NEW EXTENSION OF THE VARIATIONAL MCSHANE INTEGRAL
OF VECTOR-VALUED FUNCTIONS

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Dedicated to the memory of Professor Štefan Schwabik (1941–2009)

Abstract. We define the Hake-variational McShane integral of Banach space valued functions defined on an open and bounded subset $G$ of $m$-dimensional Euclidean space $\mathbb{R}^m$. It is a “natural” extension of the variational McShane integral (the strong McShane integral) from $m$-dimensional closed non-degenerate intervals to open and bounded subsets of $\mathbb{R}^m$.

We will show a theorem that characterizes the Hake-variational McShane integral in terms of the variational McShane integral. This theorem reduces the study of our integral to the study of the variational McShane integral. As an application, a full descriptive characterization of the Hake-variational McShane integral is presented in terms of the cubic derivative.

Keywords: Hake-variational McShane integral; variational McShane integral; Banach space; $m$-dimensional Euclidean space

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1. INTRODUCTION AND PRELIMINARIES

In paper [4], Fremlin studies, in a $\sigma$-finite outer regular quasi-Radon measure space, a method of integration of vector-valued functions, which is an essential generalization of the McShane process of integration (see [11]). The method involves (infinite) McShane partitions which are formed by sequences of disjoint measurable sets of finite measure. For a Banach-space valued function defined on a closed interval endowed with the Lebesgue measure, the variational McShane integral has been investigated in [17] by Wu and Xiaobo (who called the integral the strong McShane integral). Wu and Xiaobo showed that a Banach-space valued function is variationally McShane integrable if and only if it is Bochner integrable. Di Piazza and Musial
have proved a surprising result that, in the case of an arbitrary (even finite) quasi-Radon measure space, the class of variationally McShane integrable functions can be significantly larger, see [2], Theorem 1.

In paper [8], the Hake-Henstock-Kurzweil and the Hake-McShane integrals are defined. These are extensions of the Henstock-Kurzweil and the McShane integrals from \( m \)-dimensional closed non-degenerate intervals to open and bounded subsets of \( \mathbb{R}^m \).

In this paper, we define the Hake-variational McShane integral which is an extension of the variational McShane integral from \( m \)-dimensional closed non-degenerate intervals to open and bounded subsets of \( \mathbb{R}^m \). Our goal is not a generalization for the sake of generalization. Indeed, Theorem 2.1 reduces the study of our integral to the study of the variational McShane integral. As an application, a full descriptive characterization of the Hake-variational McShane integral is presented in terms of the cubic derivative, see Theorem 2.2.

Throughout this paper, \( X \) denotes a real Banach space with its norm \( \| \cdot \| \). The Euclidean space \( \mathbb{R}^m \) is equipped with the maximum norm. \( B_m(t, r) \) denotes an open ball in \( \mathbb{R}^m \) with center \( t \) and radius \( r > 0 \). We denote by \( \mathcal{L}(\mathbb{R}^m) \) the \( \sigma \)-algebra of Lebesgue measurable subsets of \( \mathbb{R}^m \) and by \( \lambda \) the Lebesgue measure on \( \mathcal{L}(\mathbb{R}^m) \). \( |A| \) denotes the Lebesgue measure of \( A \in \mathcal{L}(\mathbb{R}^m) \). \( G \) denotes an open and bounded subset of \( \mathbb{R}^m \). We put \( L(A) = \{ A \cap L : L \in \mathcal{L}(\mathbb{R}^m) \} \) for any \( A \in \mathcal{L}(\mathbb{R}^m) \).

