ON MINIMAL SPECTRUM OF MULTIPLICATION LATTICE MODULES

SACHIN BALLAL, Vilas Kharat, Pune

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Abstract. We study the minimal prime elements of multiplication lattice module $M$ over a $C$-lattice $L$. Moreover, we topologize the spectrum $\pi(M)$ of minimal prime elements of $M$ and study several properties of it. The compactness of $\pi(M)$ is characterized in several ways. Also, we investigate the interplay between the topological properties of $\pi(M)$ and algebraic properties of $M$.

Keywords: prime element; minimal prime element; Zariski topology

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1. Introduction

The notion of minimal prime elements of a lattice module is a generalization of minimal prime elements of a multiplicative lattice. The prime and minimal prime elements of multiplicative lattice were introduced and studied by Thakare, Manjarekar and Maeda [12], Thakare and Manjarekar [11], and the minimal prime ideals of 0-distributive lattices by Pawar and Thakare [9]. Keimel [7] unified the study of minimal prime ideals for various structures, e.g. commutative rings, distributive lattices, lattice ordered groups, $f$-rings. In this paper, we have carried out investigations leading to the study of generalizations of notions in commutative rings and multiplicative lattices along the lines of Dilworth (see [6]).

A complete lattice $L$ with the least element 0 and the greatest element 1 is said to be a multiplicative lattice if a binary operation “$\cdot$” called multiplication on $L$ satisfying the following conditions is defined:

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(1) $a \cdot b = b \cdot a$ for all $a, b \in L$,
(2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in L$,
(3) $a \cdot \bigvee_{a} b_{a} = \bigvee_{a} (a \cdot b_{a})$ for all $a, b_{a} \in L$,
(4) $a \cdot 1 = a$ for all $a \in L$.

Henceforth, $a \cdot b$ will be simply denoted by $ab$.

An element $p \neq 1$ of a multiplicative lattice $L$ is said to be prime if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. A prime element $p \in L$ is said to be a minimal prime over an element $a \in L$ if $a \leq p$ and whenever there is a prime element $q \in L$ with $a < q \leq p$, then $q = p$. In $L$, a minimal prime element over 0 will be called a minimal prime element of $L$. For $a \in L$, its radical is denoted by $\sqrt{a}$ and defined as $\sqrt{a} = \bigvee\{x \in L : x^{n} \leq a \text{ for some } n \in \mathbb{Z}^{+}\}$. An element $a \in L$ is called semiprime or radical if $\sqrt{a} = a$.

An element $a \in L$ is said to be compact if $a \leq \bigvee X$, $X \subseteq L$ implies that there exists a finite number of elements $x_{1}, x_{2}, \ldots, x_{n} \in X$ such that $a \leq \bigvee_{i=1}^{n} x_{i}$. We denote the set of all compact elements of a multiplicative lattice $L$ by $L_{c}$. In a multiplicative lattice $L$, an element $a \in L$ is said to be nilpotent if $a^{n} = 0$ for some $n \in \mathbb{Z}^{+}$ and is said to be reduced if the only nilpotent element of $L$ is 0.

An element $e \in L$ is said to be meet principal or join principal if it satisfies the identity $a \wedge be = ((a : e) \wedge b)e$ or $(ae \lor b) : e = (b : e) \lor a$, respectively, for $a, b \in L$. Also, $e$ is said to be principal if it is both join and meet principal. A multiplicative lattice $L$ is said to be principally generated (PG) if every element of $L$ is a join of principal elements of $L$. A multiplicative lattice $L$ is said to be compactly generated (CG) if every element of $L$ is the join of compact elements of $L$. According to Alarcon et al. [1], if $L$ is a compactly generated multiplicative lattice with 1 compact, then maximal elements exist in $L$ and every maximal element is a prime element. Further, in a compactly generated multiplicative lattice, if every finite product of compact elements is a compact element, then prime elements and minimal primes over $a \in L$ exist (see [1]).

By a $C$-lattice we mean a multiplicative lattice $L$ with the greatest element 1, which is compact as well as multiplicative identity, that is, generated under joins by a multiplicatively closed subset $C$ of compact elements of $L$.

A complete lattice $M$ with the smallest element $0_{M}$ and the greatest element $1_{M}$ is said to be a lattice module over the multiplicative lattice $L$ or $L$-module if there is a multiplication between elements of $M$ and $L$, denoted by $aN$ for $a \in L$ and $N \in M$, which satisfies the following properties:

(1) $(ab)N = a(bN)$;
(2) $\bigvee_{a, \alpha} \bigvee_{N, \beta} N_{\beta} = \bigvee_{a, \alpha} (a_{\alpha}N_{\beta})$;
Let $M$ be a lattice module over a multiplicative lattice $L$. For $N \in M$ and $b \in L$, denote $(N : b) = \bigvee\{X \in M : aX \leq N\}$. If $a, b \in L$, we write $(a : b) = \bigvee\{x \in L : bx \leq a\}$. If $A, B \in M$, then $(A : B) = \bigvee\{x \in L : xB \leq A\}$.

An element $A \in M$ is called weak meet principal if $(B : A)A = B \wedge A$ for all $B \in M$; $A$ is called weak join principal if $bA = b \lor (0 : A)$ for all $b \in L$; and $A$ is weak principal if $A$ is both weak meet principal and weak join principal. Lattice module $M$ over a multiplicative lattice $L$ is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$.

