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# REGULATED FUNCTIONS WITH VALUES IN BANACH SPACE

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*Cordially dedicated to the memory of Štefan Schwabik*

*Abstract.* This paper deals with regulated functions having values in a Banach space. In particular, families of equiregulated functions are considered and criteria for relative compactness in the space of regulated functions are given.

*Keywords*: regulated function; bounded variation; function with values in a Banach space; $\varphi$-variation; relative compactness; equiregulated function

*MSC 2010*: 26A45, 46E40

## Introduction

This paper is an extension of the previous one (see [2]), where regulated functions with values in Euclidean spaces were considered. Here, we deal with regulated functions having values in a Banach space. We discuss some of the properties of the space of such regulated functions, including compactness theorems.

Classic results of mathematical analysis are being used (see [4]) and some ideas from previous works on the topic of regulated functions appear here (see [3], [5]).

## 1. Notation and definitions

(i) The symbol $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{R}^N$ (where $N \in \mathbb{N}$) is the $N$-dimensional Euclidean space with the usual norm $|\cdot|_N$. We write $\mathbb{R}$ and $|\cdot|$ instead of $\mathbb{R}^1$ and $|\cdot|_1$.

(ii) Throughout the paper, the symbol $X$ will denote a Banach space with a norm $\|\cdot\|_X$ and $\mathcal{C}([a,b];X)$ is the set of all continuous functions $f\colon [a,b] \to X$.

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(iii) We say that a function $h\colon [a,b] \to \mathbb{R}$ is increasing if $a \leqslant s < t \leqslant b$ implies $h(s) < h(t)$; the function $h$ is non-decreasing if $a \leqslant s < t \leqslant b$ implies $h(s) \leqslant h(t)$.

(iv) We say that $g\colon [a,b] \to X$ is a finite step function, or shortly step function, if it is piecewise constant; i.e., there is a division $a = a_0 < a_1 < \ldots < a_k = b$ such that the function $g$ is constant on each of the intervals $(a_{i-1}, a_i)$, $i = 1, 2, \ldots, k$.

(v) We denote by $\mathcal{D}_{a,b}$ the set of divisions $\{a_0, \ldots, a_k\}$ such that $a = a_0 < a_1 < \ldots < a_k = b$.

(vi) For any function $f\colon [a,b] \to X$, we write $\|f\|_\infty = \sup\{\|f(t)\|_X \colon t \in [a,b]\}$. If $\|f\|_\infty < \infty$, we say that the function $f$ is bounded; $\|\cdot\|_\infty$ is called the sup-norm.

(vii) We say that a sequence of functions $f_n\colon [a,b] \to X$, $n \in \mathbb{N}$, is uniformly convergent to a function $f_0 : [a,b] \to X$ (or that $f_0$ is the uniform limit of $\{f_n\}_{n\in\mathbb{N}}$) if $\|f_n - f_0\|_\infty \to 0$ with $n \to \infty$; we denote $f_n \rightrightarrows f_0$.

## 2. Basic properties of a regulated function

**Definition 2.1.** We say that a function $f\colon [a,b] \to X$ is *regulated* if the limit $f(t-) = \lim\limits_{\tau \to t-} f(\tau)$ exists for every $t \in (a,b]$, and the limit $f(t+) = \lim\limits_{\tau \to t+} f(\tau)$ exists for every $t \in [a,b)$. We denote by $G([a,b];X)$ the set of all regulated functions $f\colon [a,b] \to X$.

Obviously, any finite step function on $[a,b]$ and any continuous function on $[a,b]$ are regulated on $[a,b]$. Moreover, any function with bounded variation on $[a,b]$ and any monotone real valued function are regulated on $[a,b]$.

**Proposition 2.2.** *Assume that $f_n\colon [a,b] \to X$, $n \in \mathbb{N}$, are regulated functions and $f_0\colon [a,b] \to X$ is a function such that $f_n \rightrightarrows f_0$. Then the function $f_0$ is regulated and $f_n(t-) \to f_0(t-)$ for each $t \in (a,b]$, $f_n(t+) \to f_0(t+)$ for each $t \in [a,b)$.*

Proof. The proof follows easily from the classical Moore-Osgood theorem on exchanging the order of limits, cf. e.g. [4]. $\square$

**Theorem 2.3.** *The following properties of a function $f\colon [a,b] \to X$ are equivalent:*

(i) *The function $f$ is regulated.*

(ii) *The function $f$ is the uniform limit of a sequence of step functions.*

(iii) *For every $\varepsilon > 0$ there is a step function $g\colon [a,b] \to X$ such that $\|f - g\|_\infty < \varepsilon$.*

(iv) *For every $\varepsilon > 0$ there is a division $a = a_0 < a_1 < \ldots < a_k = b$ such that if $a_{i-1} < t' < t'' < a_i$ for some $i \in \{1, 2, \ldots, k\}$ then $\|f(t'') - f(t')\|_X < \varepsilon$.*

P r o o f. (i) $\Rightarrow$ (iv): Let $\varepsilon > 0$ be given. For every $x \in (a, b]$, define

$$s_x = \inf\Big\{s \in (a, x)\colon \text{ if } \tau', \tau'' \in (s, x) \text{ then } \|f(\tau') - f(\tau'')\|_X < \frac{\varepsilon}{2}\Big\}.$$

For every $x \in [a, b)$, define

$$t_x = \sup\Big\{t \in (x, b)\colon \text{ if } \tau', \tau'' \in (x, t) \text{ then } \|f(\tau') - f(\tau'')\|_X < \frac{\varepsilon}{2}\Big\}. \tag{2.1}$$

It follows from the existence of the limits $f(x-)$, $f(x+)$ that $s_x < x$ and $t_x > x$.

Obviously,

$$[a, t_a) \cup \bigcup_{x \in (a,b)} (s_x, t_x) \cup (s_b, b] = [a, b]$$

and, since $[a, b]$ is compact, there are $k \in \mathbb{N}$ and a finite set $\{a_1, \ldots, a_{k-1}\}$ of points in $(a, b)$ such that $a_1 < a_2 < \ldots < a_{k-1}$,

$$[a, t_a) \cup \bigcup_{i=1}^{k-1} (s_{a_i}, t_{a_i}) \cup (s_b, b] = [a, b]. \tag{2.2}$$

We shall verify that $s_{a_i} < t_{a_{i-1}}$ for $i \in \{1, 2, \ldots, k\}$. On the contrary, assume that there is $\sigma$ such that $t_{a_{i-1}} \leqslant \sigma \leqslant s_{a_i}$. Thanks to (2.2), there is $j \notin \{i-1, i\}$ such that $\sigma \in (s_{a_j}, t_{a_j})$. If $j < i - 1$ then by (2.1) we have $\|f(\tau') - f(\tau'')\|_X < \frac{1}{2}\varepsilon$ for all $\tau', \tau'' \in (a_j, t_{a_j})$, which specifically holds also for all $\tau', \tau'' \in (a_{i-1}, t_{a_j})$. Hence $t_{a_j} \leqslant t_{a_{i-1}} \leqslant \sigma < t_{a_j}$ which is a contradiction. Similarly, if $j > i$ we find that this leads to a contradiction as well.

Consequently, for any $i \in \{1, 2, \ldots, k\}$, the intersection $(s_{a_i}, t_{a_{i-1}}) \cap (a_{i-1}, a_i)$ is nonempty and we choose $b_i \in (s_{a_i}, t_{a_{i-1}}) \cap (a_{i-1}, a_i)$.

Now, if $a_{i-1} < t' < t'' < a_i$ for some $i \in \{1, \ldots, k\}$, there are three possibilities: either $a_{i-1} < t' < t'' \leqslant b_i$ or $b_i \leqslant t' < t'' < a_i$ or $a_{i-1} < t' \leqslant b_i \leqslant t'' < a_i$. In the first case, both $t'$, $t''$ are in $(a_{i-1}, t_{a_{i-1}})$, and thanks to (2.1)

$$\|f(t'') - f(t')\|_X < \frac{\varepsilon}{2}.$$

Similarly, if $b_i \leqslant t' < t'' < a_i$ for some $i$ then $t', t'' \in (s_{a_i}, a_i)$ and

$$\|f(t'') - f(t')\|_X < \frac{\varepsilon}{2};$$

and, if $a_{i-1} < t' \leqslant b_i \leqslant t'' < a_i$ for some $i$ then $t', b_i \in (a_{i-1}, t_{a_{i-1}})$, and $b_i, t'' \in (s_{a_i}, a_i)$ and hence

$$\|f(t'') - f(t')\|_X \leqslant \|f(t'') - f(b_i)\|_X + \|f(b_i) - f(t')\|_X < \varepsilon.$$

To summarize, (iv) is true.