Let \( \alpha = (a_1, \ldots, a_m) \) and \( \beta = (b_1, \ldots, b_m) \) with \( -\infty < a_j < b_j < \infty \) for \( j = 1, \ldots, m \). The set \( [\alpha, \beta] = \prod_{j=1}^{m} [a_j, b_j] \) is called a closed non-degenerate interval in \( \mathbb{R}^m \). In particular, if \( b_1 - a_1 = \ldots = b_m - a_m \), then \( I = [\alpha, \beta] \) is called a cube. Two closed non-degenerate intervals \( I \) and \( J \) in \( \mathbb{R}^m \) are said to be non-overlapping if \( I^\circ \cap J^\circ = \emptyset \), where \( I^\circ \) denotes the interior of \( I \). The family of all closed non-degenerate subintervals in \( \mathbb{R}^m \) is denoted by \( \mathcal{I} \) and the family of all closed non-degenerate subintervals in \( E \subset \mathbb{R}^m \) is denoted by \( \mathcal{I}_E \).

An interval function \( F : \mathcal{I}_E \to X \) is said to be an additive interval function if for each two non-overlapping intervals \( I, J \in \mathcal{I}_E \) such that \( I \cup J \in \mathcal{I}_E \) we have

\[
F(I \cup J) = F(I) + F(J).
\]

A pair \( (t, I) \) of a point \( t \in E \) and an interval \( I \in \mathcal{I}_E \) is called an \( \mathcal{M} \)-tagged interval in \( E \), \( t \) is the tag of \( I \). A finite collection \( \{ (t_i, I_i) : i = 1, \ldots, p \} \) of \( \mathcal{M} \)-tagged intervals in \( E \) is called an \( \mathcal{M} \)-partition in \( E \) if \( \{ I_i : i = 1, \ldots, p \} \) is a collection of pairwise non-overlapping intervals in \( \mathcal{I}_E \). Given \( Z \subset E \), a positive function \( \delta : Z \to (0, \infty) \) is called a gauge on \( Z \). We say that an \( \mathcal{M} \)-partition \( \pi = \{ (t_i, I_i) : i = 1, \ldots, p \} \) in \( E \) is
\(\mathcal{M}\)-partition of \(E\) if \(\bigcup_{i=1}^{p} I_i = E\),

\(Z\)-tagged if \(\{t_1, \ldots, t_p\} \subset Z\),

\(\delta\)-fine if for each \(i = 1, \ldots, p\) we have \(I_i \subset B_m(t_i, \delta(t_i))\).

We now fix an interval \(W \in \mathcal{I}\) and let \(f : W \to X\) be a function. The function \(f\) is said to be *McShane integrable* on \(W\) if there is a vector \(x_f \in X\) such that for every \(\varepsilon > 0\) there exists a gauge \(\delta\) on \(W\) such that for every \(\delta\)-fine \(\mathcal{M}\)-partition \(\pi\) of \(W\) we have

\[
\left\| \sum_{(t, I) \in \pi} f(t)|I| - x_f \right\| < \varepsilon.
\]

In this case, vector \(x_f\) is said to be the *McShane integral* of \(f\) on \(W\) and we set

\[
x_f = \left(\int W f \, d\lambda\right)
\]

Function \(f\) is said to be *variationally McShane integrable* (or *strongly McShane integrable*) on \(W\) if there exists an additive interval function \(F : \mathcal{I}_W \to X\) such that for every \(\varepsilon > 0\) there exists a gauge \(\delta\) on \(W\) such that for every \(\delta\)-fine \(\mathcal{M}\)-partition \(\pi\) of \(W\) we have

\[
\sum_{(t, I) \in \pi} \|f(t)|I| - F(I)\| < \varepsilon.
\]

Function \(F\) is said to be the primitive of \(f\). Clearly, if \(f\) is variationally McShane integrable with the primitive \(F\), then \(f\) is McShane integrable, and by Proposition 3.6.16 in [13] we also have

\[
F(I) = \left(\int I f \, d\lambda\right) \text{ for every } I \in \mathcal{I}_W.
\]

For more information about the McShane integral we refer to [17], [2], [4], [5]–[7], [11], [10], [14] and [13].