An element $N \neq 1_M$ in $M$ is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e. $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$. An element $N \neq 1_M$ of $M$ is called a maximal element if for every element $B$ of $M$ such that $N \leq B$, either $N = B$ or $B = 1_M$. Let $M$ be an $L$-module. An element $N$ in $M$ is called compact if $N \leq \bigvee_{a \in I} A_a$ ($I$ is an indexed set) implies $N \leq \bigvee_{a \in I} A_{a_1} \vee A_{a_2} \vee \ldots \vee A_{a_n}$ for some subset $\{a_1, a_2, \ldots, a_n\}$ of $I$.

In this paper, a lattice module $M$ will be a multiplication lattice module, which is compactly generated with the largest element $1_M$ being compact and $L$ will be a $C$-lattice.

For general background and terminology of multiplicative lattice and multiplication lattice module, the reader may consult [1], [2], [4]–[6], [12], [11].

### 2. The Zariski topology

In [3], the Zariski topology over the prime spectrum $\text{Spec}(M)$ of a lattice module $M$ over a $C$-lattice $L$ has been studied by Ballal and Kharat. In [10], Phadatare et al. introduced and studied the concept of quasi-prime elements as a generalization of prime elements and also the Zariski topology on the quasi-prime spectrum of a lattice module $M$ over a $C$-lattice $L$.

In this paper most of the results in [12] and [11] are generalized.

**Definition 2.1.** Let $M$ be a lattice module over a multiplicative lattice $L$. An element $P \in M$ is called a minimal prime over an element $N \in M$ if $N \leq P$ and there is no other prime element $Q$ of $M$ such that $N \leq Q < P$.

**Lemma 2.2.** Let $M$ be a multiplication lattice module over a $C$-lattice $L$ and $(0_M : 1_M)$ be a radical element. Then for $x \in L$, $(0_M : x) = (0_M : x^n)$ for every integer $n \geq 1$. 

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Proof. Note that \((0_M : x) = \sqrt{\{N \in M : xN \leq 0_M\}}\) and as \(x^n \leq x\), we have \((0_M : x) \leq (0_M : x^n)\) for every integer \(n \geq 1\). Let \(N_1 = (0_M : x^n)\). Since \(M\) is a multiplication lattice module, \(N_1 = a_1M\) for some \(a \in L\). So \(x^n a^n 1_M \leq x^n a 1_M = 0_M\). Hence \(xa \leq \sqrt{(0_M : 1_M)} = (0_M : 1_M)\). So \(xa 1_M \leq 0_M\), i.e. \(N_1 \leq (0_M : x)\) and consequently \((0_M : x) = (0_M : x^n)\) for each integer \(n \geq 1\). \(\square\)

**Theorem 2.3** ([8]). Let \(M\) be a multiplication lattice module over a C-lattice \(L\) and \(a \in L\) be proper. A prime element \(P \in M\) with \(a_1M \leq P\) is minimal if and only if for \(x \in L_*\) with \(x 1_M \leq P\) there is an element \(y \in L_*\) such that \(y 1_M \not\leq P\) and \(x^n y 1_M \leq a_1M = N\) for some positive integer \(n\).

The following result characterizes a prime element to be a minimal prime.

**Theorem 2.4.** Let \(M\) be a multiplication lattice module over a C-lattice \(L\) and \((0_M : 1_M)\) be a radical element. A prime element \(P \in M\) is a minimal prime if and only if for \(x \in L_*\), \(P\) contains precisely one of \(x 1_M\) and \((0_M : x)\).

Proof. Suppose that the condition is true for prime element \(P \in M\). Let \(x \in L_*\) be such that \(x 1_M \leq P\) and \((0_M : x) \not\leq P\). Then there exists \(y \in L_*\) such that \(y 1_M \leq (0_M : x)\) but \(y 1_M \not\leq P\). Thus, \(xy 1_M \leq 0_M\) and hence \(x^n y 1_M \leq 0_M\) for every integer \(n \geq 1\). This shows that for each \(x \in L_*\) with \(x 1_M \leq P\) there exists an element \(y \in L_*\) such that \(y 1_M \not\leq P\) and \(x^n y 1_M \leq 0_M\). By Theorem 2.3, it follows that \(P\) is minimal.

Conversely, suppose that a prime element \(P \in M\) is minimal and also that \(x 1_M \leq P\) for \(x \in L_*\). Then by Theorem 2.3, there exists \(y \in L_*\) such that \(y 1_M \not\leq P\) and \(x^n y 1_M = 0_M\) for some positive integer \(n\). Consequently, \(y 1_M \leq (0_M : x^n)\). By Lemma 2.2, we have \((0_M : x^n) = (0_M : x)\) and hence \(y 1_M \leq (0_M : x)\). This implies that \((0_M : x) \not\leq P\).

Now, if \(x 1_M \not\leq P\) and \((0_M : x) \not\leq P\), then there exists \(y \in L_*\) such that \(y 1_M \leq (0_M : x)\) but \(y 1_M \not\leq P\). Hence, we have \(xy 1_M \leq 0_M\) and so \(xy 1_M \not\leq P\). But \(x 1_M \not\leq P\) and \(y 1_M \not\leq P\) together contradict the fact that \(P\) is a prime. This shows that \(P\) contains precisely one of \(x 1_M\) and \((0_M : x)\). \(\square\)

Let \(\sigma(M)\) be the set of prime elements of a lattice module \(M\). For an element \(N \in M\) we set \(V(N) = \{P \in \sigma(M) : N \leq P\}\). Taking the sets \(\{V(N) : N \in M\}\) as a base for closed sets, \(\sigma(M)\) becomes a topological space and this topology is called the Zariski topology (see [3]).