(iv) $\Rightarrow$ (iii): Given $\varepsilon > 0$ we can find the described division $a = a_0 < a_1 < \ldots < a_k = b$; choose points $\tau_i \in (a_{i-1}, a_i)$ and define $g(\tau) = f(\tau_i)$ for $\tau \in (a_{i-1}, a_i)$, $i = 1, 2, \ldots, k$; $g(a_i) = f(a_i)$, $i = 0, 1, \ldots, k$. Then $g$ is a step function and $\|g(\tau) - f(\tau)\|_X < \varepsilon$ for every $\tau \in [a, b]$.

(iii) $\Rightarrow$ (ii): For $\varepsilon = 1/n$, we can find a step function $g_n$ such that $\|f - g_n\|_\infty < 1/n$. Hence, $g_n \rightrightarrows f$.

(ii) $\Rightarrow$ (i): This implication follows from Proposition 2.2. $\square$

Let us notice that the equivalences contained in Theorem 2.3 have been already proved in [3] in a slightly different way. The following result also can be found in [3], but no detailed proof is provided therein.

**Proposition 2.4.** *If a function $f\colon [a, b] \to X$ is regulated, then*

(i) *for any $c > 0$, the sets $\{t \in [a, b)\colon \|f(t+) - f(t)\|_X \geqslant c\}$ and $\{t \in (a, b]\colon \|f(t-) - f(t)\|_X \geqslant c\}$ are finite;*

(ii) *the sets $J^+ = \{t \in [a, b)\colon f(t+) \neq f(t)\}$ and $J^- = \{t \in (a, b]\colon f(t-) \neq f(t)\}$ are at most countable.*

Proof. (i) By Theorem 2.3 (iv), there is a division $a = a_0 < \ldots < a_k = b$ such that

$$\|f(u) - f(t)\|_X < \frac{c}{2} \quad \text{whenever } u, t \in (a_{i-1}, a_i) \text{ for some } i.$$

Passing to the limit $u \to t+$ we get

$$\|f(t+) - f(t)\|_X \leqslant \frac{c}{2} < c \quad \text{for all } t \in [a, b] \setminus \{a_0, \ldots, a_k\}.$$

(ii) It is evident that $J^+ = \bigcup\limits_{n \in \mathbb{N}} \{t \in [a, b)\colon \|f(t+) - f(t)\|_X \geqslant 1/n\}$; this is a countable union of finite sets, therefore at most countable. Similarly for the left-sided limits. $\square$

In the following theorem we are going to use the notion of total $\varphi$-variation which appears in [1].

**Definition 2.5.** Let us denote by $\Phi$ the set of all increasing functions $\varphi\colon [0, \infty) \to [0, \infty)$ such that $\varphi(0) = \varphi(0+) = 0$, $\varphi(\infty) = \infty$. For $f\colon [a, b] \to X$, given $\varphi \in \Phi$ and a division $d = \{t_0, t_1, \ldots, t_m\}$; $d \in \mathcal{D}_{a,b}$, we define

$$\mathcal{V}_d^\varphi(f) = \sum_{j=1}^{m} \varphi(\|f(t_j) - f(t_{j-1})\|_X),$$

and the total $\varphi$-variation of $f$ by

$$\varphi\text{-}\mathop{\mathrm{Var}}_{[a,b]}(f) = \sup\bigl\{\mathcal{V}_d^\varphi(f)\colon d \in \mathcal{D}_{a,b}\bigr\}.$$

**Theorem 2.6.** *The following properties of a function $f\colon [a,b]\to X$ are equivalent:*

(i) *The function $f$ is regulated.*

(ii) *There is a continuous function $g\colon [c,d]\to X$ and a non-decreasing function $h\colon [a,b]\to[c,d]$ such that $f(t)=g(h(t))$ for every $t\in[a,b]$.*

(iii) *There is a continuous increasing function $\omega\colon [0,\infty)\to[0,\infty)$, $\omega(0+)=0$, and a non-decreasing function $h\colon [a,b]\to\mathbb{R}$ such that $\|f(t)-f(s)\|_X\leqslant\omega(|h(t)-h(s)|)$ holds for every $s,t\in[a,b]$.*

(iv) *There is a non-decreasing function $\omega\colon [0,\infty)\to[0,\infty)$, $\omega(0+)=0$, and a non-decreasing function $h\colon [a,b]\to\mathbb{R}$ such that $\|f(t)-f(s)\|_X\leqslant\omega(|h(t)-h(s)|)$ holds for every $s,t\in[a,b]$.*

(v) *There is a continuous increasing function $\varphi\colon [0,\infty)\to[0,\infty)$, $\varphi(0)=\varphi(0+)=0$, $\varphi(\infty)=\infty$, such that $\varphi\text{-}\operatorname{Var}_{[a,b]}(f)<\infty$.*

(vi) *There is an increasing function $\varphi\colon [0,\infty)\to[0,\infty)$, $\varphi(0)=\varphi(0+)=0$, $\varphi(\infty)=\infty$, such that $\varphi\text{-}\operatorname{Var}_{[a,b]}(f)\leqslant 1$.*

P r o o f. The scheme of the proof is (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i); (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (ii): According to Proposition 2.4, for any $n\in\mathbb{N}$ the sets $J_n^-$, $J_n^+$ defined by

$$J_n^-=\Big\{t\in(a,b]\colon\ \|f(t-)-f(t)\|_X\geqslant\frac{1}{n}\Big\},$$
$$J_n^+=\Big\{t\in(a,b]\colon\ \|f(t+)-f(t)\|_X\geqslant\frac{1}{n}\Big\}$$

are finite. Obviously, we can find non-decreasing functions $h_n\colon [a,b]\to\mathbb{R}$ with left- and right-hand discontinuity points in $J_n^-$ and $J_n^+$, respectively. Moreover, $h_n$ can be chosen in such a way that all of them are bounded by 1. Then we can define

$$h(t)=t+\sum_{n=1}^{\infty}2^{-n}h_n(t)$$

for $t\in[a,b]$. Denote $h(a)=c$ and $h(b)=d$. The function $h$ is increasing, and it has left-handed and right-handed discontinuities at all points of the sets $J^-=\bigcup\limits_{n\in\mathbb{N}}J_n^-$ and $J^+=\bigcup\limits_{n\in\mathbb{N}}J_n^+$, respectively.

For every $\tau\in[c,d]$, we can find a unique point $t\in[a,b]$ such that either $\tau=h(t)$ or $h(t-)\leqslant\tau<h(t)$, or $h(t)<\tau\leqslant h(t+)$. If $\tau=h(t)$, we define $g(\tau)=f(t)$. If $h(t-)\leqslant\tau<h(t)$, we define

$$g(\tau)=f(t)+\frac{h(t)-\tau}{h(t)-h(t-)}(f(t-)-f(t));$$

if $h(t) < \tau \leqslant h(t+)$, we define

$$g(\tau) = f(t) + \frac{\tau - h(t)}{h(t+) - h(t)}(f(t+) - f(t)).$$

It is obvious that $f(t) = g(h(t))$ holds for each $t \in [a,b]$; we shall verify that the function $g$ is continuous. Certainly $g$ is continuous at each interval of the form $[h(t-), h(t)]$ and $[h(t), h(t+)]$. We need to prove that $g$ is left-continuous for every $\tau = h(t-)$, and right-continuous for every $\tau = h(t+)$.

