SOME PROPERTIES OF CERTAIN SUBCLASSES OF BOUNDED MOCANU VARIATION WITH RESPECT TO $2k$-SYMMETRIC CONJUGATE POINTS

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Abstract. We introduce subclasses of analytic functions of bounded radius rotation, bounded boundary rotation and bounded Mocanu variation with respect to $2k$-symmetric conjugate points and study some of its basic properties.

Keywords: $2k$-symmetric conjugate points; bounded Mocanu variation; bounded radius rotation; bounded boundary rotation

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1. Introduction

Let $A$ be the class of analytic functions $f$ defined on the unit disc $E = \{z \in \mathbb{C}: |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E.$$

Also, let $S$, $K$, $S^*$ and $C$ denote the subclasses of $A$ which are univalent, close-to-convex, starlike and convex in $E$, respectively. Let $P_m(\gamma)$ be the class of functions $p(z)$ analytic in the unit disc $E$ satisfying the properties $p(0) = 1$ and for $z = re^{i\theta}, m \geq 2$,

$$\int_0^{2\pi} \left| \text{Re} \frac{p(z) - \gamma}{1 - \gamma} \right| \, d\theta \leq m\pi, \quad 0 \leq \gamma < 1.$$ 

The class $P_m(\gamma)$ for $\gamma = 0$ and $0 \leq \gamma < 1$ has been introduced and investigated by Pinchuk in [6], and Padmanabhan and Parvatham in [5], respectively. We note that

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$P_m(0) = P_m$ and $P_2(\gamma) = P(\gamma)$ is the class of analytic functions with positive real part greater than $\gamma$. For $m = 2$ and $\gamma = 0$ we have the class $P$ of functions with positive real part. We can write (1.2) as

\begin{equation}
(1.3) \quad p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\gamma)ze^{-it}}{1 - ze^{-it}} \, d\mu(t),
\end{equation}

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

\begin{equation}
(1.4) \quad \int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m.
\end{equation}

Also, for $p \in P_m(\gamma)$ we can write from (1.2)

\begin{equation}
(1.5) \quad p(z) = \left( \frac{m}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{m}{4} - \frac{1}{2} \right) p_2(z), \quad p_1, p_2 \in P_2(\gamma), \ z \in E.
\end{equation}

It is known [3] that $P_m(\gamma)$ is a convex set. Also $p \in P_m(\gamma)$ is in $P_2(\gamma) = P(\gamma)$ for $|z| < r_1$, where

\begin{equation}
(1.6) \quad r_1 = \frac{1}{2} \left( m - \sqrt{m^2 - 4} \right).
\end{equation}

The classes $V_m(\gamma)$ of functions of bounded boundary rotation of order $\gamma$ and $R_m(\gamma)$ of functions of bounded radius rotation of order $\gamma$ are closely related with $P_m(\gamma)$. A function $f \in A$ is in $V_m(\gamma)$ if and only if $(zf'(z))/f'(z) \in P_m(\gamma)$. Also

\begin{equation}
(1.7) \quad f \in R_m(\gamma) \Leftrightarrow \frac{zf'(z)}{f(z)} \in P_m(\gamma).
\end{equation}

It is clear that

\begin{equation}
(1.8) \quad f \in V_m(\gamma) \Leftrightarrow zf'(z) \in P_m(\gamma).
\end{equation}

When $m = 2, \gamma = 0$, then $V_2(0)$ coincides with the class $C$ and $R_2(0) = S^*$. Wang et al. in [9] introduced and investigated class $S^{(k)}(\varphi)$, which satisfies the inequality:

\[ \frac{zf'(z)}{f_k(z)} < \varphi(z), \quad z \in E, \]

where $\varphi(z) \in P, k \geq 2$ is a fixed positive integer and $f_k(z)$ is defined by the following equality:

\[ f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z), \quad \varepsilon = \exp \frac{2\pi i}{k}, \]
and a function \( f(z) \in E \) is in the class \( C_{s}^{(k)}(\varphi) \) if and only if \( zf'(z) \in S_{s}^{(k)}(\varphi) \). Also Wang and Gao (see [9]) introduced and investigated two classes \( S_{sc}^{(k)}(\varphi) \) and \( C_{sc}^{(k)}(\varphi) \) of functions starlike and convex with respect to 2k-symmetric conjugate points. Noor and Mustafa in [2] introduced and investigated class \( R_{s}^{k}(\gamma) \) of analytic functions which are of bounded radius rotation of order \( \gamma \) with respect to symmetrical points if and only if

\[
\frac{2zf'(z)}{f(z) - f(-z)} \in \mathcal{P}_{k}(z), \quad z \in E.
\]

We now define the following.