The restriction of the Zariski topology to the set of minimal prime elements \(\pi(M)\) makes it a topological space and it is called the minimal prime spectrum of \(M\).

The following results about a minimal prime spectrum are immediate.
Corollary 2.5. Let \( M \) be a multiplication lattice module over a reduced \( C \)-lattice \( L \). For \( a \in L \), \( V(0_M : a) = \pi(M) - V(a1_M) \). In particular, \( V(a1_M) \) and \( V(0_M : a) \) are disjoint open and closed sets.

Corollary 2.6. Let \( M \) be a multiplication lattice module over a reduced \( C \)-lattice \( L \) with \( 1_M \) being compact. Then \( \pi(M) \) is a Hausdorff space with a base of open and closed sets.

Definition 2.7 ([11]). A subset \( S \) of a multiplicative lattice \( L \) is said to be multiplicatively closed if \( x, y \in S \) implies \( xy \in S \), and is said to be sub-multiplicatively closed if \( x, y \in X \) implies \( a \leq xy \) for some \( a \in S \).

In order to characterize prime elements of lattice modules in terms of multiplicatively closed subset of \( L \), we need the following lemma.

Lemma 2.8 ([4]). Let \( M \) be a multiplication lattice module over a PG \( C \)-lattice \( L \) and \( N \in M \) with \( N < 1_M \). Then the following conditions are equivalent.

1. \( N \) is a prime element in \( M \).
2. \( (N : 1_M) \) is a prime element in \( L \).
3. There exists a prime element \( p \) in \( L \) with \( (0_M : 1_M) \leq p \) such that \( N = p1_M \).

For \( N \in M \) we define \( C(N) = \{ x \in L : x \not\in (N : 1_M) \} \).

Lemma 2.9. Let \( M \) be a multiplication lattice module over a PG \( C \)-lattice \( L \). An element \( P \in M \) is a prime if and only if \( C(P) \) is a multiplicatively closed subset of \( L \).

Proof. Suppose that \( P \in M \) is a prime and \( x, y \in C(P) \). Then \( x \not\in (P : 1_M) \) and \( y \not\in (P : 1_M) \). Since \( P \in M \) is a prime, by Lemma 2.8 we have that \( (P : 1_M) \in L \) is a prime. As \( x \not\in (P : 1_M) \), \( y \not\in (P : 1_M) \) and \( (P : 1_M) \) is a prime, \( xy \not\in (P : 1_M) \), i.e. \( xy \in C(P) \) and hence \( C(P) \) is multiplicatively closed.

Conversely, suppose that \( C(P) \) is a multiplicatively closed subset of \( L \) and \( xy1_M \leq P \) for \( x, y \in L \). Then \( xy \leq (P : 1_M) \) and so \( xy \not\in C(P) \). If \( x \not\in (P : 1_M) \) and \( y \not\in (P : 1_M) \), then \( x \in C(P) \), \( y \in C(P) \) and this contradicts the fact that \( C(P) \) is multiplicatively closed. Therefore \( x \leq (P : 1_M) \) or \( y \leq (P : 1_M) \), i.e. \( x1_M \leq P \) or \( y1_M \leq p \). Consequently, \( P \) is a prime. \( \square \)

Lemma 2.10 ([11]). Let \( a \) be an element of a \( C \)-lattice \( L \) and \( S \) be a multiplicatively closed subset of \( L \) satisfying the property \( s \not\leq a \) for all \( s \in S \). Then there is a multiplicatively closed subset \( S' \) of \( L \) containing \( S \) which is maximal with respect to the property \( s' \not\leq a \) for all \( s' \in S' \).
Lemma 2.11 ([11]). (Separation lemma) Let $S$ be a sub-multiplicatively closed subset of a $C$-lattice $L$. Suppose that $a \in L$ and $t \notin a$ for every $t \in S$. Then there exists a prime element $p \in L$ such that $a \leq p$ and it is maximal with respect to $t \notin p$ for each $t \in S$.

An element $a$ in a complete lattice $L$ is said to be completely join prime if $a \leq \bigvee S$, $S \subseteq L$ implies $a \leq s$ for some $s \in S$.

Lemma 2.12. Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$ and suppose every element of $L$ is a completely join prime. A prime element $P \in M$ with $a1_M \leq P$ is minimal if and only if $C(P)$ is a maximal multiplicatively closed subset of $L$ with $x \notin a$ for all $x \in C(P)$ and $a \in L$.

Proof. Suppose that $C(P)$ is a maximal multiplicatively closed subset of $L$ with $x \notin a$ for all $x \in C(P)$. By Lemma 2.11 there is a prime element $(Q : 1_M) \geq a$ that is maximal with respect to the property that $x \notin (Q : 1_M)$ for all $x \in C(P)$. Hence, by Lemma 2.9, $C(Q)$ is a multiplicatively closed subset of $L$. As $a \leq (Q : 1_M)$, we have $x \notin a$ for any $x \in C(Q)$. But $C(P)$ is a maximal multiplicatively closed subset of $L$ with the property that $x \notin a$ for all $x \in C(P)$, hence we must have $C(Q) \subseteq C(P)$.

Now, if $y \in C(P)$, then $y \notin (Q : 1_M)$ and hence $y \in C(Q)$. Consequently, we have $C(P) = C(Q)$. Now, let $z \leq (P : 1_M)$, i.e. $z \in C(P)$. Then $z \notin C(Q)$ and it implies that $z \leq (Q : 1_M)$ and it further implies $(P : 1_M) \leq (Q : 1_M)$. Similarly, we have $(Q : 1_M) \leq (P : 1_M)$ and hence $(P : 1_M) = (Q : 1_M)$. It follows that $P = Q$.