Assume that $\tau_0 = h(t_0-)$ for some $t_0 \in (a,b]$. Let $\varepsilon > 0$ be given. There is $\delta > 0$ such that

$$(t_0 - \delta, t_0) \subset [a,b] \quad \text{and} \quad \text{if } t_0 - \delta < t < t_0 \text{ then } \|f(t_0-) - f(t)\|_X < \frac{\varepsilon}{3}.$$

Obviously, if $t_0 - \delta < t < t_0$ then

$$\|f(t_0-) - f(t-)\|_X \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \|f(t_0-) - f(t+)\|_X \leqslant \frac{\varepsilon}{3}.$$

Choose a point $\sigma \in (t_0 - \delta, t_0)$ at which the function $h$ is continuous. Let $s \in (h(\sigma), h(t_0-))$ be an arbitrary point. We can find $t \in (\sigma, t_0)$ such that $h(t-) \leqslant s \leqslant h(t+)$. If $s = h(t)$, then

$$\|g(s) - g(h(t_0-))\|_X = \|f(t) - f(t_0-)\|_X < \frac{\varepsilon}{3};$$

if $h(t-) \leqslant s < h(t)$, then

$$\begin{aligned}
\|g(s) - g(h(t_0-))\|_X &\leqslant \|g(s) - g(h(t))\|_X + \|g(h(t)) - g(h(t_0-))\|_X \\
&= \frac{h(t) - s}{h(t) - h(t-)}\|f(t) - f(t-)\|_X + \|f(t) - f(t_0-)\|_X \\
&\leqslant \|f(t) - f(t-)\|_X + \|f(t) - f(t_0-)\|_X \\
&\leqslant 2\|f(t) - f(t_0-)\|_X + \|f(t-) - f(t_0-)\|_X < \varepsilon.
\end{aligned}$$

Similarly, if $h(t) < s \leqslant h(t+)$, then $\|g(s) - g(h(t_0-))\|_X < \varepsilon$. We can conclude that the function $g$ is left-continuous at the point $\tau_0 = h(t_0-)$. Analogously, it can be proved that $g$ is right-continuous at every point $\tau_0 = h(t_0+)$ for $t_0 \in [a,b)$.

(ii) $\Rightarrow$ (iii): The function $\omega$ can be defined by

$$\omega(r) = r + \sup\{\|g(\tau'') - g(\tau')\|_X;\ \tau', \tau'' \in [a,b],\ |\tau'' - \tau'| \leqslant r\}, \quad \omega(0) = 0.$$

Since a function continuous on a compact interval is uniformly continuous, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if } \tau', \tau'' \in [a,b] \text{ and } |\tau'' - \tau'| < \delta \text{ then } \|g(\tau'') - g(\tau')\|_X < \varepsilon.$$

It follows that $\lim\limits_{r\to 0+} \omega(r) = 0$.

It is obvious that the function $\omega$ is increasing, $\omega(\infty) = \infty$. If the function $\omega$ were not continuous at a point $r \in (0,\infty)$, then $\omega(r+) > \omega(r-)$ would hold.

(1) Assume that $\omega(r) > \omega(r-)$. By definition of $\omega$, there are points $\tau', \tau'' \in [a,b]$ such that

$$|\tau' - \tau''| \leqslant r \quad \text{and} \quad r + \|g(\tau'') - g(\tau')\|_X > \omega(r-).$$

We can find $r_1 \in (0,r)$ such that

$$r_1 + \|g(\tau'') - g(\tau')\|_X > \omega(r-).$$

Since $g$ is continuous, there are $s', s'' \in [a,b]$ such that

$$|s' - s''| < r \quad \text{and} \quad r_1 + \|g(s'') - g(s')\|_X \; > \omega(r-).$$

Denote $\varrho = \max\{r_1, |s' - s''|\}$. Then,

$$\varrho + \|g(s'') - g(s')\|_X \geqslant r_1 + \|g(s'') - g(s')\|_X > \omega(r-) \geqslant \omega(\varrho),$$

which is in contradiction with the definition of $\omega$.

(2) Assume that $\omega(r+) > \omega(r)$. We can fix a point $c$ such that $\omega(r+) > c > \omega(r)$. For any $n \in \mathbb{N}$, we have $\omega(r+1/n) > c$. There are $\tau_n', \tau_n'' \in [a,b]$ such that $|\tau_n'' - \tau_n'| \leqslant r + 1/n$ and

$$\omega\Big(r + \frac{1}{n}\Big) \geqslant r + \frac{1}{n} + \|g(\tau_n'') - g(\tau_n')\|_X > c.$$

We can find convergent subsequences $\tau_{n_k}' \to \tau'$, $\tau_{n_k}'' \to \tau''$; considering that the function $g$ is continuous, we obtain limits at both sides:

$$\omega(r+) \geqslant r + \|g(\tau'') - g(\tau')\|_X \geqslant c > \omega(r);$$

at the same time, $r + \|g(\tau'') - g(\tau')\|_X \leqslant \omega(r)$ because $|\tau' - \tau''| \leqslant r$, which is a contradiction.

(iii) $\Rightarrow$ (iv): This is obvious.

(iv) $\Rightarrow$ (i): For $\varepsilon > 0$ given, we can find $r > 0$ such that $\omega(r) < \varepsilon$; considering that the non-decreasing function $h$ is regulated, we can find a division $a = x_0 < x_1 < \ldots < x_k = b$ such that if $x_{i-1} < s < t < x_i$ then $|h(t) - h(s)| < r$. Then we have

$$\|f(t) - f(s)\|_X \leqslant \omega(|h(t) - h(s)|) \leqslant \omega(r) < \varepsilon.$$

Using Theorem 2.3, we conclude that the function $f$ is regulated.

(iii) $\Rightarrow$ (v): We can assume that $\omega(\infty) = \infty$, otherwise $\omega(r)$ can be replaced by $\omega(r) + r$. Let us define $\varphi = \omega^{-1}$. Then $\varphi \in \Phi$ and for any division $d \in \mathcal{D}_{a,b}$, $d = \{t_0, t_1, \ldots, t_k\}$, we have

$$\begin{aligned} \mathcal{V}_d^{\varphi}(f) &= \sum_{j=1}^{k} \varphi(\|f(t_j) - f(t_{j-1})\|_X) \leqslant \sum_{j=1}^{k} \varphi(\omega(h(t_j) - h(t_{j-1})) \\ &= \sum_{j=1}^{k} [h(t_j) - h(t_{j-1})] = h(b) - h(a). \end{aligned}$$

Then $\varphi\text{-}\operatorname{Var}_{[a,b]}(f) \leqslant h(b) - h(a)$.

(v) $\Rightarrow$ (vi): Denote $\alpha = \varphi\text{-}\operatorname{Var}_{[a,b]}(f)$; if $\alpha = 0$ then $\alpha \leqslant 1$ is satisfied; if $\alpha > 0$, we can define $\psi(x) = \varphi(x)/\alpha$, $x \in [0, \infty)$; then for any division $d \in \mathcal{D}_{[a,b]}$, $d = \{t_0, t_1, \ldots, t_k\}$, we have

$$\mathcal{V}_d^{\psi}(f) = \sum_{j=1}^{k} \psi(\|f(t_j) - f(t_{j-1})\|_X) = \sum_{j=1}^{k} \frac{1}{\alpha} \varphi(\|f(t_j) - f(t_{j-1})\|_X) = \frac{1}{\alpha} \mathcal{V}_d^{\varphi}(f),$$

consequently, $\psi\text{-}\operatorname{Var}_{[a,b]}(f) = 1$.

(vi) $\Rightarrow$ (iv): Define $h(t) = \varphi\text{-}\operatorname{Var}_{[a,t]}(f)$ for all $t \in [a, b]$; the function $h$ is non-decreasing. For any $t', t''$ such that $a \leqslant t' < t'' \leqslant b$, we have

$$h(t'') - h(t') \geqslant \varphi(\|f(t'') - f(t')\|_X)$$

because $d = \{t', t''\}$ is a division of the interval $[t', t'']$.

Keeping in mind that the function $\varphi$ is increasing and $\varphi(0) = \varphi(0+) = 0$, $\varphi(\infty) = \infty$, we can define a function $\omega\colon [0, \infty) \to [0, \infty)$ so that

$$\omega(0) = 0; \quad \omega(r) = x \quad \text{if } r = \varphi(x) \text{ for some } x \in (0, \infty);$$

and

$$\text{if } r \in (\varphi(x-), \varphi(x+)) \text{ for some } x \in [0, \infty) \text{ then } \omega(r) = x.$$

Apparently $\omega(\varphi(x)) = x$ for every $x \in [0, \infty)$ and the function $\omega$ is non-decreasing, $\omega(0+) = 0$ (actually, $\omega$ is continuous, however that is not needed here).