Given an additive interval function \(F : \mathcal{I}_W \to X\), a subset \(Z \subset W\) and a gauge \(\delta\) on \(Z\), we define

\[
V_M F(Z, \delta) = \sup \left\{ \sum_{(t, I) \in \pi} \|F(I)\| : \pi \text{ is a } Z\text{-tagged } \delta\text{-fine } \mathcal{M}\text{-partition in } W \right\}.
\]

Then we set

\[
V_M F(Z) = \inf \{V_M F(Z, \delta) : \delta \text{ is a gauge on } Z\}.
\]
The set function $V_{M}F(\cdot)$ is said to be the McShane variational measure generated by $F$. It is a Borel metric outer measure on $W$, see [1] or [15]. The McShane variational measure has been used extensively for studying the primitives (indefinite integrals) of real functions. See e.g. paper [1] by Di Piazza, [12] by Pfeffer for relations to integration and the fundamental general work [16] by Thomson.

An additive interval function $F: I_{G} \to X$ is said to be strongly absolutely continuous (sAC) on $E \subset G$ if for each $\varepsilon > 0$ there exists $\eta > 0$ such that for each finite collection $\{I_{1}, \ldots, I_{p}\}$ of pairwise non-overlapping subintervals in $I_{E}$ we have

$$\sum_{i=1}^{p} |I_{i}| < \eta \Rightarrow \sum_{i=1}^{p} \|F(I_{i})\| < \varepsilon.$$ 

Assume that a point $t \in G$ is given. We set

$$I_{G}(t) = \{I \in I_{G}: t \in I, I \text{ is a cube}\}.$$

We say that $F$ is cubic differentiable at $t$ if there exists a vector $F'_{c}(t) \in X$ such that

$$\lim_{I \in I_{G}(t) \atop |I| \to 0} \frac{F(I)}{|I|} = F'_{c}(t);$$

$F'_{c}(t)$ is said to be the cubic derivative of $F$ at $t$.

A sequence $(I_{k})$ of pairwise non-overlapping intervals in $I_{G}$ is said to be a division of $G$ if

$$\bigcup_{k=1}^{\infty} I_{k} = G.$$

We denote by $D_{G}$ the family of all divisions of the set $G$. By Lemma 2.43 in [3], the family $D_{G}$ is not empty.

An additive interval function $F: I_{G} \to X$ is said to be a strong-Hake-function if for each division $(I_{k})$ of $G$ we have:

$\triangleright$ the series $\sum_{\{k: |I \cap I_{k}| > 0\}} \|F(I \cap I_{k})\|$ converges in $\mathbb{R}$ for each $I \in I$,

$\triangleright$ $F(I) = \sum_{\{k: |I \cap I_{k}| > 0\}} F(I \cap I_{k})$ for all $I \in I_{G}$.

We say that the additive interval function $F: I_{G} \to X$ has the strong-$M$-negligible variation over a subset $Z \subset \mathbb{R}^{m}$ if for each $\varepsilon > 0$ there exists a gauge $\delta_{\varepsilon}$ on $Z$ such that for each $Z$-tagged $\delta_{\varepsilon}$-fine $M$-partition $\pi$ in $\mathbb{R}^{m}$ we have:

$\triangleright$ the series $\sum_{\{k: |I \cap I_{k}| > 0\}} F(I \cap I_{k})$ is unconditionally convergent in $X$ for each $(t, I) \in \pi$,
\[
\sum_{(t,I) \in \pi} \left\| \left( \sum_{\{k: |I \cap I_k| > 0\}} F(I \cap I_k) \right) \right\| < \varepsilon
\]

whenever \((I_k)\) is a division of \(G\). We say that \(F\) has \textit{strong-}\(M\)-\textit{negligible variation outside} of \(G\) if \(F\) has the strong-\(M\)-negligible variation over \(G^c = \mathbb{R}^m \setminus G\).