Now we show that $P$ is a minimal prime. Suppose that $P' \in M$ is a prime with $a \leq (P' : 1_M) < (P : 1_M)$. Then by Lemma 2.9, $C(P')$ is a multiplicatively closed subset of $L$ with $x \notin a$ for all $x \in C(P')$ and $C(P) \subseteq C(P')$. This contradicts the maximality of $C(P)$. Hence, $P$ is a minimal prime element of $M$ with $a1_M \leq P$.

Conversely, suppose that $P \in M$ is a minimal prime with $a1_M \leq P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of $L$ with $x \notin a$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed subset $S$ which contains $C(P)$ and $x \notin a$ for all $x \in S$. We show that $S = C(P')$, where $P' = p1_M$ and $p = \bigvee(L - S)$. Let $y \in C(P') = \{z \in L : z \notin \bigvee(L - S)\}$. This gives $y \notin \bigvee(L - S)$, i.e. $y \in S$ and $C(P') \subseteq S$. On the other hand, if $s \in S$, then $s \notin L - S$ and $s \notin \bigvee(L - S)$. As each element of $L$ is a completely join prime, we have $s \in C(P')$ and therefore $C(P) = C(P')$.

By the first part, as $S$ is a maximal multiplicatively closed subset of $L$ with respect to $x \notin a$ for all $x \in S$, we conclude that $P'$ is a minimal prime with $a1_M \leq P'$. Clearly, $C(P) \subseteq S = C(P')$ gives that $P' \leq P$ and since $P$ is minimal, we must have $P = P'$. Hence, $C(P) = S = C(P')$ is the required maximal multiplicatively closed subset of $L$ with $x \notin a$ for all $x \in M$ and $a \in L$. 

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For $N \in M$ define $\sqrt[N]{N} = \sqrt\{x \in L: x^n1_M \leq N\}1_M$.

**Theorem 2.13.** Let $L$ be a PG $C$-lattice in which every element is completely join prime and let $M$ be a multiplication lattice module over $L$. For $N \in M$, the radical $\sqrt[N]{N} = \bigwedge\{P: P$ is a minimal prime element of $M$ with $N \leq P\}$.

**Proof.** Observe that for a prime element $P \in M$ with $N \leq P$ we have $\sqrt[N]{N} \leq P$. Therefore $\sqrt[N]{N} \leq \bigwedge\{P: P$ is a minimal prime element of $M$ with $N \leq P\}$.

Now, let $x \in L_*$ be such that $x1_M \notin \sqrt[N]{N}$ and let $S = \{x^i: x^i \notin (N : 1_M)$ and $i$ is an integer$\}$. Observe that $S$ is a multiplicatively closed subset of $L$. By Lemma 2.10, there is a maximal multiplicatively closed set $S'$ such that $y \notin (N : 1_M)$ for $y \in S'$. Let $p' = \sqrt(L - S')$. Then $S' = C(p'1_M) = C(P')$. By Lemma 2.12, $P'$ is a minimal prime element of $M$ with $N \leq P'$. Clearly, $x \in C(P')$ and as such $x \notin (P : 1_M)$. This gives that $\bigwedge\{P: P$ is a minimal prime element of $M$ with $N \leq P\} \leq \sqrt[N]{N}$. Consequently, $\sqrt[N]{N} = \bigwedge\{P: P$ is a minimal prime element of $M$ with $N \leq P\}$. □

**Corollary 2.14.** Let $M$ be a lattice module over a reduced PG $C$-lattice $L$ and $N \in M$. Then for a prime element $P \in M$ with $N \leq P$ there exists a minimal prime element $Q \in M$ such that $N \leq Q \leq P$.

**Proof.** Suppose $P \in M$ is a prime element with $N \leq P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of $L$ with $x \notin (N : 1_M)$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed set $S$ such that $y \notin (N : 1_M)$ for all $y \in S$. Also, $C(Q) = S$, where $Q = p1_M = \sqrt(L - S)1_M$ is a minimal prime element of $M$ with $N \leq Q$ and $C(P) \subseteq C(Q) = S$ implies that $Q \leq P$. □

**Lemma 2.15** ([12]). Let $L$ be a $C$-lattice. Then each nonzero element of $L$ is contained in a maximal multiplicatively closed subset of $L$ not containing zero.

For $N \in M$ we set $U(N) = \{P \in \pi(M): N \notin P\}$.

**Theorem 2.16.** Let $L$ be a PG $C$-lattice in which every element is completely join prime and let $M$ be a multiplication lattice module over $L$. Then $(0_M : a) = \bigwedge U(a1_M) = \{P \in \pi(M): a1_M \notin P\}, a \in L$.

**Proof.** Suppose $P \in M$ is a minimal prime. Then by Theorem 2.4 we have $(0_M : a) \leq P$ when $a1_M \notin P$ and therefore $(0_M : a) \leq \bigwedge\{P \in \pi(M): a1_M \notin P\} = Q$. If $(0_M : a) < Q$, then there exists $x \in L_*$ such that $x1_M \in Q$ and $x1_M \notin (0_M : a)$. Clearly, $ax1_M \notin 0_M$ and so $ax \neq 0$. By Lemma 2.15, $ax$ is contained in some maximal multiplicatively closed subset $S$ of $L$ not containing 0. As proved in Lemma 2.12, $S = C(P)$, where $P = p1_M$ and $p = \sqrt(L - S)$ is a minimal prime element of $L$. Now $ax \in S$ implies $ax \notin (P : 1_M)$ and hence $ax1_M \notin P$. 91
Since $P$ is a minimal prime and $a_1 M \not\subseteq P$, we have $x_1 M \not\subseteq P$. Therefore $x_1 M \not\subseteq Q$, a contradiction and consequently, $(0_M : a) = \bigwedge\{P \in \pi(M) : a_1 M \not\subseteq P\}$.