For any $t', t''$ such that $a \leqslant t' < t'' \leqslant b$, we have

$$\|f(t'') - f(t')\|_X = \omega(\varphi(\|f(t'') - f(t')\|_X)) \leqslant \omega\Big(\varphi\text{-}\operatorname*{Var}_{[t',t'']}(f)\Big) = \omega(h(t'') - h(t')).$$

□

The function $g$ as defined in the proof is called the linear prolongation of the function $f$ along the increasing function $h$ (see [2]).

**Proposition 2.7.** *Assume that a function $f\colon [a,b] \to X$ is regulated. Then*

(i) *the function $f$ is bounded,*
(ii) *the image $\operatorname{Im}(f) = \{f(t)\colon t \in [a,b]\}$ is a relatively compact subset of $X$,*
(iii) *there is a sequence of step functions $g_n\colon [a,b] \to X$ such that $g_n \rightrightarrows f$ and $\operatorname{Im}(g_n) \subset \operatorname{Im}(f)$ for every $n \in \mathbb{N}$.*

Proof. (i) According to Theorem 2.3, we can find a step function $g\colon [a,b] \to X$ such that $\|f - g\|_\infty < 1$; then $\|f\|_\infty < \|g\|_\infty + 1$ and a step function is obviously bounded.

(ii) For $\varepsilon > 0$, we can find a step function $g\colon [a,b] \to X$ such that $\|f - g\|_\infty < \varepsilon$. The step function $g$ has finitely many values, i.e., $C = \operatorname{Im}(g) \subset X$ is a finite set. For any $t \in [a,b]$, there is a point $c \in C$ such that $\|c - f(t)\|_X < \varepsilon$ (namely, $c = g(t)$). This means that $C$ is a finite $\varepsilon$-net for the set $\operatorname{Im}(f)$; consequently, $\operatorname{Im}(f)$ is a relatively compact subset of $X$.

(iii) We can see in the proof of Theorem 2.3 that the step functions can be constructed with values from $\operatorname{Im}(f)$. □

## 3. Uniform convergence of regulated functions

**Definition 3.1.** We say that a set $\mathcal{T} \subset G([a,b]; X)$ is *equiregulated* if for every $t \in (a,b]$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $(t-\delta, t) \subset [a,b]$ and if $\tau \in (t-\delta, t)$, then $\|f(t-) - f(\tau)\|_X < \varepsilon$ holds for all $f \in \mathcal{T}$; moreover, for every $t \in [a,b)$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $(t, t+\delta) \subset [a,b]$ and if $\tau \in (t, t+\delta)$, then $\|f(t+) - f(\tau)\|_X < \varepsilon$ holds for all $f \in \mathcal{T}$.

**Proposition 3.2.** *A set of functions $\mathcal{T} \subset G([a,b]; X)$ is equiregulated if and only if for every $\varepsilon > 0$ there is a division $a = a_0 < a_1 < \ldots < a_k = b$ such that if $a_{i-1} < t' < t'' < a_i$ for some $i \in \{1, 2, \ldots, k\}$ then $\|f(t'') - f(t')\|_X < \varepsilon$ holds for all $f \in \mathcal{T}$.*

P r o o f. It can be obtained in the same way as the proof of Theorem 2.3 (i) $\Leftrightarrow$ (iv). $\square$

**Theorem 3.3.** *Assume that a sequence of regulated functions* $f_n\colon [a,b]\to X$, $n\in\mathbb{N}$, *is given, and there is a function* $f_0\colon [a,b]\to X$ *such that* $f_n(t)\to f_0(t)$ *for every* $t\in[a,b]$. *Then the function* $f_0$ *is the uniform limit of the sequence* $\{f_n\}_{n\in\mathbb{N}}$ *if and only if the set* $\{f_n\colon n\in\mathbb{N}\}$ *is equiregulated.*

P r o o f. Assume that $f_n\rightrightarrows f_0$. According to Proposition 2.2, the function $f_0$ is regulated. Let $t\in(a,b]$ be given. For any given $\varepsilon>0$, we can find $n_0\in\mathbb{N}$ such that $\|f_n-f_0\|_\infty<\frac13\varepsilon$ for all $n\geqslant n_0$. For every $n=0,1,\ldots,n_0$, there is $\delta_n>0$ such that $(t-\delta_n,t)\subset[a,b]$ and if $\tau\in(t-\delta_n,t)$, then $\|f_n(t-)-f_n(\tau)\|_X<\frac13\varepsilon$.

Denote $\delta=\min\{\delta_0,\delta_1,\ldots,\delta_{n_0}\}$. If $\tau\in(t-\delta,t)$, then $\|f_n(t-)-f_n(\tau)\|_X<\frac13\varepsilon$ for $n=1,\ldots,n_0$; and if $n\geqslant n_0$ then

$$\|f_n(t-)-f_n(\tau)\|_X\leqslant\|f_n(t-)-f_0(t-)\|_X+\|f_0(t-)-f_0(\tau)\|_X+\|f_0(\tau)-f_n(\tau)\|_X<\varepsilon.$$

The proof for right-sided limits is analogous.

Now, assume that the set $\{f_n\colon n\in\mathbb{N}\}$ is equiregulated. Let $\varepsilon>0$ be given. By Proposition 3.2, there is a division $a=a_0<a_1<\ldots<a_k=b$ such that if $a_{i-1}<t'<t''<a_i$ then $\|f_n(t'')-f_n(t')\|_X<\frac14\varepsilon$ holds for all $n\in\mathbb{N}$. Choose a point $b_i\in(a_{i-1},a_i)$ for each $i=1,2,\ldots,k$. We have $f_n(a_i)\to f_0(a_i)$, $f_n(b_i)\to f_0(b_i)$; we can find $n_0\in\mathbb{N}$ such that if $n\geqslant n_0$ then

$$\begin{aligned}\|f_n(a_i)-f_0(a_i)\|_X&<\varepsilon\quad&&\text{for } i=0,1,\ldots,k,\\ \|f_n(b_i)-f_0(b_i)\|_X&<\frac{\varepsilon}{4}\quad&&\text{for } i=1,2,\ldots,k.\end{aligned}$$

For any $t\in[a,b]$ given, either $t=a_i$ for some $i$, then $\|f_n(t)-f_0(t)\|_X<\varepsilon$; or $t\in(a_{i-1},a_i)$ for some $i\in\{1,2,\ldots,k\}$; since $f_n(t)\to f_0(t)$, there is a fixed $m\geqslant n_0$ such that $\|f_m(t)-f_0(t)\|_X<\frac14\varepsilon$. For any $n\geqslant n_0$ we have

$$\begin{aligned}\|f_n(t)-f_0(t)\|_X\leqslant{}&\|f_n(t)-f_n(b_i)\|_X+\|f_n(b_i)-f_0(b_i)\|_X+\|f_0(b_i)-f_m(b_i)\|_X\\ &+\|f_m(b_i)-f_m(t)\|_X+\|f_m(t)-f_0(t)\|_X<2\varepsilon.\end{aligned}$$

Consequently $f_n\rightrightarrows f_0$. $\square$

**Proposition 3.4.** *Assume that a set* $\mathcal{T}\subset G([a,b];X)$ *is equiregulated. Then*

(i) *for any* $c>0$, *the sets*

$$\begin{aligned}J_c^+&=\{t\in[a,b);\ \text{there is } f\in\mathcal{T} \text{ such that } \|f(t+)-f(t)\|_X\geqslant c\},\\ J_c^-&=\{t\in(a,b];\ \text{there is } f\in\mathcal{T} \text{ such that } \|f(t-)-f(t)\|_X\geqslant c\}\end{aligned}$$

*are finite;*

(ii) *the sets defined by*

$$J^+ = \{t \in [a,b);\ \text{there is } f \in \mathcal{T} \text{ such that } f(t+) \neq f(t)\},$$
$$J^- = \{t \in (a,b];\ \text{there is } f \in \mathcal{T} \text{ such that } f(t-) \neq f(t)\} \tag{3.1}$$

