TOTAL BLOW-UP OF A QUASILINEAR HEAT EQUATION WITH SLOW-DIFFUSION FOR NON-DECAYING INITIAL DATA

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Abstract. We consider solutions of quasilinear equations \( u_t = \Delta u^m + u^p \) in \( \mathbb{R}^N \) with the initial data \( u_0 \) satisfying \( 0 < u_0 < M \) and \( \lim_{|x| \to \infty} u_0(x) = M \) for some constant \( M > 0 \). It is known that if \( 0 < m < p \) with \( p > 1 \), the blow-up set is empty. We find solutions \( u \) that blow up throughout \( \mathbb{R}^N \) when \( m > p > 1 \).

Keywords: quasilinear heat equation; total blow-up; blow-up only at space infinity

MSC 2010: 35B44, 35K59

1. Introduction

We consider the nonlinear diffusion equation:

\[
\begin{cases}
    u_t = \Delta u^m + u^p, \quad x \in \mathbb{R}^N, \ t > 0, \\
    u(x, 0) = u_0(x) > 0, \quad x \in \mathbb{R}^N
\end{cases}
\]

with \( m > p > 1 \) and \( u_0 \in C(\mathbb{R}^N) \) for \( N \geq 1 \). This problem is known to admit a local time solution (see [6], [8]), but it may cease to exist in a finite time. We say that the solution of (1.1) blows up in finite time if there is some \( T = T(u_0) < \infty \) such that

\[
\limsup_{t \searrow T} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = \infty
\]

and \( T(u_0) \) is called the blow-up time of the solution \( u \) with the initial value \( u_0 \). We define the blow-up set by

\[
B(u_0) = \left\{ a \in \mathbb{R}^N : \limsup_{x \to a, t \searrow T} |u(x, t)| = \infty \right\}.
\]

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Each element of $B(u_0)$ is called a blow-up point of $u$. We say that the solution $u$ of (1.1) blows up only at space infinity if, in addition to (1.2), $B(u_0) = \emptyset$. In this case, the global blow-up profile $u(x, T) := \lim_{t \to T} u(x, t)$ is defined for every $x \in \mathbb{R}^N$.

Let us recall known results on the blow-up at space infinity. Lacey in [5] considered a one-dimensional problem $u_t = \Delta u + f(u)$ on the half-line and constructed examples of solutions that blow up only at space infinity. He also obtained results of the global blow-up profile. Giga and Umeda in [4] considered the equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$ and showed that the blow-up at space infinity occurs if the initial data $u_0$ satisfies

$$0 < u_0 < M \quad \text{and} \quad \lim_{|x| \to \infty} u_0(x) = M$$

for some constant $M > 0$. Shimojō in [12] considered semilinear heat equations on $\mathbb{R}^N$ and calculated the shape of global blow-up profile of solutions at the blow-up time. It is also proved that such blow-up is always complete, that means that the solution cannot extend as a weak solution after blow-up time.

For the case $0 < m < 1$, the heat conductivity $mu^{m-1}$ becomes small as $u$ increases. Hence, we can see that diffusion is very slow when $u$ is large. Thus, the blow-up at space infinity must occur as the result for semilinear heat equation of [3]. This is proved by Seki for $0 < m \leq 1 < p$ (see [10]). He also discusses the generalization of the nonlinearity of the form $u_t = \Delta k(u) + f(u)$ including the case $0 < m \leq 1 < p$. On the other hand, if $m > 1$, diffusion is very fast when $u$ is just as large. Hence, the speed of heat propagation, from the space infinity to the origin near the blow-up time, becomes much larger compared to the semilinear problem. Thus, a natural question is: “If $m \in (1, \infty)$ is sufficiently large, does the blow-up only at space infinity fail or not?” Partial answer of this problem was obtained by Seki-Suzuki-Umeda (see [11]). Their result implies that if $1 \leq m < p$, the blow-up only at space infinity occurs. Motivated by these results, we consider the following problem: Can the blow-up be confined to space infinity even if diffusion is so large that $m > p > 1$?

In this paper, we give a partial answer to this problem and show that the total blow-up, which means that $B(u_0) = \mathbb{R}^N$, occurs.