\textbf{Theorem 2.17.} Let $L$ be a PG $C$-lattice in which every element is a completely join prime and let $M$ be a multiplication lattice module over $L$. Then $a_1 M = (0_M : (0_M : a_1 M))$ if and only if $a_1 M = \bigwedge\{P \in \pi(M) : a_1 M \not\subseteq P\}$, $a \in L$.

\textbf{Proof.} Suppose $a_1 M = (0_M : (0_M : a_1 M))$, $a \in L$. By Theorem 2.4 we have $\bigwedge\{P \in \pi(M) : (0_M : a) \not\subseteq P\} = \bigwedge\{P \in \pi(M) : a_1 M \not\subseteq P\}$. But $(0_M : (0_M : a_1 M)) = \bigwedge\{P \in \pi(M) : (0_M : a) \not\subseteq P\}$ gives that $a_1 M = \bigwedge\{P \in \pi(M) : a_1 M \not\subseteq P\}$.

Conversely, suppose that $a_1 M = \bigwedge\{P \in \pi(M) : a_1 M \not\subseteq P\}$. By Theorem 2.16 we have $(0_M : (0_M : a_1 M)) = \bigwedge\{P \in \pi(M) : (0_M : a) \not\subseteq P\}$. Now, by Theorem 2.4 we have $\bigwedge\{P \in \pi(M) : (0_M : a) \not\subseteq P\} = \bigwedge\{P \in \pi(M) : a_1 M \not\subseteq P\}$ and by assumption, $a_1 M = (0_M : (0_M : a_1 M))$.

\textbf{Theorem 2.18.} Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$. Then $(0_M : a) = \bigwedge\{V(0_M : a)\}$, $a \in L$.

\textbf{Proof.} Note that $(0_M : a) \subseteq \bigwedge\{V(0_M : a)\}$, $a \in L$ follows immediately. Now, let $x \in L_*$ be such that $x_1 M \not\subseteq (0_M : a)$. Then $ax_1 M \not\subseteq 0_M$ and so $ax \neq 0$. Therefore $ax$ is contained in some maximal multiplicatively closed subset $S$ of $L$. Then $S = V(P) = V(p_1 M)$, where $p = \bigvee(L - S)$ and $p$ is a minimal prime element of $L$. Now $ax \in C(P)$ implies $ax \not\subseteq (P : 1_M)$ and hence $ax_1 M \not\subseteq P$. Since $P$ is a minimal prime, we have $x_1 M \not\subseteq P$ and $a_1 M \not\subseteq P$. By Theorem 2.4 we have $(0_M : a) \subseteq P$ and hence $P \in V(0_M : a)$. As $x_1 M \not\subseteq P$, we have $x_1 M \not\subseteq \bigwedge\{V(0_M : a)\}$. Thus, $x_1 M \not\subseteq (0_M : a)$ implies $x_1 M \not\subseteq \bigwedge\{V(0_M : a)\}$, i.e. $\bigwedge\{V(0_M : a)\} \subseteq (0_M : a)$.

We now show that the minimal prime spectrum $\pi(M)$ is a completely regular Hausdorff space, i.e. a Tychonoff space.

\textbf{Theorem 2.19.} Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$. Then the topology on $\pi(M)$ for which the collection $\{U(a_1 M) : a \in L\}$ is a base for open sets is Tychonoff.

\textbf{Proof.} Suppose that $P_1, P_2 \in \pi(M)$ with $P_1 \neq P_2$. Clearly $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. Let $x \in L_*$ with $x_1 M \subseteq P_1$ be such that $x_1 M \not\subseteq P_2$. By Theorem 2.3, there is $y \in L_*$ with $y_1 M \subseteq P_1$ and $x^n y_1 M = 0_M$ for some integer $n$. If $y_1 M \not\subseteq P_2$, then this together with $x_1 M \not\subseteq P_2$ gives $x^n y_1 M \not\subseteq P_2$, which is a contradiction to the fact that $0_M \subseteq P_2$. Therefore $y_1 M \subseteq P_2$. Clearly, $P_1 \in U(y_1 M)$, $P_2 \in U(x_1 M)$ and
\[ U(x_1M) \cap U(y_1M) = \{ P \in \pi(M) : x_1M \not\leq P, y_1M \not\leq P \} = U(xy_1M) = U(x^n y_1M) = U(0_M) = \varphi. \] Consequently, \( \pi(M) \) is a Hausdorff space and hence singletons are closed.

Now, let \( Q \in \pi(M) \) and \( F \) be a closed subset of \( \pi(M) \) such that \( Q \notin F \). Then \( Q \in \pi(M) - F \) and \( \pi(M) - F \) is open in \( \pi(M) \). Then there is an open set \( U(s_1M) \) for some \( s \in L \) such that \( Q \in U(s_1M) \subseteq \pi(M) - F \). Define a function \( f \) on \( \pi(M) \) as \( f(Q) = 0_M \) if \( Q \in U(s_1M) \) and \( f(Q) = 1_M \) otherwise. Then \( f(Q) = 0_M \) and \( f(F) = 1 \). Note that \( f \) is continuous and hence \( \pi(M) \) is completely regular. Consequently, \( \pi(M) \) is a completely regular Hausdorff space, i.e. a Tychonoff space. \( \square \)

**Corollary 2.20.** \( \pi(M) \) is totally disconnected and zero dimensional space.

**Theorem 2.21.** Let \( M \) be a multiplication lattice module over a PG C-lattice \( L \). Let \( x, y \in L \). Then \( U(x_1M) \subseteq U(y_1M) \) if and only if \( 0_M : (0_M : x_1M) \leq 0_M : (0_M : y_1M) \). In addition, \( U(x_1M) = U(y_1M) \) if and only if \( (0_M : x) \leq (0_M : y) \).

