\| = 1$, $f_n \xrightarrow{L_{\mathrm{loc}}} f$. We are looking for conditions under which $z_n \to z \neq 0$, $z \in L$ and $z = Kz + f$. Note that if $f = 0$, we will have a nonzero solution of homogeneous equation $z = Kz$. Denote

$$(P_\alpha^\beta z)(x) := \begin{cases} z(x), & \alpha \leqslant x \leqslant \beta, \\ 0, & x \notin [\alpha, \beta], \end{cases}$$

and[5]

$$K^{(0,\nu)} := P_0^\nu K P_0^\nu.$$

Below we sometimes write K instead of $K^{(0,\nu)}$ for simplicity of notation.

[5] In the case of an integral operator with a kernel $K(x, s)$, the $K^{(0,\nu)}$ is the restriction for the segment $[0, \nu]$.

▷ Let $\varrho_\nu = \varrho(K^{(0,\nu)})$ be the spectral radius of operator $K^{(0,\nu)}$.

Assume that the operators K, K_n satisfy the following three conditions; we call them *continuity, convergence, local compactness*:

Contin: Operators $K, K_n\colon L \to L_{\mathrm{loc}}$ are continuous.
Conver: $K_n z \xrightarrow{L_{\mathrm{loc}}} Kz$ for all $z \in L$.
LocCompact: For any $\nu > 0$ operators $K^{(0,\nu)}$ and $K_n^{(0,\nu)}$ are compact.

Definition A.1. A set $\Omega \subset L$ is said to be (L_{loc}, L)-*regular* if

$$\big(z_n \in \Omega \text{ and } z_n \xrightarrow{L_{\mathrm{loc}}} z\big) \quad \Rightarrow \quad \big(z \in L \text{ and } z_n \xrightarrow{L} z\big).$$

Definition A.2. A set $\Omega \subset L$ is said to be uniformly vanishing at ∞ if

$$\lim_{\nu\to\infty} \sup_{z\in\Omega} \int_\nu^\infty |z(s)|\,\mathrm{d}s = 0. \tag{A.3}$$

The same can be said about a sequence z_n, that is,

$$\lim_{\nu\to\infty} \sup_{n} \int_\nu^\infty |z_n(s)|\,\mathrm{d}s = 0.$$

Lemma A.1. *A necessary and sufficient condition for relative compactness of a set $\Omega \subset L_{\mathrm{loc}}$ is if for any $\nu > 0$ the set of restrictions to $L(0,\nu)$ of elements from Ω is relatively compact.*

Proof. The necessity is evident. Consider the sufficiency. Suppose $(z_n) \subset \Omega$ and $\nu_m \to \infty$. For each $m \geqslant 1$ there exists a sequence $z_k^{(m)}$ which is a subsequence of $z_k^{(m-1)}$, $z_k^{(0)} = z_k$, and it converges on $[0, \nu_m]$. It is clear that the diagonal sequence $z_k^{(k)}$ converges in L_{loc}. □

Lemma A.2. *If a set $\Omega \subset L$ is uniformly vanishing at ∞, then Ω is (L_{loc}, L)-regular.*

Proof. Let $z_n \in \Omega$ be convergent, $z_n \xrightarrow{L_{\mathrm{loc}}} z$. From (A.3) we can conclude that z_n is bounded. Therefore, inclusion $z \in L$ follows from the inequality

$$\int_0^\nu |z(s)|\,\mathrm{d}s \leqslant \int_0^\nu |z_n(s) - z(s)|\,\mathrm{d}s + \int_0^\nu |z_n(s)|\,\mathrm{d}s.$$

The convergence in L can be obtained from

$$\int_0^\infty |z_n(s) - z(s)|\,\mathrm{d}s \leqslant \int_0^\nu |z_n(s) - z(s)|\,\mathrm{d}s + \int_\nu^\infty |z_n(s)|\,\mathrm{d}s + \int_\nu^\infty |z(s)|\,\mathrm{d}s.$$