*are at most countable.*

Proof. The proof is analogous to the proof of Proposition 2.4. □

**Lemma 3.5.** *Assume that sets $\mathcal{J} \subset G([a,b];X)$ and $\mathcal{T} \subset G([a,b];X)$ are equiregulated. Then the set $\{f+g\colon f \in \mathcal{J}, g \in \mathcal{T}\}$ is equiregulated.*

Proof. Let $t \in (a,b]$ be given. For any $\varepsilon > 0$ we can find $\delta_1 > 0$ such that $(t-\delta_1, t) \subset [a,b]$ and if $\tau \in (t-\delta_1, t)$ then

$$\|f(t-) - f(\tau)\|_X < \frac{\varepsilon}{2} \quad \text{holds for all } f \in \mathcal{J};$$

and we can find $\delta_2 > 0$ such that $(t-\delta_2, t) \subset [a,b]$ and if $\tau \in (t-\delta_2, t)$ then

$$\|g(t-) - g(\tau)\|_X < \frac{\varepsilon}{2} \quad \text{holds for all } g \in \mathcal{T}.$$

Then we put $\delta = \min\{\delta_1, \delta_2\}$ and if $\tau \in (t-\delta, t)$ then

$$\|(f+g)(t-) - (f+g)(\tau)\|_X \leqslant \|f(t-) - f(\tau)\|_X + \|g(t-) - g(\tau)\|_X < \varepsilon.$$

Similarly for right-sided limits. □

**Proposition 3.6.** *Assume that sequences of regulated functions $f_n\colon [a,b] \to X$, $g_n\colon [a,b] \to X$, $n \in \mathbb{N}$, are given such that $\|g_n - f_n\|_\infty \to 0$. If the set $\{f_n\colon n \in \mathbb{N}\}$ is equiregulated, then the set $\{g_n\colon n \in \mathbb{N}\}$ is equiregulated.*

Proof. Denote $h_n = g_n - f_n$. We have a sequence of regulated functions $\{h_n\}_{n\in\mathbb{N}}$ which is uniformly convergent to the zero function. According to Theorem 3.3, the set $\{h_n\colon n \in \mathbb{N}\}$ is equiregulated. Now we can use Lemma 3.5 to conclude that the set $\{g_n\colon n \in \mathbb{N}\} = \{f_n + h_n\colon n \in \mathbb{N}\}$ is equiregulated. □

**Definition 3.7.** We say that a set of regulated functions $\mathcal{T} \subset G([a,b];X)$ has *bounded jumps* if for each $t \in (a,b]$ the set $\{f(t) - f(t-)\colon f \in \mathcal{T}\}$ is bounded, and for each $t \in [a,b)$ the set $\{f(t+) - f(t)\colon f \in \mathcal{T}\}$ is bounded.

For $t \in (a,b]$ and $s \in [a,b)$, we denote

$$K_t^- = \sup\{\|f(t) - f(t-)\|_X\colon f \in \mathcal{T}\},$$
$$K_s^+ = \sup\{\|f(s) - f(s+)\|_X\colon f \in \mathcal{T}\}. \tag{3.2}$$

**Proposition 3.8.** *Assume that a set $\mathcal{T} \subset G([a,b];X)$ is equiregulated and has bounded jumps. Then there is $K > 0$ such that $\|f(t) - f(a)\|_X \leqslant K$ holds for all $f \in \mathcal{T}$, $t \in [a,b]$.*

*Moreover, if the set $\{f(a)\colon\ f \in \mathcal{T}\}$ is bounded, then the set $\mathcal{T}$ is bounded.*

Proof. Using Proposition 3.2, we can find a division $a = a_0 < a_1 < \ldots < a_k = b$ such that $\|f(t'') - f(t')\|_X < 1$ holds for any $f \in \mathcal{T}$, $a_{i-1} < t' < t'' < a_i$.

Let $K^+_{a_{i-1}}$, $K^-_{a_i}$ be given by (3.2). We have

$$\begin{aligned}\|f(a_i) - f(a_{i-1})\|_X \\ &\leqslant \|f(a_i) - f(a_i-)\|_X + \|f(a_i-) - f(a_{i-1}+)\|_X + \|f(a_{i-1}+) - f(a_{i-1})\|_X \\ &\leqslant K^-_{a_i} + 1 + K^+_{a_{i-1}};\end{aligned}$$

then $\|f(a_j) - f(a)\|_X \leqslant \sum\limits_{i=1}^{j} \|f(a_i) - f(a_{i-1})\|_X \leqslant j + \sum\limits_{i=1}^{j} (K^-_{a_i} + K^+_{a_{i-1}})$.

If $t \in (a_j, a_{j+1})$ then

$$\|f(t) - f(a)\|_X \leqslant \|f(t) - f(a_j+)\|_X + K^+_{a_j} + \|f(a_j) - f(a)\|_X;$$

we can conclude that

$$\|f(t) - f(a)\|_X \leqslant K := k + \sum_{i=0}^{k-1} K^+_{a_i} + \sum_{i=1}^{k} K^-_{a_i}$$

holds for all $f \in \mathcal{T}$, $t \in [a,b]$.

The latter part of the proposition is evident. □

**Proposition 3.9.** *If the set $\mathcal{T} \subset G([a,b];X)$ is equiregulated and for every $t \in [a,b]$ the set $\{f(t)\colon\ f \in \mathcal{T}\}$ is bounded, then the set $\mathcal{T}$ is bounded.*

Proof. According to Proposition 3.2, we can find a division $a = a_0 < a_1 < \ldots < a_k = b$ such that if $a_{i-1} < t' < t'' < a_i$ then $\|f(t'') - f(t')\|_X < 1$ holds for any $f \in \mathcal{T}$, $i = 1, 2, \ldots, k$. For each $i = 1, 2, \ldots, k$, choose a point $b_i \in (a_{i-1}, a_i)$. The set

$$\{f(a_i)\colon\ f \in \mathcal{T},\ i = 0, 1, \ldots, k\} \cup \{f(b_i)\colon\ f \in \mathcal{T},\ i = 1, 2, \ldots, k\}$$

is bounded by a constant $K$.

Let any $t \in [a,b]$ be given, and $f \in \mathcal{T}$. Either $t = a_i$ for some $i \in \{0, 1, \ldots, k\}$, then $\|f(t)\|_X = \|f(a_i)\|_X \leqslant K$; or $t \in (a_{i-1}, a_i)$ for some $i \in \{1, 2, \ldots, k\}$, then

$$\|f(t)\|_X \leqslant \|f(t) - f(b_i)\|_X + \|f(b_i)\|_X < 1 + K,$$

concluding the proof. □

**Theorem 3.10.** *For any set of regulated functions $\mathcal{T} \subset G([a,b];X)$, the following properties are equivalent:*

(i) *$\mathcal{T}$ is equiregulated and has bounded jumps;*

(ii) *there is a non-decreasing function $h\colon [a,b] \to [c,d]$ and an equicontinuous set $\mathcal{B} \subset \mathcal{C}([c,d];X)$ such that for any $f \in \mathcal{T}$ there is a continuous function $g \in \mathcal{B}$ satisfying $f(t) = g(h(t))$ for $t \in [a,b]$;*

(iii) *there is a non-decreasing function $\omega\colon [0,\infty) \to [0,\infty)$, $\omega(0+) = 0$, and a non-decreasing function $h\colon [a,b] \to \mathbb{R}$ such that $\|f(t'') - f(t')\|_X \leqslant \omega(|h(t'') - h(t')|)$ holds for all $f \in \mathcal{T}$, $a \leqslant t' < t'' \leqslant b$.*

Proof. (i) $\Rightarrow$ (ii): It follows from Proposition 3.4 that the sets $J^+$, $J^-$ are at most countable. As was proved in Theorem 2.6, there exists a non-decreasing function $h\colon [a,b] \to \mathbb{R}$ such that

$$J^- = \{t \in (a,b]\colon\ h(t-) \neq h(t)\},$$
$$J^+ = \{t \in [a,b)\colon\ h(t+) \neq h(t)\}.$$

We can assume that the function $h$ is increasing (if not, it can be replaced by $\tilde{h}(t) = h(t) + t$).