**Proof.** Suppose that \( U(x_1M) \subseteq U(y_1M) \) for \( x, y \in L \). By Theorem 2.16 we have \( (0_M : y) \leq (0_M : x) \) and hence \( (0_M : y) \not\leq P \) which implies \( (0_M : x) \not\leq P \) and so \( \{ P \in \pi(M) : (0_M : y) \not\leq P \} \subseteq \{ P \in \pi(M) : (0_M : x) \not\leq P \} \). By Theorem 2.4 we have \( (0_M : x_1M) \leq (0_M : y_1M) \).

Conversely, suppose that \( (0_M : x_1M) \leq (0_M : y_1M) \). Therefore \( \{ P \in \pi(M) : (0_M : y) \not\leq P \} \subseteq \{ P \in \pi(M) : (0_M : x) \not\leq P \} \) and so \( \{ P \in \pi(M) : x_1M \leq P \} \subseteq \{ P \in \pi(M) : y_1M \leq P \} \) by Theorem 2.4. This gives \( \{ P \in \pi(M) : x_1M \not\leq P \} \subseteq \{ P \in \pi(M) : y_1M \not\leq P \} \) and therefore \( U(x_1M) \subseteq U(y_1M) \).

For the second part, suppose that \( U(x_1M) = U(y_1M) \). Then \( U(x_1M) \subseteq U(y_1M) \) implies \( 0_M : (0_M : x_1M) \leq 0_M : (0_M : y_1M) \) and \( U(y_1M) \subseteq U(x_1M) \) implies \( 0_M : (0_M : y_1M) \leq 0_M : (0_M : x_1M) \). Hence, \( 0_M : (0_M : y_1M) = 0_M : (0_M : x_1M) \) and \( 0_M : (0_M : y_1M) \leq 0_M : (0_M : x_1M) \). Consequently, \( (0_M : x) = (0_M : y) \).

Conversely, suppose that \( (0_M : x) = (0_M : y) \). Then \( 0_M : (0_M : x_1M) = 0_M : (0_M : y_1M) \), i.e. \( 0_M : (0_M : x_1M) \leq 0_M : (0_M : y_1M) \) and \( 0_M : (0_M : y_1M) \leq 0_M : (0_M : x_1M) \) and the result follows by the first part. \( \square \)

**Theorem 2.22.** Let \( M \) be a multiplication lattice module over a PG C-lattice \( L \). Let \( I \) be an indexing set and \( S = \{ x_r : r \in I \} \) be a set of points in \( L \) such that the collection of sets \( \{ U(x_r1M) : r \in I \} \) has the finite intersection property. Then the intersection of all \( \{ U(x_r1M) : r \in I \} \) is nonempty.

**Proof.** We have \( \bigcap_{r=1}^{n} U(x_r) = U(y_1M) \), where \( y = x_1 x_2 \ldots x_n \). Note that the multiplication of finite number of nonzero \( x_r, r \in I \) is nonzero. The collection of
all nonzero \(x_r, r \in I\) together with finite multiplication of \(x_r \in S\) is multiplicatively closed subset of \(L\) not containing 0. By Lemma 2.10, there is a maximal multiplicatively closed subset \(S'\) of \(L\) containing \(S\) and not containing 0. We have \(S' = C(P) = C(p1_M)\), where \(p = \sqrt{(L - S')}\) and \(p\) is a minimal prime element of \(L\). Clearly, \(P \in U(x_r1_M)\) for all \(x_r \in S'\). As \(S \subseteq S'\), we have \(P \in U(x_r1_M)\) for all \(x_r \in S\). Thus, \(P \in \bigcap_{r \in I} U(x_r1_M)\), which implies that \(\bigcap_{r \in I} U(x_r1_M) \neq \varnothing\). \(\square\)

If the family \(\{V(x1_M) : x \in L\}\) is considered as an open basis for \(\pi(M)\), the resulting topology is called the dual topology and denoted by \(\tau^d\). We denote the topology for which \(\{U(x1_M) : x \in L\}\) is an open basis by \(\tau\).

**Theorem 2.23.** Let \(M\) be a multiplication lattice module over a PG \(C\)-lattice \(L\). The topology \(\tau\) on \(\pi(M)\) for which \(\{U(x1_M) : x \in L\}\) is a basis for open sets is finer than the topology \(\tau^d\) on \(\pi(M)\) for which \(\{V(x1_M) : x \in L\}\) is a basis for open sets and moreover \(\tau = \tau^d\).

**Proof.** We know that \(\{V(x1_M) : x \in L\}\) is a basis for open sets for the topology on \(\pi(M)\) denoted by \(\tau^d\). Clearly, \(V(x1_M) = \pi(M) - U(x1_M)\) for all \(x \in L\). Note that for \(x \in L\), \(U(x1_M)\) is closed in \(\pi(M)\). Hence, \(V(x1_M)\) is open in the topology \(\tau\) for \(\pi(M)\), i.e. \(\tau\) is finer than \(\tau^d\).