□

Lemma A.3. *If a set $\Omega \subset L$ is bounded and uniformly vanishing at ∞, then the set $\bigcup_n K_n\Omega$ is relatively compact in L_{loc}.*

Proof. By virtue of Lemma A.1 it is sufficient to show for any $\nu > 0$ the compactness of the set of restrictions $P_0^\nu \cup K_n\Omega$ to $[0,\nu]$. Given $\varepsilon > 0$ find $\delta > 0$ such that

$$\|z\| < \delta \Rightarrow \sup_n \|K_n z\|_{L(0,\nu)} < \varepsilon.$$

Find now $\alpha \geqslant \nu$ such that

$$\int_\alpha^\infty |z(s)|\,\mathrm{d}s < \delta \quad \forall\, z \in \Omega.$$

The set $\bigcup_n P_0^\nu K_n P_0^\alpha \Omega$ is a compact ε-net for $\bigcup_n P_0^\nu K_n \Omega$. Indeed, let $y = K_n\eta$, $\eta \in \Omega$, and $\widetilde{y} = K_n P_0^\alpha \eta$. Then $y - \widetilde{y} = K_n P_\alpha^\infty \eta$, $\|P_\alpha^\infty \eta\| < \delta$ and

$$\|y - \widetilde{y}\|_{L(0,\nu)} = \|K_n P_\alpha^\infty \eta\|_{L(0,\nu)} < \varepsilon.$$

□

Proposition A.1. *Suppose there exists a sequence z_n of solutions of equation* (A.2) *satisfying the conditions $\|z_n\| = 1$ and $f_n \to f$ in L_{loc}. If z_n is uniformly vanishing at ∞, then a subsequence z_{n_k} converges to a solution of equation* (A.1), *and $\|z\| = 1$.*

Proof. By virtue of Lemma A.3 there exists a subsequence z_{n_k} such that $K_{n_k} z_{n_k} \to y$ in L_{loc}. From (A.2) we have $z_{n_k} \xrightarrow{L_{\mathrm{loc}}} z = y + f$. From Lemma A.2, $z \in L$ and $z_{n_k} \xrightarrow{L} z$. Since $\|z_n\| = 1$, $\|z\| = 1$. Now

$$z = \lim z_{n_k} = \lim(K_{n_k} z_{n_k} + f_{n_k}) = Kz + f$$

since $\|K_n z_n - K_n z\|_{L(0,\nu)} \leqslant \sup \|K_n\|_\nu \cdot \|z_n - z\|_{L(0,\nu)}$. Here $\|K_n\|_\nu$ stands for the norm of operator $K_n\colon L(0,\nu) \to L(0,\nu)$. Note that the sequence $\|K_n\|_\nu$ is bounded by virtue of the Banach (Uniform Boundedness Principle) theorem. □

Corollary A.1. *Let $\omega(x)$ be a weight function positive on $[0,\infty)$. Assume that operator $K_\omega := \omega K \omega^{-1}$ acts in L and $\varrho(K_\omega) \leqslant 1$. Suppose that $0 < \lambda_n < 1$, $\lambda_n \to 1$, $f_n \xrightarrow{L_{\mathrm{loc}}} 0$ such that the solution z_n of equation*

$$z_n = \lambda_n K z_n + f_n$$

has the norm $\|z_n\| = 1$. *If the sequence* z_n *is uniformly vanishing on* ∞, *then there exists a subsequence* z_{n_k} *which converges to a nontrivial solution of equation*

$$z = Kz$$

and $\|z\| = 1$.

If $\varrho(K_\omega) < 1$, *then the solution* z *is non-finite.*

Proof. It is evident that $K_n = \lambda_n K$ satisfies conditions Contin, Conver and LocCompact.