For each $f \in \mathcal{T}$, we can define its linear prolongation $g_f$ as in the proof of Theorem 2.6:

If $\tau = h(t)$, we define

$$g_f(\tau) = f(t).$$

If $h(t-) \leqslant \tau < h(t)$, we define

$$g_f(\tau) = f(t) + \frac{h(t) - \tau}{h(t) - h(t-)}(f(t-) - f(t)). \tag{3.3}$$

If $h(t) < \tau \leqslant h(t+)$, we define

$$g_f(\tau) = f(t) + \frac{\tau - h(t)}{h(t+) - h(t)}(f(t+) - f(t)).$$

Then $g_f(h(t)) = f(t)$; $g_f(h(t-)) = f(t-)$; $g_f(h(t+)) = f(t+)$. All these functions $g_f$ are continuous and we denote $\mathcal{B} = \{g_f\colon\ f \in \mathcal{T}\}$. We will prove that the set $\mathcal{B}$ is equicontinuous.

Let $t \in (a,b]$ be given such that $h(t-) < h(t)$. It is assumed that

$$\|f(t) - f(t-)\|_X \leqslant K_t^-$$

for all $f \in \mathcal{T}$, where $K_t^- < \infty$ is given by (3.2). We have

$$\|g_f(h(t)) - g_f(h(t-))\|_X = \|f(t) - f(t-)\|_X \leqslant K_t^-,$$

hence for any $\tau', \tau'' \in [h(t-), h(t)]$ we have

$$\|g_f(h(\tau'')) - g_f(h(\tau'))\|_X \leqslant \frac{|\tau'' - \tau'|K_t^-}{h(t) - h(t-)};$$

the functions $g_f$ are equicontinuous on $[h(t-), h(t)]$. Analogously, they are equicontinuous on each interval $[h(t), h(t+)]$ where $h(t) \neq h(t+)$.

Now assume that $s_0 = h(t_0-)$ for some $t_0 \in (a, b]$ (regardless if $h$ if left-continuous at $t_0$ or not); we will prove that the functions in $\mathcal{B}$ are equicontinuous from the left at $s_0$. For given $\varepsilon > 0$ we can find $\delta > 0$ such that $t_0 - \delta > a$, and if $t_0 - \delta < \tau < t_0$ then $\|f(t_0-) - f(\tau)\|_X < \frac{1}{3}\varepsilon$. It is evident that

$$\|f(t_0-) - f(\tau+)\|_X \leqslant \frac{\varepsilon}{3}, \quad \|f(t_0-) - f(\tau-)\|_X \leqslant \frac{\varepsilon}{3}$$

holds for any $\tau \in (t_0 - \delta, t_0)$. Fix a point $\tau \in (t_0 - \delta, t_0)$ and denote $\eta = h(t_0-) - h(\tau)$. We have $\eta > 0$ because the function $h$ is increasing. Let $s \in (s_0 - \eta, s_0) = (h(\tau), h(t_0-))$ be an arbitrary point. Considering that $h$ is an increasing function, there is a unique point $t \in (\tau, t_0)$ such that $h(t-) \leqslant s \leqslant h(t+)$.

The first case is $h(t-) \leqslant s \leqslant h(t)$; then for any $f \in \mathcal{T}$ we have

$$\begin{aligned}
\|g_f(s) - g_f(s_0)\|_X &\leqslant \|g_f(s) - g_f(h(t))\|_X + \|g_f(h(t)) - g_f(h(t_0-))\|_X \\
&= \frac{s - h(t)}{h(t-) - h(t)}\|f(t-) - f(t)\|_X + \|f(t) - f(t_0-)\|_X \\
&\leqslant \|f(t-) - f(t_0-)\|_X + 2\|f(t) - f(t_0-)\|_X < \varepsilon
\end{aligned}$$

or in the case $h(t) \leqslant s \leqslant h(t+)$, again we obtain $\|g_f(s) - g_f(s_0)\| < \varepsilon$. This proves the equicontinuity at $h(t_0-)$ from the left; equicontinuity at $h(t_0+)$ from the right can be proved similarly.

(ii) $\Rightarrow$ (iii): Define

$$\omega(r) = \sup\{\|g(s'') - g(s')\|_X;\ s', s'' \in [c, d],\ |s'' - s'| \leqslant r;\ g \in \mathcal{B}\}, \quad \omega(0) = 0.$$

It is well-known that an equicontinuous set of functions is uniformly continuous; therefore $w(0+) = 0$. We have

$$\|g(s'') - g(s')\|_X \leqslant \omega(|s'' - s'|) \quad \text{for any } g \in \mathcal{B},\ s', s'' \in [c, d].$$

It follows that

$$\|f(t'') - f(t')\|_X = \|g_f(h(t'')) - g_f(h(t'))\|_X \leqslant \omega(|h(t'') - h(t')|)$$
$$\text{for all} \quad f \in \mathcal{T},\ t', t'' \in [a,b].$$

(iii) ⇒ (i): It is well-known that any non-decreasing function is regulated. Let $\varepsilon > 0$ be given; there is $r > 0$ such that $\omega(r) < \varepsilon$. For any $t \in [a,b)$ there is $\delta > 0$ such that $h(t+\delta) - h(t+) < r$. If $f \in \mathcal{T}$ and $\tau \in (t, t+\delta)$, then

$$\|f(\tau) - f(t+)\|_X \leqslant \omega(h(\tau) - h(t+)) \leqslant \omega(r) < \varepsilon;$$

similarly for the left-sided limits. Further, for any $t \in [a,b)$ and $f \in \mathcal{T}$ we have

$$\|f(t+) - f(t)\|_X \leqslant \omega(h(t+) - h(t));$$

similarly, for any $t \in (a,b]$ and $f \in \mathcal{T}$ we have

$$\|f(t) - f(t-)\|_X \leqslant \omega(h(t) - h(t-)).$$

Consequently, the set $\mathcal{T}$ has bounded jumps. □

**Proposition 3.11.** *Assume that a sequence of regulated functions* $\{f_n\}_{n\in\mathbb{N}} \subset G([a,b];X)$ *is given such that:*

- ▷ *there is a non-decreasing function* $\omega\colon [0,\infty) \to [0,\infty)$, $\omega(0+) = 0$, *and*
- ▷ *there is a bounded sequence of non-decreasing functions* $h_n\colon [a,b] \to \mathbb{R}$, $n \in \mathbb{N}_0$ *such that*
$$\|f_n(t'') - f_n(t)\|_X \leqslant \omega(h_n(t'') - h_n(t'))$$
*for every* $n \in \mathbb{N}$, $a \leqslant t' < t'' \leqslant b$.

*The following conditions are sufficient for the set* $\{f_n\colon n \in \mathbb{N}\}$ *to be equiregulated:*

(i) *the set* $\{h_n\colon n \in \mathbb{N}\}$ *is equiregulated;*

(ii) $\limsup\limits_{n\to\infty}[h_n(t'') - h_n(t')] \leqslant h_0(t'') - h_0(t')$ *holds for any* $a < t' < t'' < b$ *and the function* $h_0$ *is continuous;*

(iii) $\lim\limits_{n\to\infty} h_n(t) = h_0(t)$ *for every* $t \in [a,b]$ *and the function* $h_0$ *is continuous.*

Proof. (i) Assume that the set $\{h_n\colon n \in \mathbb{N}\}$ is equiregulated. According to Theorem 3.10, we can find a non-decreasing function $\vartheta\colon [0,\infty) \to [0,\infty)$, $\vartheta(0+) = 0$ and a non-decreasing function $h\colon [a,b] \to \mathbb{R}$ such that

$$|h_n(t'') - h_n(t')| \leqslant \vartheta(|h(t'') - h(t')|)$$

holds for any $n \in \mathbb{N}$, $a \leqslant t' < t'' \leqslant b$. Then

$$\|f_n(t'') - f_n(t')\|_X \leqslant \omega(|h_n(t'') - h_n(t')|) \leqslant \omega(\vartheta(h(t'') - h(t')));$$

using Theorem 3.10, we conclude that the set $\{f_n\colon n \in \mathbb{N}\}$ is equiregulated.