Now, for any \(x \in L\) we have \(U(x1_M) = V(0_M : x)\). Thus, every basic open set in \(\tau\) is open in \(\tau^d\) and so we conclude that \(\tau = \tau^d\). \(\square\)

**Theorem 2.24.** Let \(M\) be a multiplication lattice module over a PG \(C\)-lattice \(L\). The following statements are equivalent in \(M\).

1. (1) \(\pi(M)\) is compact.
2. (2) The poset \(\{U(x1_M) : x \in L\}\), under set inclusion, is a Boolean lattice.
3. (3) For \(x \in L\) there exist \(N_1 = y_11_M, N_2 = y_21_M, \ldots, N_n = y_n1_M \in M\) with \(y_i1_M = N_i \leq (0_M : x)\) for \(i = 1, 2, \ldots, n\) and \((0_M : x) \land \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M\).
4. (4) For \(x \in L\) there exist \(N_1 = y_11_M, N_2 = y_21_M, \ldots, N_n = y_n1_M \in M\) such that \(0_M : (0_M : x1_M) = \bigwedge_{i=1}^{n} (0_M : y_i)\).
5. (5) \(\tau = \tau^d\).
6. (6) \(\{U(x1_M) : x \in L\}\) is a subbasis for open sets of \(\pi(M)\) with respect to the topology \(\tau\).
7. (7) \(\{V(x1_M) : x \in L\}\) is a subbasis for open sets of \(\pi(M)\) with respect to the topology \(\tau^d\).

**Proof.** (1) \(\Rightarrow\) (2): Clearly the set \(\{U(x1_M) : x \in L\}\) is partially ordered under set inclusion.
Now, we first show that
(i) \( U(x_1M) \cup U(y_1M) = U(x_1M \lor y_1M) \);
(ii) \( U(x_1M) \cap U(y_1M) = U(xy_1M) \).

Let \( P \in U(x_1M) \cup U(y_1M) \), then \( P \in U(x_1M) \) or \( P \in U(y_1M) \) and so \( x_1M \not\in P \) or \( y_1M \not\in P \). Therefore \( x_1M \lor y_1M \not\in P \) and this implies \( P \in U(x_1M \lor y_1M) \). Now, let \( Q \in U(x_1M \lor y_1M) \), then \( x_1M \lor y_1M \not\in Q \) and this implies that \( x_1M \not\in Q \) or \( y_1M \not\in Q \). Therefore \( Q \in U(x_1M) \cup U(y_1M) \). Consequently, \( U(x_1M) \cup U(y_1M) = U(x_1M \lor y_1M) \). Similarly, \( U(x_1M) \cap U(y_1M) = U(xy_1M) \).

From this we conclude that \( \{U(x_1M) : x \in L\} \cup \cap \) is a lattice.

Now, \( U(0_1M) = U(0_M) \) and \( U(1_1M) = U(1_M) = \pi(M) \). This shows that \( \{U(x_1M) : x \in L\} \cup \cap \) is a bounded lattice. Again, observe that \( U(x_1M) \cup U(y_1M) \cap U(z_1M) \) = \( (U(x_1M) \cup U(y_1M)) \cap (U(x_1M) \cup U(z_1M)) \) and \( U(x_1M) \cap (U(y_1M) \cup U(z_1M)) \) = \( (U(x_1M) \cap U(y_1M)) \cup (U(x_1M) \cap U(z_1M)) \). This shows that \( \{U(x_1M) : x \in L\} \cup \cap \) is a distributive lattice.

Finally, we show that \( \{U(x_1M) : x \in L\} \cup \cap \) is complemented. Note that for \( x \in L \) we have \( V(x_1M) \cap V(0_M : x) = \varphi \). Then \( V(x_1M) \cap \{V(N) : N \leq (0_M : x)\} = \varphi \). Since \( \pi(M) \) is compact, there exist \( N_1, N_2, \ldots, N_n \leq (0_M : x) \) such that \( V(x_1M) \cap \{V(N_i) : N_i \leq (0_M : x), i = 1, 2, \ldots, n\} = \varphi \). By taking complements in \( \pi(M) \), we get \( \pi(M) = U(x_1M) \cup U(N_1) \cup \cdots \cup U(N_n) \). Since each \( N_i \leq (0_M : x) \) for \( i = 1, 2, \ldots, n \), we have \( U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) = \varphi \). For, if \( P \in U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) \), then \( x_1M \not\in P \), which implies \( (0_M : x) \not\in P \). Therefore \( N_i \leq P \) for \( i = 1, 2, \ldots, n \), a contradiction as \( P \in \bigcup_{i=1}^{n} U(N_i) \) and so \( N_k \not\in P \) for some \( k, 1 \leq k \leq n \). Thus, we have \( \pi(M) = U(x_1M) \cup \bigcup_{i=1}^{n} U(N_i) \) and \( U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) = \varphi \). Consequently, \( \{U(x_1M) : x \in L\} \cup \cap \) is a Boolean lattice.