**Theorem 1.1.** Let $p > 1$ and $m - p > 2(p - 1)/N$. Then problem (1.1) has a total blow-up solution with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying

$$0 < u_0 < M \quad \text{and} \quad \lim_{|x| \to \infty} u_0(x) = M$$

for a certain positive constant $M \in \mathbb{R}$. 288
This paper is organized as follows. In Section 2, we discuss the condition $m - p > 2(p - 1)/N$ of Theorem 1.1 from the point of asymptotic expansion. The rigorous proof of Theorem 1.1 is given in Section 3 by constructing backward self-similar solution.

Remark 1.1. For problem (1.1) with nonnegative initial data satisfying the condition $\lim_{|x| \to \infty} u_0(x) = 0$, it is known that if $p > m > 1$, the blow-up set reduces to finite number of points (see [1], [13]). For $1 < p < m$, total blow-up occurs (see [2]). There is also a third possibility, $B(u_0)$ is a bounded domain for $p = m$. See also Mochizuki and Suzuki [7] for higher dimensional problem. They consider the case when the support of the initial data is compact, and that the support of the solution remains bounded if $p > m$ and it spreads out the whole space if $p < m$ at the blow-up time. The precise behavior of such solutions in one dimensional case is considered in the book [9].

2. Formal asymptotics

We shall explain why the condition $m - p > 2(p - 1)/N$ yields total blow-up. We will achieve that by a formal asymptotic calculation. Let $f(u) = u^p$, then the solution of the ODE

\begin{equation}
U' = f(U), \quad U(0) = M, \quad M > 0
\end{equation}

is written as $U(t) = \varphi(T(M) - t)$, where $\varphi(s) := \kappa s^{1/(p-1)}$ and $\kappa := (p - 1)^{-1/(p-1)}$. Here $T = T(M)$ is the blow-up time for the initial data $U(0) = M$. Substituting $t = 0$ gives $M = \varphi(T(M))$. Furthermore, by a simple calculation, we have

\begin{equation}
\varphi'(s) = -f(\varphi(s)), \quad \lim_{s \to +0} \varphi(s) = \infty.
\end{equation}

Let us consider (1.1) with initial data $u_0(x) = M - \varepsilon q_0(x)$, where $q$ is a positive function satisfying $\lim_{|x| \to \infty} q_0(x) = 0$ and $\varepsilon > 0$ is a small constant. The first approximation at space infinity must be the flat solution $\varphi(T - t)$. In order to calculate the second term, we shall consider a formal outer expansion

$$u(x, t) = \sum_{i=0}^{\infty} u^{(i)}(x, t) \varepsilon^i$$

and substitute this into $u_t = \Delta k(u) + f(u)$, where $k(u) = u^m$. Then

$$u_t^{(0)} = \Delta k(u^0) + f(u^{(0)}),$$

$$u_t^{(1)} = k'(u^{(0)}) \Delta u^{(1)} + f'(u^{(0)}) u^{(1)}.$$
Observing the initial condition at space infinity, we assume \( u^{(0)}(x, t) = \varphi(T - t) \) as the first approximation of the solution, hence

\[
(2.3) \quad u_t^{(1)} = k'(\varphi(T - t))\Delta u^{(1)} + f'(\varphi(T - t))u^{(1)}.
\]

Let \( q(x, t) = e^{\Phi(t)}\Delta q_0 \) be a solution of \( q_t = k'(\varphi(T - t))\Delta q \) with the initial condition \( q(x, 0) = q_0(x) \in L^1(\mathbb{R}^N) \). In other words,

\[
q(x, t) = e^{\Phi(t)}\Delta q_0, \quad \Phi(t) = \int_0^t k'(\varphi(T - \tau)) \, d\tau.
\]

Here we employ the notation

\[
(e^s\Delta q_0)(x) := \int_{\mathbb{R}^N} G(x - y, s)q_0(y) \, dy
\]

where \( G \) is the fundamental solution of the heat equation in \( \mathbb{R}^N \):

\[
G(x, s) := \frac{1}{(4\pi s)^{N/2}} \exp \left( -\frac{|x|^2}{4s} \right).
\]

Then the solution of (2.3) is represented as \( u^{(1)}(x, t) = -f(\varphi(T - t))q(x, t) \). This can be easily checked from the following calculation.