**Definition 1.1.** Let \( f \in \mathcal{A} \). Then \( f \) is said to be of bounded radius rotation of order \( \gamma \) with respect to 2k-symmetric conjugate points if and only if

\[
\frac{zf'(z)}{f(z)} \in \mathcal{P}_{m}(\gamma), \quad z \in E,
\]

where \( k \geq 1 \) is a fixed positive integer and \( f_{2k}(z) \) is defined as

\[
f_{2k}(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} (\varepsilon^{-\nu}f(\varepsilon^{\nu}z) + \varepsilon^{\nu}f(\overline{\varepsilon^{\nu}z})), \quad \varepsilon = \exp \frac{2\pi i}{k}.
\]

We shall denote the class of such functions as \( R_{s}^{m-2k}(\gamma) \). We note that \( R_{2}^{m-2}(\gamma) \) is the class \( S_{s}^{*} \) of univalent functions starlike with respect to symmetrical points defined by Sakaguchi (see [8]). Also we define the class \( V_{m}^{s-2k}(\gamma) \) as follows.

**Definition 1.2.**

\[
f \in V_{m}^{s-2k}(\gamma) \iff zf' \in R_{m}^{s-2k}(\gamma), \quad z \in E.
\]

Motivated by the above-mentioned classes we now define the following subclasses of analytic functions.

**Definition 1.3.** Let \( f \in \mathcal{A} \) and \( f(z)f'(z)z^{-1} \neq 0 \) for \( z \in E \). Then \( f \) is said to be of bounded Mocanu variation of order \( \gamma \) with respect to 2k-symmetric conjugate points if and only if

\[
\alpha \frac{zf'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{(zf'(z))'}{f_{2k}(z)} \in \mathcal{P}_{m}(\gamma), \quad z \in E,
\]

where \( 0 \leq \alpha \leq 1 \) and \( k \geq 1 \) is a fixed positive integer and \( f_{2k}(z) \) is defined by (1.10). We shall denote the class of such functions as \( M_{m}^{s-2k}(\alpha, \gamma) \).
Definition 1.4. Let $f \in A$ and $f(z)f'(z)z^{-1} \neq 0$ for $z \in E$. Then $f$ belongs to the class $\mathcal{H}_{s-2k}^{s,m,m_1}(\alpha, \gamma)$ if

\begin{equation}
\alpha \frac{zf'(z)}{g_{2k}(z)} + (1 - \alpha) \frac{(zf'(z))'}{g_{2k}(z)} \in P_m(\gamma),
\end{equation}

where $0 \leq \alpha \leq 1$ and $k \geq 1$ is a fixed positive integer and $g_{2k}(z)$ is defined as

\begin{equation}
g_{2k}(z) = \frac{1}{2k} \sum_{v=0}^{k-1} (\varepsilon^{-v}g(\varepsilon^v z) + \varepsilon^v g(\varepsilon^v z)), \quad \varepsilon = \exp^{2\pi i/k}
\end{equation}

with $g \in M_{s-2k}^m(\alpha, \gamma)$.

For simplicity, we write $\mathcal{H}_{s-2k}^{s,m,m_1}(\alpha, \gamma) = \mathcal{H}_{m}^{s-2k}(\alpha, \gamma)$.

In our investigation of the classes $R_{s-2k}^{s-2k}(\gamma)$, $V_{s-2k}^{s-2k}(\gamma)$, $M_{s-2k}^{s-2k}(\alpha, \gamma)$ and $\mathcal{H}_{s-2k}^{s,m,m_1}(\alpha, \gamma)$ we need the following lemmas.