We say that a function \(f : G \to X\) is \textit{Hake-variationally McShane integrable} on \(G\) with the primitive \(F : \mathcal{I}_G \to X\) if we have:

\(\triangleright\) for each \(\varepsilon > 0\) there exists a gauge \(\delta\) on \(G\) such that for each \(\delta\)-fine \(M\)-partition \(\pi\) in \(G\) we have

\[
\sum_{(t,I) \in \pi} \|f(t)|I| - F(I)\| < \varepsilon,
\]

\(\triangleright\) \(F\) is the strong-Hake-function,

\(\triangleright\) \(F\) has the strong-\(M\)-negligible variation outside of \(G\).

In this case, we define the Hake-variational McShane integral of \(f\) over \(I\) as

\[
(HvM) \int_I f \, d\lambda = F(I) \quad \forall I \in \mathcal{I}_G.
\]

2. The main results

Since \(G\) is a bounded subset of \(\mathbb{R}^m\), there exists \(I_0 \in \mathcal{I}\) such that \(G \subset I_0\). Given a function \(f : G \to X\), we define a function \(f_0 : I_0 \to X\) as

\[
f_0(t) = \begin{cases} f(t) & \text{if } t \in G, \\ 0 & \text{if } t \in I_0 \setminus G. \end{cases}
\]

**Theorem 2.1.** Assume that a division \((C_k)\) of \(G\), a function \(f : G \to X\) and an additive interval function \(F : \mathcal{I}_G \to X\) are given. Define

\[
f_k = f|_{C_k} \quad \text{and} \quad F_k = F|_{\mathcal{I}_{C_k}} \quad \text{for each } k \in \mathbb{N}.
\]

Then the following statements are equivalent:

(i) \(f\) is Hake-variationally McShane integrable on \(G\) with the primitive \(F\),

(ii) \(f_0\) is variationally McShane integrable on \(I_0\) with the primitive \(F_0\) such that \(F_0(I) = F(I)\) for all \(I \in \mathcal{I}_G\),

(iii) for each \(k \in \mathbb{N}\), function \(f_k\) is variationally McShane integrable on \(C_k\) with the primitive \(F_k\), \(F\) is a strong-Hake function and has the strong-\(M\)-negligible variation outside of \(G\).
Proof. (i) ⇒ (iii): Assume that \( f \) is Hake-variationally-McShane integrable on \( G \) with the primitive \( F \). Then given \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \( G \) such that for each \( \delta \)-fine \( \mathcal{M} \)-partition \( \pi \) in \( G \) we have

\[
\sum_{(t,I) \in \pi} \| f(t) |I| - F(I) \| < \varepsilon.
\]

We can choose \( \delta(t) \) such that \( B_{\delta(t)}(t, \delta(t)) \subset G \) for all \( t \in G \).

Since \( F \) is a strong-Hake function and has the strong-\( \mathcal{M} \)-negligible variation outside of \( G \), it is enough to prove that each \( f_k \) is variationally McShane integrable on \( C_k \) with the primitive \( F_k \). Let \( \pi_k \) be a \( \delta_k \)-fine \( \mathcal{M} \)-partition of \( C_k \), where \( \delta_k = \delta \mid_{C_k} \).

Then \( \pi_k \) is a \( \delta \)-fine \( \mathcal{M} \)-partition in \( G \) and therefore

\[
\sum_{(t,I) \in \pi_k} \| f_k(t) |I| - F_k(I) \| = \sum_{(t,I) \in \pi_k} \| f(t) |I| - F(I) \| < \varepsilon.
\]

This means that \( f_k \) is variationally McShane integrable on \( C_k \) with the primitive \( F_k \).