\[
\begin{align*}
    u_t^{(1)} &= -f(\varphi(T - t))q_t - \frac{df(\varphi(T - t))}{dt}q \\
    &= -f(\varphi(T - t))q_t + f'(\varphi(T - t))\varphi'(T - t)q \\
    &= -f(\varphi(T - t))k'(\varphi(T - t))\Delta q - f'(\varphi(T - t))f(\varphi(T - t))q \\
    &= k'(\varphi(T - t))\Delta u^{(1)} + f'(\varphi(T - t))u^{(1)},
\end{align*}
\]

where we applied (2.2) and substitute \( s = T - t \). By a formal asymptotic expansion, together with \( \varphi'(T - t) = -f(\varphi(T - t)) \) again, we get

\[
u(x, t) = \varphi(T - t) - \varepsilon f(\varphi(T - t))q(x, t) + O(\varepsilon^2) = \varphi(T - t + \varepsilon q(x, t))
\]

provided that \(|x|\) is sufficiently large so that \( T - t \gg q(x, t) \). We shall discuss a sufficient condition for this approach. Note that \( \Phi(t) \) is proportional to \( (T - t)^{(p-m)/(p-1)} - T^{(p-m)/(p-1)} \), which implies \( \Phi(T) = \infty \) if \( m > p \). Assume, for simplicity, that the support of \( q_0 \) is compact. Then by applying the inequality

\[
\sup_{x \in \mathbb{R}^N} |q(x, t)| \leq \frac{1}{(4\pi \Phi(t))^{N/2}} \int_{\mathbb{R}^N} q_0(x) \, dx,
\]
we get the following sufficient condition for $T - t \gg q(x, t)$:

$$T - t \gg O((T - t)^{N(m-p)/(2(p-1))}) = O(\Phi(t)^{-N/2}) \gg q(x, t).$$

Since we are interested in what happens as $t \to T_-$, we need the restriction below, which appeared in Theorem 1.1.

$$1 < \frac{N(m-p)}{2(p-1)} \iff m - p > \frac{2}{N}(p-1).$$

Under this condition, we obtain the following approximation:

$$u(x, t) \approx \varphi(T - t + \varepsilon e^{\Phi(t) \Delta q_0}) \quad \text{if } t \approx T$$

provided that $|x|$ is sufficiently large so that $T - t \gg q(x, t)$. Here $a \approx b$ means that there exist two constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$, where $a$ and $b$ are two positive functions. Taking a limit $t \to T$ and regarding $e^{\Phi(T) \Delta q_0} \equiv 0$, we expect that the total blow-up occurs when $m - p > 2(p - 1)/N$. On the other hand, the above formal calculation suggests that $m - p < 2(p - 1)/N$ yields the blow-up only at space infinity, and the global profile must be

$$(2.4) \quad u(x, T) \approx \varphi(\varepsilon e^{\Phi(T) \Delta q_0}) \quad \text{if } t \approx T.$$

Note that $\Phi(T) < \infty$ if $m - p < 2(p - 1)/N$. This conjecture (2.4) is proved rigorously in [12] for the semi-linear problem ($m = 1$), by constructing suitable sub-super solutions.

### 3. Total blow-up for quasilinear equation

Our aim of this section is to construct a backward self-similar total blow-up solution of problem (1.1) with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying (1.3).

Assume the solution $u$ of (1.1) blows up in finite time and let $T > 0$ be its blow-up time. We introduce a simple change of variable as described in Section 2:

$$(3.1) \quad u(x, t) = \varphi(T - t + h(x, t)).$$

From this and $\lim_{s \to 0} \varphi(s) = \infty$, we can see that the blow-up of the solution $u(x, t)$ for (1.1) as $t \to T$ corresponds to the extinction of the solution $h(x, t)$ as $t \to T$. By a simple calculation together with (3.1) and (2.2),

$$\partial_t \varphi(T - t + h) = \varphi'(T - t + h)(h_t - 1), \quad f(\varphi(T - t + h)) = -\varphi'(T - t + h).$$
By substituting (3.1) into $\Delta u^m = m(m-1)u^{m-2}|\nabla u|^2 + mu^{m-1}\Delta u$, we have

$$
\Delta \varphi^m(T - t + h) \\
= m(m-1)\varphi^{m-2}(T - t + h)|\varphi'(T - t + h)\nabla h|^2 \\
+ m\varphi^{m-1}(T - t + h)(\varphi'(T - t + h)\Delta h + \varphi''(T - t + h)|\nabla h|^2) \\
= m(m-1)\varphi^{m-2}(T - t + h)|\varphi'(T - t + h)\nabla h|^2 \\
+ m\varphi^{m-1}(T - t + h)(\Delta h - f'(\varphi(T - t + h))|\nabla h|^2)\varphi'(T - t + h).
$$