(ii) Let $\varepsilon > 0$ be given. The continuous function $h_0$ is uniformly continuous on $[a, b]$; then there is $\delta > 0$ such that if $a \leqslant t' < t'' \leqslant b$ and $t'' - t' < \delta$ then $h_0(t'') - h_0(t') < \varepsilon$. We can find a division $a = b_0 < b_1 < \ldots < b_k = b$ such that

$$h_0(b_j) - h_0(b_{j-1}) < \frac{\varepsilon}{2} \quad \text{for any } i = 1, 2, \ldots, k.$$

There is $n_0 \in \mathbb{N}$ such that if $n \geqslant n_0$ and $j = 1, 2, \ldots, k$ then

$$0 \leqslant h_n(b_j) - h_n(b_{j-1}) < \frac{\varepsilon}{2} + h_0(b_j) - h_0(b_{j-1}).$$

Considering that the functions $h_n$ are non-decreasing, we get

$$0 \leqslant h_n(t'') - h_n(t') \leqslant h_n(b_j) - h_n(b_{j-1}) < \frac{\varepsilon}{2} + h_0(b_j) - h_0(b_{j-1}) < \varepsilon$$

for any $t'$, $t''$ such that $b_{j-1} \leqslant t' < t'' \leqslant b_j$, $n \geqslant n_0$.

The functions $h_1, h_2, \ldots, h_{n_0}$ are regulated, therefore, for each interval $[b_{j-1}, b_j]$ we can find a subdivision $b_{j-1} = a_{0,j} < a_{1,j} < \ldots < a_{l_j,j} = b_j$ such that $0 \leqslant h_n(t'') - h_n(t') < \varepsilon$ holds for $n \leqslant n_0$, $a_{i-1,j} \leqslant t' < t'' \leqslant a_{i,j}$; it follows that the conditions of Proposition 3.2 are satisfied, and therefore, the set $\{h_n \colon n \in \mathbb{N}\}$ is equiregulated. Now, we can use part (i).

Finally, (iii) is a consequence of (ii). □

## 4. Sup-norm topology

**Proposition 4.1.** *The linear space of regulated functions $G([a, b]; X)$ with the norm $\|\cdot\|_\infty$ is a Banach space.*

Proof. Obviously $G([a, b]; X)$ is a linear space and $\|\cdot\|_\infty$ is a norm. We shall prove that it is a complete normed linear space.

Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence of regulated functions. For any $t \in [a, b]$, the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ has the Cauchy property, therefore its limit in the Banach space $X$ exists, and it can be denoted by $f_0(t)$. For each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\|f_n(t) - f_m(t)\|_X < \varepsilon \quad \text{for all } t \in [a, b] \text{ and all } m, n \geqslant n_0.$$

Passing to the limit $m \to \infty$, we get

$$\|f_n(t) - f_0(t)\|_X \leqslant \varepsilon \quad \text{for all } t \in [a, b] \text{ and all } n \geqslant n_0.$$

We have $f_n \rightrightarrows f_0$ and it follows from Proposition 2.2 that the function $f_0$ is regulated. □

**Theorem 4.2.** *A set of regulated functions $\mathcal{T} \subset G([a,b];X)$ is relatively compact in the Banach space $G([a,b];X)$ if and only if the set $\mathcal{T}$ is equiregulated and satisfies the following condition:*

(4.1) *for every $t \in [a,b]$, the set $\{f(t)\colon f \in \mathcal{T}\}$ is relatively compact in $X$.*

P r o o f. (i) Assume that $\mathcal{T}$ is relatively compact. Then for every $\varepsilon > 0$ there is a finite $\varepsilon$-net, i.e., a finite set $\mathcal{P} \subset G([a,b];X)$ such that for each $f \in \mathcal{T}$ there is $g \in \mathcal{P}$ satisfying $\|f-g\|_\infty < \varepsilon$. For any fixed $t \in [a,b]$, denote $\mathcal{P}_t = \{g(t)\colon g \in \mathcal{P}\}$; this is a finite subset of $X$ and for any $f \in \mathcal{T}$ we can find $p \in \mathcal{P}_t$ ($p = g(t)$) such that $\|f(t)-p\|_X < \varepsilon$; this means that $\mathcal{P}_t$ is a finite $\varepsilon$-net for the set $\{f(t)\colon f \in \mathcal{T}\}$. Consequently, this is a relatively compact subset of $X$.

Now, we shall prove that the functions in $\mathcal{T}$ have uniform one-sided limits. Let $\tau \in [a,b]$ and $\varepsilon > 0$ be given. We can find a finite $\frac{1}{3}\varepsilon$-net $\mathcal{P} \subset G([a,b];X)$ for $\mathcal{T}$.

Let us denote the elements of $\mathcal{P}$ as $\{g_1, g_2, \ldots, g_n\}$. These are regulated functions, therefore we can find $\delta > 0$ such that if $t \in (\tau-\delta,\tau) \cap [a,b]$ then $\|g_j(t) - g_j(\tau-)\|_X < \frac{1}{3}\varepsilon$ and if $t \in (\tau,\tau+\delta) \cap [a,b]$ then $\|g_j(t) - g_j(\tau+)\|_X < \frac{1}{3}\varepsilon$ for any $j \in \{1,2,\ldots,n\}$.

Let $f \in \mathcal{T}$ be given, then we can find $j$ such that $\|f - g_j\|_\infty < \frac{1}{3}\varepsilon$. For any $t \in (\tau-\delta,\tau) \cap [a,b]$ we have

$$\|f(t) - f(\tau-)\|_X \leqslant 2\|f-g_j\|_\infty + \|g_j(t) - g_j(\tau-)\|_X < \varepsilon;$$

and for any $t \in (\tau,\tau+\delta) \cap [a,b]$ we have

$$\|f(t) - f(\tau+)\|_X \leqslant 2\|f-g_j\|_\infty + \|g_j(t) - g_j(\tau+)\|_X < \varepsilon.$$

(ii) Assume that the set $\mathcal{T}$ is equiregulated and satisfies condition (4.1). Let $\varepsilon > 0$ be given. According to Proposition 3.2, there is a division $a = a_0 < a_1 < \ldots < a_k = b$ such that if $a_{i-1} < t' < t'' < a_i$ for an index $i \in \{1,2,\ldots,k\}$ and $f \in \mathcal{T}$ then $\|f(t'') - f(t')\|_X < \varepsilon$.

Let us choose a point $b_i \in (a_{i-1}, a_i)$ for each $i \in \{1,2,\ldots,k\}$. Due to (4.1) the set $Y = \{f(a_i),\ i = 0,1,\ldots,k; f \in \mathcal{T}\} \cup \{f(b_i),\ i = 1,2,\ldots,k; f \in \mathcal{T}\}$ is relatively compact in the Banach space $X$; consequently, it has a finite $\frac{1}{2}\varepsilon$-net $Z \subset X$.

Let us define a set $Q$ of all step functions with values in $Z$ which are constant on each of the intervals $(a_{i-1}, a_i)$, $i = 1,2,\ldots,k$. The set $Q$ is finite. For a given $f \in \mathcal{T}$, we have $f(a_i) \in Y$, $f(b_i) \in Y$, hence there are $\alpha_i \in Z$, $\beta_i \in Z$ such that

$$\|f(a_i) - \alpha_i\|_X < \frac{\varepsilon}{2}, \quad i = 0,1,\ldots,k,$$
$$\|f(b_i) - \beta_i\|_X < \frac{\varepsilon}{2}, \quad i = 1,2,\ldots,k.$$

Define $g(a_i) = \alpha_i$, $g(t) = \beta_i$ for $t \in (a_{i-1}, a_i)$; then $g \in Q$ and we have

$$\|f(a_i) - g(b_i)\|_X < \frac{\varepsilon}{2}, \quad \|f(t) - g(t)\|_X \leqslant \|f(t) - f(b_i)\|_X + \|f(b_i) - g(b_i)\|_X < \varepsilon$$

for all $t \in (a_{i-1}, a_i)$. This means that for an arbitrary $f \in \mathcal{T}$ a function $g \in Q$ was found such that $\|f - g\|_\infty < \varepsilon$; the set $Q$ is a finite $\varepsilon$-net for $\mathcal{T}$.