(iii) ⇒ (ii): Assume that (iii) holds and an arbitrary \( \varepsilon > 0 \) is given. Then, since each function \( f_k \) is variationally McShane integrable on \( C_k \) with the primitive \( F_k \), by Lemma 3.6.15 in [13] there exists a gauge \( \delta_k \) on \( C_k \) such that for each \( \delta_k \)-fine \( \mathcal{M} \)-partition \( \pi_k \) in \( C_k \) we have

\[
\sum_{(t,I) \in \pi_k} \| f_k(t) |I| - F_k(I) \| \leq \frac{\varepsilon}{2^k}.
\]

Note that for \( t \in G \) we have the following possible cases:

1. There exists \( i_0 \in \mathbb{N} \) such that \( t \in (C_{i_0})^0 \);
2. There exists \( j_0 \in \mathbb{N} \) such that \( t \in C_{j_0} \setminus (C_{j_0})^0 \). In this case, there exists a finite set
   \[ \mathcal{N}_t = \{ j \in \mathbb{N} : t \in C_j \setminus (C_j)^0 \} \]
   such that \( t \in \bigcap_{j \in \mathcal{N}_t} C_j \) and \( t \notin C_k \) for all \( k \in \mathbb{N} \setminus \mathcal{N}_t \).

   Hence, \( t \in \left( \bigcup_{j \in \mathcal{N}_t} C_j \right)^0 \), where \( \left( \bigcup_{j \in \mathcal{N}_t} C_j \right)^0 \) is the interior of \( \bigcup_{j \in \mathcal{N}_t} C_j \).

   We can choose each \( \delta_k \) so that \( B_{m}(t, \delta_k(t)) \subset C_k \) if \( t \in (C_k)^0 \), and

   \[ B_{m}(t, \delta_k(t)) \subset \bigcup_{j \in \mathcal{N}_t} C_j \quad \text{if} \ t \in C_k \setminus (C_k)^0. \]

Since \( F \) has the strong-\( \mathcal{M} \)-negligible variation outside of \( G \), there exists a gauge \( \delta_v \) on \( G^c \) such that for each \( G^c \)-tagged \( \delta_v \)-fine \( \mathcal{M} \)-partition \( \pi_v \) in \( \mathbb{R}^m \) we have

\[
\sum_{(t,I) \in \pi_v} \left\| \left( \sum_{\{k, |I \cap C_k| > 0\}} F(I \cap C_k) \right) \right\| < \varepsilon.
\]
By hypothesis, we have also that $F$ is a strong-Hake-function. Therefore we can define an additive interval function $F_0 : \mathcal{I}_{I_0} \to X$ as

$$F_0(I) = \sum_{\{k : |I \cap C_k| > 0\}} F(I \cap C_k) \quad \forall I \in \mathcal{I}_{I_0}.$$ 

Clearly, $F_0(I) = F(I)$ for all $I \in \mathcal{I}_G$. We will show that $f_0$ is variationally McShane integrable on $I_0$ with the primitive $F_0$. To see this, we first define a gauge $\delta_0 : I_0 \to (0, \infty)$ as follows. For any $t \in G$ we choose $\delta_0(t) = \delta_i(t)$ if $t \in (C_i)_0$, and $\delta_0(t) = \min\{\delta_j(t) : j \in N_i\}$ otherwise. If $t \in I_0 \setminus G$, then we choose $\delta_0(t) = \delta_v(t)$. Let $\pi$ be an arbitrary $\delta_0$-fine $\mathcal{M}$-partition of $I_0$. Then $\pi = \pi_a \cup \pi_b \cup \pi_c$, where

$$\pi_a = \{(t, I) \in \pi : (\exists i_0 \in \mathbb{N}) [t \in (C_{i_0})^o]\}$$
and

$$\pi_b = \{(t, I) \in \pi : (\exists j_0 \in \mathbb{N})[t \in C_{j_0} \setminus (C_{j_0})^o]\};$$

$$\pi_c = \{(t, I) \in \pi : t \in I_0 \setminus G\}.$$ 

Therefore

$$\sum_{(t, I) \in \pi} \|f_0(t)|I| - F_0(I)|| = \sum_{(t, I) \in \pi_a} \|f(t)|I| - F(I)|| + \sum_{(t, I) \in \pi_b} \|f(t)|I| - F(I)|| + \sum_{(t, I) \in \pi_c} \|F_0(I)||.$$ 