Here we apply the relation $\varphi''(s) = -f'(\varphi(s))\varphi'(s)$, which can be shown by differentiating (2.2). Substituting (3.1) into (1.1) and dividing it by $\varphi'(T - t + h)$, we obtain

$$
h_t = m\varphi^{m-1}(T - t + h)\left(\Delta h + \left((m-1)\frac{\varphi'(T - t + h)}{\varphi(T - t + h)} - f'(\varphi(T - t + h))\right)|\nabla h|^2\right).
$$

Applying $\varphi'(s)/\varphi(s) = -s^{-1}/(p-1)$ and $f'(\varphi(s)) = ps^{-1}/(p-1)$, we get the equation

$$
(3.2) \quad h_t = \frac{mk^{m-1}}{(T - t + h)(m-1)/(p-1)}\left(\Delta h - \frac{(m+p-1)|\nabla h|^2}{(p-1)(T - t + h)}\right)
$$

with the initial data $h(\cdot, 0) = \varphi^{-1}(u_0) - T$.

Next we introduce new space and time variables and a function

$$
w(y, \sigma) := \frac{h(x, t)}{T - t}, \quad y := (T - t)^{\beta}x, \quad \sigma = \log\frac{1}{T - t},
$$

where $\beta := (m-p)/(2(p-1))$ and $h$ is the solution of (3.2). By the chain rule, together with

$$
y_t(x, t) = -e^\sigma \beta y(x, t), \quad y_x(x, t) = e^{-\beta}e^\sigma, \quad \sigma_t(t) = e^\sigma,
$$

we obtain

$$
h_t(x, t) = \partial_t((T - t)w(y, \sigma)) = -\beta y \cdot \nabla w(y, \sigma) + w_\sigma(y, \sigma) - w(y, \sigma)
$$

and

$$
\nabla h(x, t) = e^{-(\beta+1)\sigma} \nabla w(y, \sigma), \quad \Delta h(x, t) = e^{-(2\beta+1)\sigma} \Delta w(y, \sigma).
$$

Substituting these into (3.2), we have

$$
-\beta y \cdot \nabla w(y, \sigma) + w_\sigma(y, \sigma) - w(y, \sigma)
$$

$$
= \frac{mk^{m-1}}{(1 + w(y, \sigma))^{(m-1)/(p-1)}}e^{((m-1)/(p-1) - (2\beta+1)\sigma}
$$

$$
\times \left(\Delta w(y, \sigma) - \frac{m+p-1}{p-1} \frac{|\nabla w(y, \sigma)|^2}{1 + w(y, \sigma)}\right).
$$
Therefore, the function \( w \) satisfies the rescaled equation

\[
(3.3) \quad w_\sigma = \frac{m\kappa^{m-1}}{(1+w)^{2\beta+1}} \left( \Delta w - \frac{m+p-1}{p-1} \frac{|\nabla w|^2}{1+w} \right) + (\beta y \cdot \nabla w + w)
\]

for \( y \in \mathbb{R}^N \) and \( s > 0 \). We can easily see that

\[
(3.4) \quad \lim_{\sigma \to \infty} \|e^{-\sigma}w(\cdot, \sigma)\|_{L^\infty(\mathbb{R}^N)} = 0 \quad \text{if and only if} \quad B(u_0) = \mathbb{R}^N.
\]

The simplest example of a solution of (3.3) is a constant \( w \equiv 0 \), which corresponds to a flat solution \( u(x, t) = U(t) \) of the original problem (1.1). Here \( U(t) \) is the solution of (2.1). Another typical example is the self-similar solution. In our case, it has the form \( h(x, t) = (T-t)^{-\beta}g((T-t)^{\beta}x) \), where \( g(y) \) satisfies

\[
(3.5) \quad \Delta g - \frac{m+p-1}{p-1} \frac{|\nabla g|^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}} (\beta y \cdot \nabla g + g) = 0
\]

with \( y = (T-t)^{\beta}x \). In other words, a solution \( h \) is self-similar if its rescaled function \( w(y, \sigma) \) is independent of \( \sigma \). If we assume that \( g(y) \) is a radial function, \( g = g(r) \) is the solution of the following ordinary differential equation:

\[
(3.6) \quad g_{rr} + \frac{N-1}{r} g_r - \frac{m+p-1}{p-1} \frac{g_r^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}} (\beta rg_r + g) = 0,
\]

\[
(3.7) \quad g(0) = \mu, \quad g_r(0) = 0,
\]

where \( r = |y| \) and \( \mu > 0 \) is a constant.