(2) \( \Rightarrow \) (3): Suppose that the finite union of \( \{U(x_1M) : x \in L\} \) forms a Boolean lattice and suppose that the complement of \( U(x_1M) \) is \( \bigcup_{i=1}^{n} U(N_i) \). As \( U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) = \varphi \), we get \( U(x_1M) \cap U(N_i) = \varphi \), \( i = 1, 2, \ldots, n \). Therefore \( \{P \in \pi(M) : xN_i \not\in P \} = \varphi \), \( i = 1, 2, \ldots, n \), i.e. \( U(xN_i) = \varphi \) for \( i = 1, 2, \ldots, n \), which implies \( xN_i = 0_M \) for \( i = 1, 2, \ldots, n \). Thus \( N_i \leq (0_M : x) \) for \( i = 1, 2, \ldots, n \). Also, \( \pi(M) = U(x_1M) \cup \bigcup_{i=1}^{n} U(N_i) \) gives \( \bigwedge(\pi(M)) = \bigwedge\left(U(x_1M) \cup \bigcup_{i=1}^{n} U(N_i)\right) \), i.e. \( 0_M = \bigwedge(\pi(M)) = \bigwedge\left(U(x_1M) \lor \bigvee_{i=1}^{n} N_i\right) \). Note that \( \bigwedge\left(U(x_1M) \lor \bigvee_{i=1}^{n} N_i\right) = \bigwedge(U(x_1M)) \land \bigwedge_{i=1}^{n} (U(N_i)) \). Then by Theorem 2.16 we have \( (0_M : x) \land \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M \).

(3) \( \Rightarrow \) (4): Suppose that (3) holds. Then for any \( x \in L \) there exist \( N_1 = y_11_M, N_2 = y_21_M, \ldots, N_n = y_n1_M \in M \) with \( y_i1_M = N_i \leq (0_M : x) \) for \( i = 1, 2, \ldots, n \).
and \((0_M : x) \land \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M\). This implies \((0_M : x_1M) \land \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M\)

i.e. \(\bigwedge_{i=1}^{n} (0_M : y_i) \subseteq (0_M : (0_M : x_1M))\). Also note that \((0_M : (0_M : x_1M)) \leq (0_M : y_i)\)

for \(i = 1, 2, \ldots, n\). Hence \((0_M : (0_M : x_1M)) \leq \bigwedge_{i=1}^{n} (0_M : y_i)\) and consequently,

\((0_M : (0_M : x_1M)) = \bigwedge_{i=1}^{n} (0_M : y_i)\).

\((4) \Rightarrow (5)\): Let \(x\) be an element of \(L\). By \((4)\), there exist \(N_1 = y_11M, N_2 = y_21M, \ldots, N_n = y_n1M \in M\) such that \((0_M : (0_M : x_1M)) = \bigwedge_{i=1}^{n} (0_M : y_i)\). Hence we have

\[V(0_M : (0_M : x_1M)) = V\left(\bigwedge_{i=1}^{n} (0_M : y_i)\right) = \bigcup_{i=1}^{n} V(0_M : y_i) = \bigcup_{i=1}^{n} U(y_i1M) = V(x1M)\]

Taking complements in \(\pi(M)\), we have \(\pi(M) - V(x1M) = \pi(M) - \bigcup_{i=1}^{n} U(y_i1M)\),

i.e. \(U(x1M) = \bigcap_{i=1}^{n} V(y_i1M)\). It follows that \(U(x1M)\) is a finite intersection of open sets in dual topology \(\tau^d\). Hence, \(U(x1M)\) is open in \(\tau^d\), which implies \(\tau^d\) is finer than \(\tau\), and \(\tau\) is finer than \(\tau^d\) follows by Theorem 2.23.

\((5) \Rightarrow (1)\): Suppose that \(\tau = \tau^d\). Then \(\{U(x1M) : x \in L\}\) is also a base for closed sets in \(\pi(M)\). Let \(\{U(y1M) : y \in K\}\) be a family of closed sets with finite intersection property in \(\pi(M)\), where \(K \subseteq L\). Then \(\bigcap_{i=1}^{n} U(y_i1M) = U(y_1y_2\ldots y_n1M) \neq \varnothing\) and so \(y_1y_2\ldots y_n1M \neq 0_M\) for any finite number of elements \(y_1, y_2, \ldots, y_n \in K\). All the nonzero elements in \(K\) together with the finite multiplication of elements in \(K\)

form a multiplicatively closed set not containing 0. This multiplicatively closed set is again contained in some maximal multiplicatively closed set \(S\) not containing 0.

As proved in Lemma 2.12, \(S = C(P) = C(p1M)\), where \(p = \sqrt{(L - S)}\) is a minimal prime element of \(L\). Note that \(K \subseteq C(P)\) and therefore \(P \in U(y1M)\) for all \(y \in K\). Thus, \(p \in \bigcap\{U(y1M) : y \in K\} \neq \varnothing\) and so \(\pi(L)\) is compact.

\((5) \Rightarrow (6)\): The implication follows immediately as \(\{V(x1M) : x \in L\}\) is a basis for open sets in \(\tau^d\).

\((6) \Rightarrow (5)\): Let \(\{U(x1M) : x \in L\}\) be any basis for open sets in \(\tau\). Then we have \(U(x1M) = \bigcap_{i=1}^{n} V(x_i)\) as \(\{V(x1M) : x \in L\}\) is a subbasis for open sets in \(\pi(M)\) with respect to \(\tau\). This implies that \(\{U(x1M) : x \in L\}\) is open in \(\tau^d\) and hence \(\tau \subseteq \tau^d\)

and the result follows by Theorem 2.23.

\((6) \Rightarrow (7)\): If \(\{V(x1M) : x \in L\}\) is a subbasis for open sets in \(\tau\), then \(\{\pi(M) - V(x1M) : x \in L\}\) = \(\{U(x1M) : x \in L\}\) forms a subbasis for open sets in \(\tau^d\) and conversely. \(\square\)

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Authors’ address: Sachin Ballal, Vilas Kharat, Department of Mathematics, Savitribai Phule Pune University, Pune-411 007, India, e-mail: ballalshyam@gmail.com, laddoo1@yahoo.com.