AN OBSERVATION ON SPACES WITH A ZEROSET DIAGONAL

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Abstract. We say that a space $X$ has the discrete countable chain condition (DCCC for short) if every discrete family of nonempty open subsets of $X$ is countable. A space $X$ has a zeroset diagonal if there is a continuous mapping $f: X^2 \to [0,1]$ with $\Delta_X = f^{-1}(0)$, where $\Delta_X = \{(x, x): x \in X\}$. In this paper, we prove that every first countable DCCC space with a zeroset diagonal has cardinality at most $c$.

Keywords: first countable; discrete countable chain condition; zeroset diagonal; cardinal

MSC 2010: 54D20, 54E35

1. Introduction

All topological spaces in this paper are assumed to be Hausdorff unless otherwise stated. The cardinality of a set $X$ is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of $X$. We write $\omega$ for the first infinite cardinal, $\omega_1$ for the first uncountable cardinal and $c$ for the cardinality of the continuum.

In 1977, Ginsburg and Woods proved that the cardinality of a $T_1$-space with countable extent and a $G_\delta$-diagonal is at most $c$ (see [5]). In the same paper, Ginsburg and Woods asked if it was true that a regular CCC-space (here CCC denotes the countable chain condition) with a $G_\delta$-diagonal has cardinality at most $c$. This question was also posted by Arhangel’skii independently. In 1984, Shakhmatov showed that cardinalities of such spaces may not have an upper bound (see [8]). And later, Uspenskij proved that an upper bound still does not exist even assuming Fréchet property (see [9]). Regular $G_\delta$-diagonal is a property stronger than $G_\delta$-diagonal. Arhangel’skii asked what if “$G_\delta$-diagonal” is replaced by “regular $G_\delta$-diagonal”.

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In 2005, Buzyakova proved that the cardinality of a CCC-space with a regular $G_{\delta}$-diagonal is at most $c$ (see [3]). In 2015, Gotchev in [6] proved that the cardinality of a weakly Lindelöf space with a regular $G_{\delta}$-diagonal is at most $2^c$.

**Definition 1.1.** We say that a space $X$ has the *discrete countable chain condition* (DCCC for short) if every discrete family of nonempty open subsets of $X$ is countable.

By Definition 1.1, it follows immediately that every CCC space is DCCC. In fact, every weakly Lindelöf space is DCCC, but the converse is not true. For example, $\omega_1$ with the ordered topology is a first countable and countably compact (hence, DCCC) space which is not weakly Lindelöf, because the open cover $U = \{[0, \alpha]: \alpha < \omega_1\}$ of $\omega_1$ does not have a countable subfamily whose union is dense in $\omega_1$.

**Definition 1.2 ([2]).** A space $X$ has a *zeroset diagonal* if there is a continuous mapping $f: X^2 \to [0, 1]$ with $\Delta_X = f^{-1}(0)$, where $\Delta_X = \{(x, x): x \in X\}$.

It is well-known and easy to prove that every submetrizable space has a zeroset diagonal and every zeroset diagonal is a regular $G_{\delta}$-diagonal. The converses are not true (see [1], [10]).

In this paper, we prove that every first countable DCCC space with a zeroset diagonal has cardinality at most $c$.

All notations and terminology not explained in the paper are given in [4].

2. Results

We will use the following countable version of a set-theoretic theorem due to Erdős and Radó (see [7], page 8).

**Lemma 2.1.** Let $X$ be a set with $|X| > c$ and suppose $[X]^2 = \bigcup\{P_n: n \in \omega\}$. Then there exist $n_0 < \omega$ and a subset $S$ of $X$ with $|S| > \omega$ such that $[S]^2 \subset P_{n_0}$.

**Theorem 2.2.** Every first countable DCCC space $X$ with a zeroset diagonal has cardinality at most $c$.

**Proof.** Assume the contrary, i.e. that $|X| > c$. Fix a continuous function $f: X^2 \to [0, 1]$ with $\Delta_X = f^{-1}(0)$. Let $B(x) = \{B_n(x): n \in \omega\}$ be a local decreasing base for each $x \in X$. Since for any distinct $x, y \in X$ there is some $n_1 \in \omega$ such that $(x, y) \in f^{-1}((1/(n_1 + 2019), 1])$ and since $f$ is continuous, there are $n_2, n_3 \in \omega$ such that $B_{n_2}(x) \times B_{n_3}(y) \subset f^{-1}\left(\left(\frac{1}{n_1 + 2019}, 1\right]\right)$.  

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Let $n^* = \max\{n_1, n_2, n_3\}$. Then by our hypothesis, we can deduce that

$$B_{n^*}(x) \times B_{n^*}(y) \subset f^{-1}\left(\left[\frac{1}{n^* + 2019}, 1\right]\right).$$

Thus, the following sets $P_n$ are well defined. For each $n \in \omega$ let

$$P_n = \left\{\{x, y\} \in [X]^2 : B_n(x) \times B_n(y) \subset f^{-1}\left(\left[\frac{1}{n + 2019}, 1\right]\right)\right\}.$$  

It is clear that $[X]^2 = \bigcup\{P_n : n \in \omega\}$. (Note that $[X]^2$ is the set of two-element subsets of $X$). We can apply Lemma 2.1 to conclude that there exists an uncountable subset $S$ of $X$ such that $[S]^2 \subset P_{n_0}$. It follows immediately that $\mathcal{U} = \{B_{n_0}(x) : x \in S\}$ is an uncountable family of nonempty open sets of $X$. Since $X$ is DCCC, the family $\mathcal{U}$ must have a cluster point $x \in X$. Pick any neighbourhood $O_x$ of $x$ such that $O_x \times O_x \subset f^{-1}\left(\left[0, \frac{1}{n_0 + 2019}\right]\right)$. Obviously, $O_x$ meets infinitely many members of $\mathcal{U}$. Thus, there exist two distinct (at least) $y, z \in S$ such that $O_x \cap B_{n_0}(y) \neq \emptyset$ and $O_x \cap B_{n_0}(z) \neq \emptyset$. Take any $y' \in O_x \cap B_{n_0}(y)$ and $z' \in O_x \cap B_{n_0}(z)$. Hence, $f(y', z') < 1/(n_0 + 2019)$ since $y', z' \in O_x$. On the other hand, $f(y', z') > 1/(n_0 + 2019)$ since $y' \in B_{n_0}(y), z' \in B_{n_0}(z)$ and $\{y, z\} \in P_{n_0}$. This gives a contradiction and we prove that $|X| \leq c$. □

If we drop the condition “DCCC”, or “zeroset diagonal” in Theorem 2.2, the conclusion is no longer true, which can be seen in the following examples.

**Example 2.3.** Let $D$ be a discrete space with $|D| = 2^c$. It is evident that $D$ is first countable and has a zeroset diagonal, but $D$ is not DCCC.

**Example 2.4.** Let $X$ be the subspace of $[0, 2^c]$, consisting of all ordinals of countable cofinality, equipped with the ordered topology. Then $X$ is a first countable and countably compact (hence DCCC) space of cardinality $2^c$, but it does not have a zeroset diagonal.

We finish the paper with the following question.

**Question 2.5.** Is it true that every DCCC (or weakly Lindelöf) space with a zeroset diagonal has cardinality at most $c$?

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