Let us note that equation (3.6) has a trivial solution \( g \equiv 0 \), as well as the spatially homogeneous solution \( g \equiv -1 \). Let us also note that problem (3.6)–(3.7) admits a solution \( g(r) \) with asymptotic behavior:

\[
(3.8) \quad g(r) = \mu - \frac{\mu(1+\mu)^{2\beta+1}}{2m\kappa^{m-1}N} r^2 + o(r^2) \quad \text{as} \ r \to 0.
\]

This asymptotics is obtained by solving an approximated ordinary differential equation:

\[
g_{rr} + \frac{(1+\mu)^{2\beta+1}}{m\kappa^{m-1}} g \approx 0 \quad \text{for} \ r \approx 0,
\]

which comes from the even symmetric assumption \( g_r(0) = 0 \) and \( g(0) = \mu \).

We must find a value \( \mu \) with the corresponding solution of the above problem (3.6)–(3.7) that is nonnegative and decreasing at space infinity.

**Proposition 3.1.** Let \( p > 1 \) and \( m - p > 2(p - 1)/N \). Then problem (3.6)–(3.7) has a strictly positive monotone solution satisfying \( g(\infty) = 0 \) if \( \mu > 0 \) is sufficiently small.
If we assume this Proposition, by (3.1), the corresponding solution \( u \) of problem (1.1) is written in the form:

\[
 u_s(x, t) = \varphi((T - t)(1 + g((T - t)\beta x))), \quad \beta > 0.
\]

Combining this with \( \varphi(0) = \infty \), we obtain \( u_s(x, T) = \infty \) for any \( x \in \mathbb{R}^N \). Thus \( B(u_s(\cdot, 0)) = \mathbb{R}^N \). Furthermore, condition (1.3) of the initial value can be easily checked and our result is obtained. Now we shall prove the existence of strictly positive solution \( g = g(r) \) for problem (3.6)–(3.7).

**Lemma 3.1.** Let \( g = g(r) \) be the solution of problem (3.6)–(3.7). If \( g > 0 \) on an interval \([0, R_0)\), then \( g \) is strictly decreasing on \([0, R_0)\).

**Proof.** Define

\[
 r_0 = \sup\{ r > 0 : g \text{ is strictly decreasing on } [0, r]\}
\]

and assume \( r_0 < R_0 \). Then the definition of \( r_0 \) implies \( g_r(r_0) = 0 \) (both \( g_r(r_0) > 0 \) and \( g_r(r_0) < 0 \) easily lead to a contradiction) and (3.6) implies \( g_{rr}(r_0) < 0 \). This in turn means that \( g \) is strictly decreasing on a right neighborhood of \( r_0 \), a contradiction with the definition of \( r_0 \). Hence \( r_0 \geq R_0 \). 

By Lemma 3.1, one can distinguish the following two cases:

(a) \( g > 0 \) on \([0, \infty)\) and \( g \) is strictly decreasing on \([0, \infty)\).

(b) There exists \( R \in (0, \infty) \) such that \( g > 0 \) on \([0, R)\) and \( g(R) = 0 \). This implies that \( g \) is strictly decreasing on \([0, R)\); thus, by continuity, it is strictly decreasing on \([0, R]\). In particular, \( g_r(R) < 0 \).

Now we exclude the second case (b) using the following lemma.

**Lemma 3.2.** Assume that \( \beta N > (1 + \mu)^{2\beta + 1} \). Let \( g = g(r) \) be the solution of problem (3.6)–(3.7). Then \( g > 0 \) on \([0, \infty)\).

**Proof.** The decay rate of the solution is given by the solution of \( \beta r \overline{\gamma}_r + \overline{\gamma} = 0 \), which is the dominant term of the ODE (3.6). Thus, we introduce a function

\[
(3.9) \quad \nu := -\frac{\beta rg_r}{g} : [0, R) \to [0, \infty).
\]