Lemma 1.1 ([1]). Let $p$ be an analytic function in the unit disc with $P(0) = a$, where $\Re a > 0$. Let $P: E \to \mathbb{C}$ be a function such that $\Re P(z) > 0$ for $z \in E$. Then

$$\Re[p(z) + P(z)zp'(z)] > 0 \Rightarrow \Re p(z) > 0.$$ 

Lemma 1.2 ([1]). Let $\beta, \gamma \in \mathbb{C}$ and $h$ be convex and univalent function in $E$ with

$$h(0) = 1 \quad \text{and} \quad \Re(\beta h(z) + \gamma) > 0, \quad z \in E.$$ 

If $p$ is analytic in $E$ with $p(0) = 1$, then subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that

$$p(z) \prec h(z).$$
2. Basic properties of $R_{m}^{s-2k}(\gamma)$, $V_{m}^{s-2k}(\gamma)$, $M_{m}^{s-2k}(\alpha, \gamma)$ and $H_{m,m_{1}}^{s-2k}(\alpha, \gamma)$

**Theorem 2.1.** Let $f \in M_{m}^{s-2k}(\alpha, \gamma)$. Then the function

\begin{equation}
\psi(z) = f_{2k}(z)
\end{equation}

belongs to $M_{m}^{s-2k}(\alpha, \gamma)$.

**Proof.** Let $f \in M_{m}^{s-2k}(\alpha, \gamma)$. Then from Definition 1.3 we have

$$
\alpha \frac{zf'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{(zf'(z))'}{f_{2k}'(z)} \in P_{m}(\gamma), \quad z \in E;
$$

or

\begin{equation}
\alpha \frac{zf'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{zf''(z)}{f_{2k}'(z)} \in P_{m}(\gamma), \quad z \in E.
\end{equation}

Replacing $z$ by $\varepsilon^{v}z$, $v = 0, 1, 2, \ldots, k - 1$ in (2.2) leads to

\begin{equation}
\alpha \frac{\varepsilon^{v}zf'(\varepsilon^{v}z)}{f_{2k}(\varepsilon^{v}z)} + (1 - \alpha) \frac{\varepsilon^{v}zf''(\varepsilon^{v}z)}{f_{2k}'(\varepsilon^{v}z)} \in P_{m}(\gamma).
\end{equation}

We note that

\begin{equation}
\frac{f_{2k}(\varepsilon^{v}z)}{f_{2k}(\varepsilon^{v}z)} = \varepsilon^{v}f_{2k}(z), \quad \frac{f_{2k}'(\varepsilon^{v}z)}{f_{2k}(\varepsilon^{v}z)} = f_{2k}'(z), \quad \psi_{2k}(z) = f_{2k}(z).
\end{equation}

Thus, in view of (2.3) and (2.4) we obtain

\begin{equation}
\alpha \frac{zf'(\varepsilon^{v}z)(\varepsilon^{v}zf'(\varepsilon^{v}z))'}{f_{2k}(\varepsilon^{v}z)} + (1 - \alpha) \frac{zf'(\varepsilon^{v}z)}{f_{2k}(\varepsilon^{v}z)} \in P_{m}(\gamma)
\end{equation}

and

\begin{equation}
\alpha \frac{zf'(\varepsilon^{v}z)}{f_{2k}(\varepsilon^{v}z)} + (1 - \alpha) \frac{zf'(\varepsilon^{v}z)}{f_{2k}(\varepsilon^{v}z)} \in P_{m}(\gamma).
\end{equation}