Note that if we define

$$\pi_a^k = \{(t, I) : (t, I) \in \pi_a, t \in (C_k)^o\}$$
and

$$\pi_b^k = \{(t, I \cap C_k) : (t, I) \in \pi_b, t \in C_k \setminus (C_k)^o, |I \cap C_k| > 0\},$$

then $\pi_a^k$ and $\pi_b^k$ are $\delta_k$-fine $\mathcal{M}$-partitions in $C_k$. Therefore by (2.2) it follows that

$$\sum_{(t, I) \in \pi_a} \|f(t)|I| - F(I)|| = \sum_k \left( \sum_{(t, I) \in \pi_a} \|f(t)|I| - F_k(I)|| \right)$$

$$= \sum_k \left( \sum_{(t, I) \in \pi_a^k} \|f(t)|I| - F_k(I)|| \right) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

143
\[\sum_{(t,I) \in \pi_b} \|f(t)|I| - F(I)\|\]

\[= \sum_{(t,I) \in \pi_b} \left\| \left( \sum_{j \in \mathcal{N}_t} (f(t)|I \cap C_j| - F(I \cap C_j)) \right) \right\|\]

\[= \sum_{(t,I) \in \pi_b} \left\| \left( \sum_{j \in \mathcal{N}_t} (f_j(t)|I \cap C_j| - F_j(I \cap C_j)) \right) \right\|\]

\[\leq \sum_{(t,I) \in \pi_b} \left( \sum_{j \in \mathcal{N}_t} \|f_j(t)|I \cap C_j| - F_j(I \cap C_j)\| \right)\]

\[\leq \sum_{k} \left( \sum_{(t,I) \in \pi_k} \|f_k(t)|I \cap C_k| - F_k(I \cap C_k)\| \right) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2k} = \varepsilon.\]

By (2.3), the equality

\[\sum_{(t,I) \in \pi_c} \|F_0(I)\| = \sum_{(t,I) \in \pi_c} \left\| \left( \sum_{\{k: |I \cap C_k| > 0\}} F(I \cap C_k) \right) \right\|\]

together with the fact that \(\pi_c\) is a \(G^c\)-tagged \(\delta_v\)-fine \(\mathcal{M}\)-partition in \(\mathbb{R}^m\) yields

\[\sum_{(t,I) \in \pi_c} \|F_0(I)\| < \varepsilon.\]

Hence, by (2.4), (2.5) and (2.6) we obtain

\[\sum_{(t,I) \in \pi} \|f_0(t)|I| - F_0(I)\| < 3\varepsilon,\]

and since \(\pi\) is an arbitrary \(\delta_0\)-fine \(\mathcal{M}\)-partition of \(I_0\), it follows that \(f_0\) is variationally McShane integrable on \(I_0\) with the primitive \(F_0\).

(ii) \(\Rightarrow\) (i): Assume that \(f_0\) is variationally McShane integrable on \(I_0\) with the primitive \(F_0\) such that \(F_0(I) = F(I)\) for all \(I \in \mathcal{I}_G\). By Lemma 3.6.15 in [13], given \(\varepsilon > 0\) there exists a gauge \(\delta_0\) on \(I_0\) such that for each \(\delta_0\)-fine \(\mathcal{M}\)-partition \(\pi\) in \(I_0\) we have

\[\sum_{(t,I) \in \pi} \|f_0(t)|I| - F_0(I)\| < \varepsilon.\]

144
We can choose $\delta_0$ so that $B_m(t, \delta_0(t)) \subset G$ for all $t \in G$. Hence, if we define $\delta = \delta_0|_G$, then for each $\delta$-fine $\mathcal{M}$-partition $\pi$ in $G$ we have

$$\sum_{(t,I) \in \pi} \|f(t)|I| - F(I)\| < \varepsilon.$$ 

Thus, it remains to show that $F$ is a strong-Hake function and has strong-$\mathcal{M}$-negligible variation outside of $G$. Let $(I_k)$ be an arbitrary division of $G$.