By the definition of \( R \), the function \( \nu \) is a nonnegative function and is well-defined. Assume that \( R < \infty \). Then case (b) of Lemma 3.1 implies that \( \lim_{r \to R} \nu(r) = \infty \).
Differentiating (3.9) and using (3.6), we get
\begin{align*}
v_r &= -\frac{\beta r}{g} \left( g_{rr} + \frac{1}{r} g_r \right) + \beta r \left( \frac{g_r}{g} \right)^2 \\
&= \beta(N-2) \frac{g_r}{g} + \beta r \left( \frac{g_r}{g} \right)^2 - \frac{m + p - 1}{p - 1} \frac{\beta rg_r^2}{g(1 + g)} + \frac{\beta r(1 + g)^{2\beta + 1}}{m \kappa^{m-1}} (1 - v) \\
&= -(N-2) \frac{v}{r} + \frac{v^2}{\beta r} - \frac{m + p - 1}{p - 1} \frac{m \kappa^{m-1}}{1 + g} \frac{v^2}{\beta r} + \frac{\beta r(1 + g)^{2\beta + 1}}{m \kappa^{m-1}} (1 - v) \\
&= -(N-2) \frac{v}{r} + \left( 1 - \frac{m + p - 1}{p - 1} \frac{m \kappa^{m-1}}{1 + g} \right) \frac{v^2}{\beta r} + \frac{\beta r(1 + g)^{2\beta + 1}}{m \kappa^{m-1}} (1 - v).
\end{align*}

From (3.8) and (3.9), we see that
\[ v(r) = \frac{\beta (1 + \mu)^{2\beta + 1}}{m \kappa^{m-1} N} r^2 + o(r^2) \quad \text{as} \quad r \to 0. \]

We will use this asymptotics in order to estimate the function \( v \) from above. Next we shall check that the function \( v(r) := \frac{\beta (1 + \mu)^{2\beta + 1}}{m \kappa^{m-1} N} r^2 \) is a super-solution of the above ODE provided that
\begin{equation}
1 \leq \beta N \frac{(1 + g)^{2\beta + 1}}{(1 + \mu)^{2\beta + 1}} + \frac{m + p - 1}{p - 1} \frac{g}{1 + g}
\end{equation}
for all \( r \in [0, R) \). In fact, under condition (3.10), we get
\begin{align*}
\overline{v}_r + (N-2) \frac{\overline{v}}{r} - \left( 1 - \frac{m + p - 1}{p - 1} \frac{g}{1 + g} \right) \overline{v}^2 \beta r - \frac{\beta r(1 + g)^{2\beta + 1}}{m \kappa^{m-1}} (1 - \overline{v}) \\
&= \frac{N \overline{v}}{r} \left( 1 - \frac{(1 + g)^{2\beta + 1}}{(1 + \mu)^{2\beta + 1}} \right) - \left( 1 - \frac{m + p - 1}{p - 1} \frac{g}{1 + g} \right) \frac{\overline{v}^2}{\beta r} - \beta N \frac{(1 + g)^{2\beta + 1}}{(1 + \mu)^{2\beta + 1}} \frac{\overline{v}}{\beta r} \\
&\geq - \left( 1 - \frac{m + p - 1}{p - 1} \frac{g}{1 + g} - \beta N \frac{(1 + g)^{2\beta + 1}}{(1 + \mu)^{2\beta + 1}} \right) \frac{\overline{v}^2}{\beta r} \geq 0.
\end{align*}

Here we used the relations \( \overline{v}_r = 2 \overline{v}/r \) together with
\[ \frac{\beta r(1 + g)^{2\beta + 1}}{m \kappa^{m-1}} = \frac{N \overline{v}}{r} \frac{(1 + g)^{2\beta + 1}}{(1 + \mu)^{2\beta + 1}} \]
and the inequality \( g(r) \leq \mu \) for \( r \in [0, R] \). Condition (3.10) is satisfied because the function \( g \) is nonnegative on \([0, R]\) and \( \beta N > (1 + \mu)^{2\beta + 1} \). Therefore, by the comparison argument, \( v \leq \overline{v} \) for all \( r \in [0, R] \) and \( \lim_{r \to R} v(r) \leq \overline{v}(R) < \infty \). This yields a contradiction. \( \square \)
Proof of Proposition 3.1. Let \( p > 1 \) and \( m - p > 2(p - 1)/N \), then \( \beta N > 1 \). By Lemma 3.2, problem (3.6)–(3.7) has a positive solution if we choose \( \mu > 0 \) sufficiently small such that \( \beta N > (1 + \mu)^{2\beta + 1} \). Lemma 3.1 implies that this solution is strictly decreasing. Furthermore, since there exists no positive spatially homogeneous solution of equation (3.6), we obtain \( g(\infty) = 0 \). Hence we obtain the result. \( \square \)

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