Since $P_{m}(\gamma)$ is a convex set, summing (2.5) and (2.6) leads to

\begin{equation}
\alpha \frac{\frac{1}{2}z(f'(\varepsilon^{v}z) + f'(\varepsilon^{v}z))}{f_{2k}(z)} + (1 - \alpha) \frac{\frac{1}{2}(f'(\varepsilon^{v}z) + f'(\varepsilon^{v}z))}{f_{2k}(z)} \in P_{m}(\gamma).
\end{equation}
Putting \( \upsilon = 0, 1, 2, \ldots, k - 1 \) in (2.7) and summing the resulting equations yields
\[
\alpha \frac{1}{2} z^{k-1} \sum_{\upsilon=0}^{k-1} \left( f'\left( \epsilon^\upsilon z \right) + f'\left( \epsilon^{1-\upsilon} \right) \right) \frac{1}{f_{2k}(z)} \\
+ (1 - \alpha) \frac{1}{2} k^{-1} \sum_{\upsilon=0}^{k-1} \left( f'\left( \epsilon^\upsilon z \right) + f'\left( \epsilon^{1-\upsilon} \right) + z \left( \epsilon^\upsilon f''\left( \epsilon^\upsilon z \right) + \epsilon^{-\upsilon} f''\left( \epsilon^{1-\upsilon} \right) \right) \right) \in P_m(\gamma)
\]
and hence \( \psi \in P_k(\gamma) \) in \( E \).

Putting \( \alpha = 0, 1 \) in Theorem 2.1 we have the following results for the classes \( R^{s-2k}_{m}\gamma \) and \( V^{s-2k}_{m}\gamma \).

**Corollary 2.1.** Let \( f \in R^{s-2k}_{m}\gamma \). Then the function \( \psi(z) = f_{2k}(z) \) belongs to \( R^{s-2k}_{m}\gamma \) in \( E \).

**Corollary 2.2.** Let \( f \in V^{s-2k}_{m}\gamma \). Then the function \( \psi(z) = f_{2k}(z) \) belongs to \( V^{s-2k}_{m}\gamma \) in \( E \).

In order to prove our next result we need the following lemma.

**Lemma 2.1.** Let \( p \) and \( \varphi \) be analytic functions in \( E \) with \( p(0) = 1 \) and \( \text{Re} \varphi(z) > 0 \) for \( z \in E \). If
\[
p(z) + \varphi(z)zp'(z) \in P_m(\gamma),
\]
then \( p(z) \in P_m(\gamma) \).

**Proof.** From the definition of \( P_m(\gamma) \) there exist \( q_1, q_2 \in P_2(\gamma) \) such that
\[
p(z) + \varphi(z)zp'(z) = mq_1(z) + (1 - m)q_2(z).
\]
Let \( p_1 \) and \( p_2 \) be the solutions of the Cauchy problems
\[
p(z) + \varphi(z)zp'(z) = q_1(z), \quad p(0) = 1
\]
and
\[
p(z) + \varphi(z)zp'(z) = q_2(z), \quad p(0) = 1,
\]
respectively. In view of (2.9) and (2.10) we rewrite (2.8) as
\[
p(z) + \varphi(z)zp'(z) = m(p_1(z) + \varphi(z)zp'_1(z)) + (1 - m)(p_2(z) + \varphi(z)zp'_2(z)),
\]
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or equivalently,

\[(2.11) \quad (p(z) - mp_1(z) - (1 - m)p_2(z)) + z\varphi(z)(p'(z) - mp_1'(z) - (1 - m)p_2'(z)) = 0.\]

Now if we define \( h(z) = p(z) - mp_1(z) - (1 - m)p_2(z) \), then \( h(0) = 0 \) and (2.11) yields

\[(2.12) \quad h(z) + \varphi(z)zh'(z) = 0, \quad h(0) = 0.\]

But it is clear that Cauchy problem (2.12) has the only solution \( h(z) = 0 \). Hence \( p(z) = mp_1(z) + (1 - m)p_2(z) \). For completing the proof we show that \( p_1, p_2 \in P_2(\gamma) \). Form equation (2.9) we can write

\[
\frac{q_1(z) - \gamma}{1 - \gamma} = \frac{p_1(z) - \gamma}{1 - \gamma} + \frac{\varphi(z)}{1 - \gamma}zp_1'(z).
\]

Since \( \text{Re} \left( \frac{q_1(z) - \gamma}{1 - \gamma} \right) > 0 \) and \( \text{Re} \varphi(z) > 0 \), applying Lemma 1.1 we obtain \( \text{Re} p_1(z) > \gamma \). Similarly, we have \( \text{Re} p_2(z) > \gamma \) and this means that \( p \in P_m(\gamma) \) and the proof is complete. \( \square \)

**Theorem 2.2.** Let \( 0 < \alpha \leq 1, \ k \geq 1 \) and \( m \geq 2 \). Then

\[
\mathcal{H}_{s,2}^{\alpha,2k}(\alpha, \gamma, g) \subseteq \mathcal{H}_{m,2}^{s,2k}(1, \gamma, g).
\]