We first show that $F$ is a strong-Hake function. Since $f_0$ is variationally McShane integrable on $I_0$, $\|f_0\|$ is McShane integrable on $I_0$. Hence, by Theorem 4.1.11 and Theorem 7.5.4 in [13] we obtain

$$(2.8) \quad (M) \int_G \|f_0\| \, d\lambda = \sum_k (M) \int_{I_k} \|f_0\| \, d\lambda = \sum_k \mathcal{M}F_0(I_k),$$

and since

$$\mathcal{M}F_0(I_k) \geq \mathcal{M}F_0(I \cap I_k) \geq \mathcal{M}F(I \cap I_k) \geq \|F(I \cap I_k)\| \quad \text{for each } I \in \mathcal{I},$$

it follows that the series $\sum_{\{k: |I \cap I_k| > 0\}} \|F(I \cap I_k)\|$ converges in $\mathbb{R}$. By hypothesis, for each $I \in \mathcal{I}_G$ we have also

$$F(I) = F_0(I) = (M) \int_I f_0 \, d\lambda = (M) \int_{\bigcup_k (I \cap I_k)} f_0 \, d\lambda = \sum_k (M) \int_{I \cap I_k} f_0 \, d\lambda$$

$$= \sum_k F_0(I \cap I_k) = \sum_{\{k: |I \cap I_k| > 0\}} F(I \cap I_k).$$

Thus, $F$ is a strong-Hake-function.

It remains to prove that $F$ has the strong-$\mathcal{M}$-negligible variation outside of $G$. To see this, we first define a gauge $\delta_v: G^c \to (0, \infty)$ as follows: $\delta_v(t) = \delta_0(t)$ if $t \in Z = I_0 \setminus G$, while for $t \notin I_0$ we choose $\delta_v(t)$ so that $B_m(t, \delta_v(t)) \cap I_0 = \emptyset$. Assume that $\pi_v$ is a $G^c$-tagged $\delta_v$-fine $\mathcal{M}$-partition in $\mathbb{R}^m$. Hence,

$$\pi_Z = \{(t, I \cap I_0): (t, I) \in \pi_v, \ t \in Z, \ |I \cap I_0| > 0\}$$

is a $\delta_0$-fine $\mathcal{M}$-partition in $I_0$. Then by (2.7), it follows that

$$\varepsilon \geq \sum_{(t,J) \in \pi_Z} \|f_0(t)|J| - F_0(J)\| = \sum_{(t,J) \in \pi_Z} \|F_0(J)\|$$

145
It follows that 

\[(\text{ii})\] exists and 

\[\lim_{|I| \to 0} \frac{F_0(I)}{|I|} = \lim_{|I| \to 0} \frac{F(I)}{|I|}.

Thus, \(F_c'(t)\) exists and \(F_c'(t) = f(t)\). Since \(k\) and \(t\) are arbitrary, the last result holds at almost all \(t \in \bigcup_k (C_k)\), and since

\[|G \setminus \bigcup_k (C_k)\| = 0,

it follows that \(F_c'(t)\) exists and \(F_c'(t) = f(t)\) at almost all \(t \in G\).
By the definition of the Hake-variationally McShane integrability, we have also that $F$ is a strong-Hake function and has the strong-$\mathcal{M}$-negligible variation outside of $G$.

(ii) $\Rightarrow$ (i): Assume that (ii) holds and define

$$f_k = f|_{C_k} \quad \text{and} \quad F_k = F|_{I_{C_k}} \quad \text{for each} \quad k \in \mathbb{N}.$$ 

Then each $F_k$ is sAC on $C_k$, $(F_k)'_c(t)$ exists and $(F_k)'_c(t) = f_k(t)$ at almost all $t \in C_k$. Therefore by Theorem 1.4 in [9], each $f_k$ is variational McShane integrable on $C_k$ with the primitive $F_k$. Therefore by Theorem 2.1, $f$ is Hake-variationally McShane integrable with the primitive $F$, and this ends the proof. □

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