A NOTE ON THE OPEN PACKING NUMBER IN GRAPHS

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Abstract. A subset $S$ of vertices in a graph $G$ is an open packing set if no pair of vertices of $S$ has a common neighbor in $G$. An open packing set which is not a proper subset of any open packing set is called a maximal open packing set. The maximum cardinality of an open packing set is called the open packing number and is denoted by $\rho_o(G)$. A subset $S$ in a graph $G$ with no isolated vertex is called a total dominating set if any vertex of $G$ is adjacent to some vertex of $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. We characterize graphs of order $n$ and minimum degree at least two with $\rho_o(G) = \gamma_t(G) = \frac{1}{2}n$.

Keywords: packing; open packing; total domination

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1. Introduction

In this paper, we follow the notations of [3], [7]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n$ and edge set $E$. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The degree of $v$ is $\deg(v) = |N(v)|$. The maximum and minimum degrees in $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex of degree one in a tree is called a leaf and its unique neighbor is called a support vertex. A pendant edge in a graph is an edge incident with a leaf. The corona graph $\text{cor}(H)$ of a graph $H$ is a graph obtained from $H$ by adding a leaf to every vertex of $H$. A matching in a graph $G$ is a set of edges no pair of which has a common vertex. For a subset $S$ of vertices of $G$, the subgraph induced by $S$ is denoted by $G[S]$. A subset $S$ of vertices of a graph $G$ is a dominating set of $G$ if every vertex $x \in V - S$ is adjacent to a vertex of $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $S$ of a graph $G$ is

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called a total dominating set if $G[S]$ has no isolated vertices. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A graph $G$ is total domination partitionable if its vertex set can be partitioned into two total domination sets. For a comprehensive study of domination and total domination see [7], [10].

A packing of a graph $G$ is a set of vertices whose closed neighborhoods are pairwise disjoint. The packing number of $G$, denoted by $\delta(G)$, is the maximum cardinality among all packings of $G$. For reference on the packing number of a graph, see for example [2], [4], [11], [15]. A set $S$ of vertices of a graph $G$ is an open packing of $G$ if the open neighborhoods of the vertices of $S$ are pairwise disjoint in $G$. The open packing number of $G$, denoted by $\delta^o(G)$, is the maximum cardinality among all open packings of $G$. The open packing number of a graph has been studied in [13], [8], [9], [14], for example.

A subset $S$ of vertices of $G$ is an efficient open dominating set if $|N(v) \cap S| = 1$ for every vertex $v \in V(G)$. An efficient open domination graph is a graph with an efficient open dominating set. The study of efficient open domination graphs has begun by Cockayene et al. [6] and further studied in, for example, [12]. Note that the efficient open domination graphs are graphs $G$ with $\delta^o(G) = \gamma_t(G)$.

Recently, Hamid and Saravanakumar in [13] continued the study of open packing in graphs, and presented several important results on the open packing number of a graph. They posed the characterization of graphs of order $n$ with $\delta(G) \geq 2$ for which $\delta^o(G) + \gamma_t(G) = n$ as an open problem. We give a characterization of graphs of order $n$ with minimum degree at least two for which $\delta^o(G) = \gamma_t(G) = \frac{1}{2}n$. We make use of the following.

**Theorem 1** ([13]). If $G$ is a connected graph of order $n \geq 2$, then $\delta^o(G) \leq n/\delta(G)$.

**Theorem 2** ([5]). If $G$ is a graph without isolated vertices of order $n \geq 3$, then $\gamma_t(G) \leq \frac{2}{3}n$.

**Theorem 3** ([1]). If $G$ is a graph of order $n$ with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{1}{2}n$.

### 2. Main result

We begin with the following.

**Lemma 4.** Let $G$ be a connected graph of order $n$ with $\delta(G) \geq 2$, and $\delta^o(G) + \gamma_t(G) = n$, and let $S$ be a $\delta^o(G)$-set. Then:

1. $\delta(G) = 2$;
(2) \(|S| \leq |V(G) - S|\);
(3) Any non-support vertex of \(G[V(G) - S]\) is adjacent to precisely one vertex of \(S\).