**Proof.** Let \( f \in \mathcal{H}_{m,2}^{s,2k}(\alpha, \gamma, g) \). Then by the definition of the class \( \mathcal{H}_{m,2}^{s,2k}(\alpha, \gamma, g) \) and applying Theorem 2.1 we know that \( g_{2k} \in M_{2}^{s,2k}(\alpha, \gamma) \), i.e.

\[
\alpha \frac{z\varphi'(z)}{\varphi(z)} + (1 - \alpha) \frac{(z\varphi'(z))'}{\varphi'(z)} \in P(\gamma),
\]

where \( \varphi = g_{2k} \).

Or equivalently,

\[
(2.13) \quad \alpha \frac{z\varphi'(z)}{\varphi(z)} + (1 - \alpha) \frac{(z\varphi'(z))'}{\varphi'(z)} \prec h(z) := \frac{1 + (1 - 2\gamma)z}{1 - z}.
\]

Set

\[
q(z) = \frac{z\varphi'(z)}{\varphi(z)},
\]

then we can rewrite (2.13) as

\[
(2.14) \quad \alpha \frac{z\varphi'(z)}{\varphi(z)} + (1 - \alpha) \frac{(z\varphi'(z))'}{\varphi'(z)} = q(z) + \frac{(1 - \alpha)zq'(z)}{q(z)} \prec h(z).
\]
Since \( h \) is convex and univalent in \( E \) with \( h(0) = 1 \) and \( \text{Re}(h(z)/(1 - \alpha)) > 0 \), applying Lemma 1.2, we obtain

\[
q(z) \prec h(z), \quad z \in E.
\]

By Setting
\[
p(z) = \frac{zf'(z)}{g_{2k}(z)},
\]
we get
\[
zp'(z) = \frac{(zf'(z))'g_{2k}(z) - g_{2k}'(z)zf'(z)}{g_{2k}^2(z)} = \frac{zf'(z)}{g_{2k}(z)} - \frac{zf'(z)q(z)}{g_{2k}(z)}.
\]

Therefore in view of \( f \in \mathcal{H}_{m, 2}^{s-2k}(\alpha, \gamma, g) \) and (2.16) we conclude that
\[
a \frac{zf'(z)}{g_{2k}(z)} + (1 - \alpha)\frac{(zf'(z))'}{g_{2k}^2(z)} = p(z) + (1 - \alpha)\frac{zp'(z)}{q(z)} \in P_m(\gamma).
\]

Now from relation (2.15) it is clear that \( \text{Re}(q(z)/(1 - \alpha)) > 0 \), so applying Lemma 2.1, we get \( p(z) \in P_m(\gamma) \) and the proof is complete. \( \Box \)

By Putting \( m = 2 \) and considering \( g = f_{2k} \) in Theorem 2.2, we have the following corollary.

**Corollary 2.3.** Let \( 0 < \alpha < 1 \) and \( k \geq 1 \). Then
\[
\mathcal{M}_2^{s-2k}(\alpha, \gamma) \subseteq \mathcal{R}_2^{s-2k}(\gamma) \subseteq K \subseteq S.
\]

**Theorem 2.3.** Let \( 0 \leq \alpha < 1 \) and \( f \in \mathcal{M}_m^{s-2k}(\alpha, \gamma) \). Then there exists a function \( p \in P_m(\gamma) \) such that

\[
f_{2k}(z) = \left( \frac{1}{1 - \alpha} \int_0^z u^{\alpha/(1 - \alpha)} \exp \left( \frac{1}{1 - \alpha} \int_0^u h(t) - 1 \, dt \right) \, du \right)^{1 - \alpha},
\]
where

\[
h(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} (p(\nu z) + \overline{p(\nu \overline{z})}).
\]
Proof. Since \( f \in \mathcal{M}_m^{s-2k}(\alpha, \gamma) \), there exists a function \( p \in P_m(\gamma) \) such that

\[
\alpha \frac{zf'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{(zf'(z))'}{f'_{2k}(z)} = p(z).
\]