Suppose $\varrho(K_\omega) < 1$ and $z(t) = 0$ on (ν, ∞) for $\nu > 0$. Then $y = \omega z \in L$ is a nonzero solution of equation $y = K_\omega y$ which contradicts the uniqueness of the solution. □

Corollary A.2. *Suppose that* $\varrho_\nu < 1$ *for any* $\nu > 0$, *and* $f_\nu \xrightarrow{L_{\mathrm{loc}}} 0$ *when* $\nu \to \infty$ *such that the solution of equation*

$$z_\nu = K^{(0,\nu)} z_\nu + f_\nu \tag{A.4}$$

has the norm $\|z_\nu\| = 1$. *If the set* z_ν *is uniformly vanishing on* ∞, *then there exists a sequence* z_{ν_k} *converging to a non-finite solution of equation* (A.1), *and* $\|z\| = 1$.

Proof. It is evident that operator $K^{(0,\nu)}$ satisfies condition Contin. To prove Conver note that $K - K^{(0,\nu)} = P_0^\nu K P_\nu^\infty + P_\nu^\infty K$. If $z \in L$, then $K^{(0,\nu)} z \xrightarrow{L_{\mathrm{loc}}} Kz$ when $\nu \to \infty$.

Suppose $z(x) = 0$ for $x > \nu$. Then z satisfies an equation $z_\nu = K^{(0,\nu)} z_\nu$. This contradicts $\varrho_\nu < 1$. Thus, z is a non-finite function. □

A.2. Integral operator. Here, consider the question of existence of a non-finite non-negative solution $z \in L$ of the equation

$$z(x) - \int_0^\infty K(x,s) z(s)\,\mathrm{d}s = 0. \tag{A.5}$$

Assume that everywhere below the function $K(x,s)$ satisfies the following conditions:

Positiv: $K(x,s) \geqslant 0$, $x, s \geqslant 0$;

Contin(I): $K(x,s)$ is measurable as function of two variables, and

$$\operatorname*{ess\,sup}_{s\in[0,\infty)} \int_0^\nu K(x,s)\,\mathrm{d}x < \infty \quad \forall \nu > 0; \tag{A.6}$$

LocCompact(I): For any $\nu > 0$ the vector function $s \mapsto K(x,s)$, $s, x \in [0,\nu]$ with values in L is essentially compact, that is, a set of values of this function for a subset of full measure from $[0,\nu]$ is compact in L.

Note that from condition Contin(I) it follows the continuity of the integral operator K with kernel $K(x,s)$ as operator from L to L_{loc} (see [9], Theorem 6.8, condition (6.39)). Condition LocCompact(I) is a necessary and sufficient condition for LocCompact (see [9], Theorem 6.6, page 116). This allows to use Proposition A.1 and Corollaries A.1 and A.2.

We assume that one more important condition is satisfied:

$$\liminf_{s\to\infty}\int_0^\infty K(x,s)\,\mathrm{d}x=\infty. \tag{A.7}$$

In accordance with the notation in Section A.1, ϱ_ν denotes the spectral radius $\varrho(K^{(0,\nu)})$ of the integral operator $K^{(0,\nu)}$ with the kernel $K(x,s)$ which is considered on the interval $[0,\nu]$.

Theorem A.1. *Suppose there exists a weight function $\omega(x)$ positive on $[0,\infty)$ such that $\varrho(K_\omega)\leqslant 1$, where K_ω is an integral operator with the kernel*

$$K_\omega(x,s)=\frac{\omega(x)}{\omega(s)}K(x,s).$$

Then equation (A.5) *has a nontrivial non-negative solution $z(x)$. If $\varrho(K_\omega)<1$, then the solution $z(x)$ is non-finite.*