We have found that the set $\mathcal{T}$ is totally bounded, and therefore it is relatively compact in the Banach space $G([a, b]; X)$. □

**Corollary 4.3.** *A set of regulated functions $\mathcal{T} \subset G([a, b]; \mathbb{R}^N)$ is relatively compact in $G([a, b]; \mathbb{R}^N)$ if and only if the set $\mathcal{T}$ is equiregulated and for every $t \in [a, b]$ the set $\{f(t); f \in \mathcal{T}\}$ is bounded.*

**Proposition 4.4.** *If a set $\mathcal{T} \subset G([a, b]; X)$ is relatively compact, then its image $\operatorname{Im}(\mathcal{T}) = \{f(t)\colon\ f \in \mathcal{T},\ t \in [a, b]\}$ is a relatively compact subset of X.*

Proof. We are going to prove that the set $\operatorname{Im}(\mathcal{T})$ is totally bounded; i.e., has a finite $\varepsilon$-net for any $\varepsilon > 0$.

Let $\varepsilon > 0$ be given. The relatively compact set $\mathcal{T}$ has a finite $\frac{1}{2}\varepsilon$-net $Q \subset G([a, b]; X)$, it means that for every $f \in \mathcal{T}$ there is $g \in Q$ satisfying $\|f - g\|_\infty < \frac{1}{2}\varepsilon$. According to Theorem 2.3, for each $g \in Q$ there is a step function $\psi_g$ such that $\|g - \psi_g\|_\infty < \frac{1}{2}\varepsilon$. The finite set of step functions $\{\psi_g\colon\ g \in Q\}$ has a finite set of values

$$Z = \{\psi_g(t)\colon\ t \in [a, b],\ g \in Q\}.$$

For any $f \in \mathcal{T}$ we can find $g \in Q$ such that $\|f - g\|_\infty < \frac{1}{2}\varepsilon$; then

$$\|f(t) - \psi_g(t)\|_X \leqslant \|f - g\|_\infty + \|g - \psi_g\|_\infty < \varepsilon$$

and $\psi_g(t) \in Z$; this means that $Z$ is a finite $\varepsilon$-net for $\operatorname{Im}(\mathcal{T})$. □

**Proposition 4.5.** *For an equiregulated set of functions $\mathcal{T} \subset G([a, b]; X)$ its relative compactness is equivalent to relative compactness of its image.*

Proof. (i) If the set $\mathcal{T}$ is equiregulated, then $\operatorname{Im}(\mathcal{T})$ is relatively compact according to Proposition 4.4.

(ii) If $\operatorname{Im}(\mathcal{T})$ is relatively compact, then condition (4.1) holds and we can use Theorem 4.2. □

**Theorem 4.6.** *For a set of regulated functions $\mathcal{T} \subset G([a,b];X)$, the following properties are equivalent:*

(i) *The set $\mathcal{T}$ is a relatively compact subset of the Banach space $(G([a,b];X;\|\cdot\|_\infty)$.*

(ii) *There is a non-decreasing function $h\colon [a,b] \to [c,d]$ and a set $\mathcal{B} \subset \mathcal{C}([c,d];X)$ of continuous functions which is relatively compact in the sup-norm $\|\cdot\|_\infty$ so that for every $f \in \mathcal{T}$ there is $g \in \mathcal{B}$ satisfying $f(t) = g(h(t))$, $t \in [a,b]$.*

(iii) *For every $t \in [a,b]$, the set $\{f(t)\colon f \in \mathcal{T}\}$ is relatively compact in $X$ and there is a non-decreasing function $h\colon [a,b] \to \mathbb{R}$ and a non-decreasing function $\omega\colon [0,\infty) \to [0,\infty)$, $w(0+) = 0$ such that $\|f(t'') - f(t')\|_X \leqslant \omega(|h(t'') - h(t')|)$ holds for every $f \in \mathcal{T}$, $t', t'' \in [a,b]$.*

P r o o f. (i) $\Rightarrow$ (ii): According to Theorem 4.2, the set $\mathcal{T}$ is equiregulated and satisfies (4.1). Then $\mathcal{T}$ has bounded jumps; using Theorem 3.10, we can find a non-decreasing function $h\colon [a,b] \to [c,d]$ where $h(a) = c$, $h(b) = d$ and an equicontinuous set $\mathcal{B} \subset C([c,d];X)$ defined by $\mathcal{B} = \{g_f\colon f \in \mathcal{T}\}$, where the function $g_f$ is the linear prolongation of $f$ defined by the formula (3.3).

We are going to prove that the set $\mathcal{B}$ is totally bounded, therefore relatively compact.

Given $\varepsilon > 0$, there is a finite $\varepsilon$-net $Q \subset G([a,b];X)$ for $\mathcal{T}$. Define $g_\zeta$ by the formula (3.3) for every $\zeta \in Q$. Then the set $\{g_\zeta\colon \zeta \in Q\}$ is a finite $\varepsilon$-net for the set $\mathcal{B}$.

(ii) $\Rightarrow$ (iii): For each $t \in [a,b]$ we have $\{f(t)\colon f \in \mathcal{T}\} \subset \{g(h(t))\colon g \in \mathcal{B}\}$; hence the set $\{f(t)\colon f \in \mathcal{T}\}$ is relatively compact in $X$. We can define

$$\omega(r) = \sup\{\|g(s') - g(s'')\|_X\colon |s' - s''| \leqslant r;\ g \in \mathcal{B}\}.$$

It follows from Arzelà-Ascoli theorem (version in Banach space) that the relatively compact set $\mathcal{B}$ is equicontinuous, consequently $w(0+) = 0$. Obviously, the function $\omega$ is non-decreasing.

For any $f \in \mathcal{T}$ we can find $g \in \mathcal{B}$ satisfying $f(t) = g(h(t))$, $t \in [a,b]$. Then, for any $t', t'' \in [a,b]$ we obtain the inequality $\|f(t'') - f(t')\|_X = \|g(h(t'')) - g(h(t'))\|_X \leqslant \omega(|h(t'') - h(t')|)$.

(iii) $\Rightarrow$ (i): Follows from Theorem 4.2. □

**Proposition 4.7.** *Assume that a set $\mathcal{T} \subset G([a,b];X)$ is equiregulated. Denote the sets $J^-$, $J^+$ as in (3.1). Then for any dense subset $M \subset [a,b]$ and any $\varepsilon > 0$ there is a division $a = c_0 < c_1 < \ldots c_k = b$ such that $\{c_1, c_2, \ldots, c_{k-1}\} \subset M \cup J^- \cup J^+$ and if $c_{i-1} < t' < t'' < c_i$ for some $i \in \{1,2,\ldots,k\}$ then $\|f(t'') - f(t')\|_X < \varepsilon$ holds for all $f \in \mathcal{T}$.*

P r o o f. We can find a division $a = a_0 < a_1 < \ldots a_k = b$ as described in Proposition 3.2. If $a_i \in M \cup J^- \cup J^+ \cup \{a,b\}$, we denote $c_i = a_i$. If $a_i \notin M \cup J^- \cup J^+ \cup \{a,b\}$ then all functions $f \in \mathcal{T}$ are continuous at $a_i$: there is $\delta > 0$ such that if $|t - a_i| < \delta$ and $f \in \mathcal{T}$ then $\|f(t) - f(a_i)\|_X < \varepsilon$. The set $M$ is dense in $[a,b]$, therefore we can find $c_i \in (a_{i-1}, a_i) \cap M$ such that $|c_i - a_i| < \delta$. In both cases, we have $\|f(c_i) - f(a_i)\|_X < \varepsilon$ for every $f \in \mathcal{T}$, $i \in \{0,1,2,\ldots k\}$. Now, if $i \in \{1,2,\ldots,k\}$ and $c_{i-1} < t' < t'' < c_i$, there are several options and it is only a technical matter to verify that $\|f(t'') - f(t')\|_X < 2\varepsilon$ in all possible cases. □

**Theorem 4.8.** *Assume that a set of functions $\mathcal{T} \subset G([a,b];X)$ is equiregulated. Denote the sets $J^-$, $J^+$ as in* (3.1). *Assume that there is a dense subset $M \subset [a,b]$ such that for every $t \in M_0 = M \cup J^- \cup J^+ \cup \{a,b\}$ the set $\{f(t)\colon f \in \mathcal{T}\}$ is relatively compact in $X$. Then the set $\mathcal{T}$ is relatively compact in the Banach space $G([a,b];X)$.*

P r o o f. The proof can be performed the same way as the second part of the proof of Theorem 4.2, where the points $b_i \in (a_{i-1}, a_i)$ can be chosen so that $b_i \in M_0$ for each $i \in \{1,2,\ldots,k\}$. □

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