Using similar arguments given in the proof of Theorem 2.1 to (2.19) we obtain

\[
\alpha \frac{zf_{2k}'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{(zf_{2k}'(z))'}{f'_{2k}(z)} = \frac{1}{2k} \sum_{v=0}^{k-1} (p\varepsilon^v z + p(-\varepsilon^v z)) = h(z).
\]

Let us define \( F \) as

\[
\alpha \frac{zf_{2k}'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{(zf_{2k}'(z))'}{f'_{2k}(z)} = \frac{zF'(z)}{F(z)},
\]

then

\[
f_{2k}(z) = \left( \frac{1}{1 - \alpha} \int_0^z \frac{(F(t))^{1/(1-\alpha)}}{t} \mathrm{d}t \right)^{1-\alpha}
\]

and the function \( F \) is analytic with \( F(0) = 0 \) and from (2.20) we can write

\[
\frac{zF'(z)}{F(z)} = h(z).
\]

Now by solving the last equation and putting its response into equality (2.21) we get the result and the proof is complete. □

**Theorem 2.4.** Let \( 0 \leq \alpha < 1 \) and \( f \in \mathcal{M}_m^{s-2k}(\alpha, \gamma) \). Then there exists a function \( p \in P_m(\gamma) \) such that

\[
f'(z) = \frac{1}{(1 - \alpha)^{1-\alpha}} \frac{\int_0^1 u^{\alpha/(1-\alpha)} \exp((1 - \alpha)^{-1} \int_0^u (h(t) - 1)t^{-1} \mathrm{d}t)p(u) \mathrm{d}u}{\left( \int_0^1 u^{\alpha/(1-\alpha)} \exp((1 - \alpha)^{-1} \int_0^u (h(t) - 1)t^{-1} \mathrm{d}t) \right)^{\alpha}},
\]

where \( h \) is given by (2.18).

Proof. Suppose that \( f \in \mathcal{M}_m^{s-2k}(\alpha, \gamma) \), we can get

\[
\alpha \frac{zf'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{(zf'(z))'}{f'_{2k}(z)} \in P_k(\gamma),
\]

so there exists a function \( p \in P_k(\gamma) \) such that

\[
\alpha \frac{zf'(z)}{f_{2k}(z)} + (1 - \alpha) \frac{(zf'(z))'}{f'_{2k}(z)} = p(z).
\]
Taking \( F(z) = z f'(z) \) and \( G(z) = f_{2k}(z) \) in the above equation yields
\[
\alpha \frac{F(z)}{G(z)} + (1 - \alpha) \frac{F'(z)}{G'(z)} = p(z),
\]
or
\[
F'(z) + \frac{\alpha}{1 - \alpha} \frac{G'(z)}{G(z)} F(z) = \frac{p(z) G'(z)}{1 - \alpha}.
\]
Now solving Cauchy problem (2.23) and considering (2.17) we get our result and the proof is complete.

\[\square\]

**Theorem 2.5.** Let \( f, g \in \mathcal{M}_{s}^{-2k}(\alpha, \gamma) \) and suppose that \( F \) is defined by
\[
F(z) = \frac{1}{\delta z^{1/\delta - 1}} \int_{0}^{z} t^{1/\delta - 2}(f_{2k}(t))^{\beta/(1+\beta)}(g_{2k}(t))^{1/(1+\beta)} \, dt,
\]
where \( z \in E, \delta > 0, \beta \geq 0 \) and \( \gamma + \delta^{-1} - 1 > 0 \). Then \( F \) belongs to \( \mathcal{M}_{s}^{-2k}(1, \gamma) \).

\[\text{Proof.} \quad \text{Since } f, g \in \mathcal{M}_{s}^{-2k}(\alpha, \gamma), \text{ by applying Theorem 2.1 and Corollary 2.3 we obtain } f_{2k}, g_{2k} \in \mathcal{M}_{s}^{-2k}(1, \gamma). \text{ Differentiating (2.24) logarithmically and setting } p(z) = z F'(z)/F(z), \text{ we have}
\]
\[
p(z) + \frac{zp'(z)}{p(z) + \delta^{-1} - 1} = \frac{\beta}{1 + \beta} \frac{zf_{2k}'(z)}{f_{2k}(z)} + \frac{1}{1 + \beta} \frac{zg_{2k}'(z)}{g_{2k}(z)}.
\]