Proof. Let us use Corollary A.1. It is sufficient to show the sequence z_n is uniformly vanishing. It follows from (A.7) and from the following inequalities for any $\nu>0$ that

$$\begin{aligned}1&=\int_0^\infty z_n(x)\,\mathrm{d}x\geqslant\int_0^\infty\biggl(\int_0^\infty K(x,s)z_n(s)\,\mathrm{d}s\biggr)\,\mathrm{d}x\\&=\int_0^\infty z_n(s)\biggl(\int_0^\infty K(x,s)\,\mathrm{d}x\biggr)\,\mathrm{d}s\geqslant\biggl(\int_\nu^\infty z_n(s)\,\mathrm{d}s\biggr)\operatorname*{ess\,inf}_{s\in[\nu,\infty)}\int_0^\infty K(x,s)\,\mathrm{d}x.\end{aligned}$$

□

Theorem A.2. *If $\varrho_\nu<1$ for any $\nu>0$, then equation* (A.5) *has a non-finite non-negative solution $z(x)$.*

Proof. Since $K^{(0,\nu)}$ is compact, there exists a non-negative nonzero solution of equation (A.4). Condition (A.7) can be used to prove the family z_ν, $\nu>0$ is uniformly vanishing in the same way as it was done in Theorem A.1.

Now from Corollary A.2 it can be concluded that in L there exists a non-finite solution of $\eta_0=K\eta_0$. This solution is non-negative as well as a limit of non-negative solutions of equation (A.4). □

Lemma A.4. *Let the kernel $K(x,s)$ be non-decreasing in s. Suppose there exists a non-finite non-negative solution of the inequality*

$$z(x) \geqslant \int_0^\infty K(x,s)z(s)\,\mathrm{d}s. \tag{A.8}$$

Then for any $\nu > 0$ the spectral radius $\varrho_\nu = \varrho(K^{(0,\nu)})$ is less than unit.

Proof. For any $\nu > 0$ operator $K^{(0,\nu)}$ is positive and compact. If $\varrho_\nu > 0$, then ϱ_ν is a positive eigenvalue of $K^{(0,\nu)}$ and of its conjugate operator (see [10]), and their eigenfunctions are non-negative. Let $f_0 \geqslant 0$ be an eigenfunction of the conjugate operator

$$\varrho_\nu f_0(s) = \int_0^\nu K(x,s)f_0(x)\,\mathrm{d}x. \tag{A.9}$$

We have

$$\begin{aligned}\int_0^\nu f_0(x)\eta(x)\,\mathrm{d}x &\geqslant \int_0^\nu f_0(x)\,\mathrm{d}x \int_0^\infty K(x,s)\eta(s)\,\mathrm{d}s = \int_0^\infty \eta(s)\,\mathrm{d}s \int_0^\nu K(x,s)f_0(x)\,\mathrm{d}x \\ &= \varrho_\nu \int_0^\nu f_0(s)\eta(s)\,\mathrm{d}s + \int_\nu^\infty \eta(s)\,\mathrm{d}s \int_0^\nu K(x,s)f_0(x)\,\mathrm{d}x.\end{aligned}$$

Since η is non-finite, $K(x,\cdot)$ is not decreasing, for $s \geqslant \nu$ the inequality

$$\int_0^\nu K(x,s)f_0(x)\,\mathrm{d}x \geqslant \int_0^\nu K(x,\nu)f_0(x)\,\mathrm{d}x = \varrho_\nu f_0(\nu)$$

is fulfilled. Thus,

$$(1-\varrho_\nu)\int_0^\nu f_0(s)\eta(s)\,\mathrm{d}s \geqslant \varrho_\nu f(\nu)\int_\nu^\infty \eta(s)\,\mathrm{d}s.$$

We see from (A.9) that $f_0(x)$ is non-decreasing on $[0,\nu]$. Thus, $f_0(\nu) > 0$ and $\varrho_\nu < 1$. □

Theorem A.3. *Let the kernel $K(x,s)$ be non-decreasing with respect to s. If there exists a non-finite non-negative solution of inequality* (A.8)*, then equation* (A.5) *has a non-finite non-negative solution $z(x)$.*

Proof. By virtue of Lemma A.4 for any $\nu > 0$ the spectral radius $\varrho_\nu < 1$. It is sufficient to refer to Theorem A.2. □

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