**Proof.** Let \(G\) be a connected graph of order \(n\) with \(\delta(G) \geq 2\) and \(\varrho^o(G) + \gamma_t(G) = n\). We consider each claim separately:

1. By Theorems 1 and 2, \(\frac{2}{3}n \geq \gamma_t(G) = n - \varrho^o(G) \geq n - n/\delta(G)\), and we obtain that \(2 \leq \delta(G) \leq 3\). If \(\delta(G) = 3\), then by Theorems 1 and 3, \(\frac{1}{2}n \geq \gamma_t(G) = n - \varrho^o(G) \geq n - n/\delta(G)\), and we obtain that \(\delta(G) = 2\), a contradiction. Thus \(\delta(G) = 2\).

2. Let \(S\) be a \(\varrho^o(G)\)-set. Then clearly \(V(G) - S\) is a \(\gamma_t(G)\)-set, since \(\delta(G) = 2\) and any component of \(G[S]\) is \(K_2\) or \(K_1\). Since no pair of vertices of \(S\) have a common neighbor in \(V(G) - S\), we have \(|S| \leq |V(G) - S|\).

3. If there is a non-support vertex \(x\) of \(G[V(G) - S]\) that is not adjacent to a vertex of \(S\), then \((V(G) - S) - \{x\}\) is a total dominating set for \(G\), a contradiction with \(\varrho^o(G) + \gamma_t(G) = n\). Since \(S\) is an open packing set, \(x\) is adjacent to precisely one vertex of \(S\).

It is known that \(\varrho^o(G) \leq \gamma_t(G)\) for any graph \(G\) with no isolated vertex (see [12], Lemma 5). Let \(\mathcal{H}_1\) be the class of all graphs \(G\) such that \(G\) is obtained from a corona \(cor(H)\), where \(H\) is a graph of even order and with no isolated vertex, by adding a perfect matching between the leaves of \(cor(H)\). Figure 1 shows a graph in the family \(\mathcal{H}_1\). It is easy to see that any graph in the family \(\mathcal{H}_1\) is total domination partitionable.

![Figure 1. A graph in \(\mathcal{H}_1\).](image)

**Theorem 5.** If \(G\) is a connected graph of order \(n\) with \(\delta(G) \geq 2\), then \(\varrho^o(G) = \gamma_t(G) = \frac{1}{2}n\) if and only if \(G \in \mathcal{H}_1\).

**Proof.** Let \(G\) be a connected graph of order \(n\) with \(\delta(G) \geq 2\) and \(\varrho^o(G) = \gamma_t(G) = \frac{1}{2}n\). By Lemma 4, \(\delta(G) = 2\). Let \(S\) be a \(\varrho^o(G)\)-set. Clearly \(V(G) - S\) is a total dominating set for \(G\), and so \(\gamma_t(G) \leq n - \varrho^o(G)\). Now, \(n = \varrho^o(G) + \gamma_t(G) \leq \varrho^o(G) + |V(G) - S| = n\), and thus \(|V(G) - S| = \gamma_t(G)\). It is evident that any component of \(G[S]\) is \(K_1\) or \(K_2\). Since \(|S| = |V(G) - S|\), any vertex of \(V(G) - S\) is adjacent to precisely one vertex of \(S\). Thus, any component of \(G[S]\) is \(K_2\). Furthermore, \(\deg(x) = 2\) for any vertex \(x \in S\). Let \(G'\) be obtained from \(G\) by removing all edges of \(G[S]\). Then clearly \(G' = cor(G[V(G) - S])\). Since \(V(G) - S\) is a total dominating set of \(G\), \(G'[V(G) - S]\) has no isolated vertex. Consequently, \(G \in \mathcal{H}_1\).
Conversely, assume that $G \in \mathcal{H}_1$. Thus $G$ is obtained from a corona $\text{cor}(H)$, where $H$ is a graph of even order and with no isolated vertex, by adding a perfect matching $M$ between the leaves of $\text{cor}(H)$. Clearly $V(H)$ is a total dominating set for $G$, and thus $\gamma_t(G) \leq |V(H)|$. Let $S$ be a total dominating set in $G$. For any edge $xy \in G[S]$, $|S \cap (N[x] \cup N[y])| \geq 2$. Thus $|S| \geq |V(H)|$. Consequently, $|V(H)| = \gamma_t(G)$. On the other hand, the vertices of the perfect matching $M$ form an open packing for $G$, and so $\varrho^o(G) \geq |V(H)|$. Since $\varrho^o(G) \leq \gamma_t(G)$, we obtain that $\varrho^o(G) = |V(H)|$, as desired.

□

References


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