Since the functions \( zf_{2k}'(z)/f_{2k}(z) \) and \( zg_{2k}'(z)/g_{2k}(z) \) belong to \( P_{2}(\gamma) \) in \( E \), and \( P_{2}(\gamma) \) is a convex set,
\[
\frac{\beta}{1 + \beta} \frac{zf_{2k}'(z)}{f_{2k}(z)} + \frac{1}{1 + \beta} \frac{zg_{2k}'(z)}{g_{2k}(z)} \in P_{2}(\gamma).
\]
We now apply Lemma 1.2 to obtain \( p(z) \in P_{2}(\gamma) \) and the proof is complete. \[\square\]

Let \( L(r, f) \) denote the length of the image of the circle \(|z| = r\) under \( f \). We prove the following.

**Theorem 2.6.** Let \( f \in \mathcal{H}_{s}^{-2k}(1, \gamma) \). Then for \( 0 < r < 1 \),
\[
L(r, f) \leq \frac{4\pi(1-\gamma)}{(1-r)^{(k+2)/k}}.
\]
Proof. Using Theorem 2.2 and in view of the definition of class \( H_{2}^{s-2k}(1, \gamma) \) there exists a function \( g \in M_{2}^{s-2k}(1, \gamma) \) such that

\[
zf'(z) = \psi(z)h(z), \quad \psi = g_{2k} \in S^{*}(\gamma), \ h \in P_{2}(\gamma).
\]

Since \( \psi \in S^{*}(\gamma) \) and \( \psi \) is a \( k \)-fold symmetric function, there exists a \( k \)-fold symmetric function \( \psi_{1}(z) \) such that

\[
\psi(z) = z\left(\frac{\psi_{1}(z)}{z}\right)^{1-\gamma}.
\]

Now for \( z = re^{i\theta} \) we have

\[
L(r, f) = \int_{0}^{2\pi} |zf'(z)| \, d\theta
= \int_{0}^{2\pi} \left| z\left(\frac{\psi_{1}(z)}{z}\right)^{1-\gamma} h(z)\right| \, d\theta
= r^{\gamma} \int_{0}^{2\pi} |(\psi_{1}(z))^{1-\gamma} h(z)| \, d\theta,
\]

and so, using Hölder’s inequality, we obtain

\[
L(r, f) \leq 2\pi r^{\gamma} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |\psi_{1}(z)|^{2} \, d\theta \right)^{1/2} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |h(z)|^{2} \, d\theta \right)^{1/2}.
\]

For \( h \in P_{2}(\gamma) \), from the Parseval’s identity it is easy to see that

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |h(z)|^{2} \, d\theta \leq \frac{1 + (4(1-\gamma)^2 - 1)r^2}{1 - r^2}.
\]

Also for \( k \)-fold symmetric function \( \psi_{1} \) it is known that (see [4])

\[
|\psi_{1}(z)| \leq \frac{|z|}{(1-|z|^{k})^{2/k}}.
\]

Using (2.29) and (2.30) in (2.28), it follows that

\[
L(r, f) \leq 2\pi r^{\gamma} \left( \frac{1 + (4(1-\gamma)^2 - 1)r^2}{1 - r^2} \right)^{1/2} \frac{r}{(1 - r^{k})^{2/k}} \leq \frac{4\pi(1-\gamma)}{(1-r)^{1+2/k}}.
\]

This completes the proof. \( \square \)

Theorem 2.7. Let \( f \in H_{2}^{s-2k}(1, \gamma) \). Then for \( 0 < r < 1 \),

\[
|a_{n}| \leq 4\pi(1-\gamma)n^{2/k}.
\]
Proof. Since with $z = re^{i\theta}$ Cauchy Theorem gives

$$na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z)e^{-in\theta} \, d\theta,$$

then

$$n|a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)| \, d\theta = \frac{1}{2\pi r^n} L(r,f).$$

Using Theorem 2.6 and putting $r = 1 - n^{-1}$, $n \to \infty$, we obtain the required result. □

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References


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