COMMON FIXED POINTS FOR FOUR NON-SELF MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. We formulate a common fixed point theorem for four non-self mappings in convex partial metric spaces. The result extends a fixed point theorem by Gajić and Rakočević (2007) proved for two non-self mappings in metric spaces with a Takahashi convex structure. We also provide an illustrative example on the use of the theorem.

Keywords: common fixed point; convex partial metric space; non-self mapping

MSC 2010: 47H10, 54H25

1. Introduction and preliminaries

Gajić and Rakočević [5] proved a common fixed point theorem for non-self mappings on a Takahashi convex metric space for a pair of mappings. In their work, they generalized the theorems by Jungck [7], Das and Naik [4], Ćirić et al. [3], Ćirić [2] and Imdad and Kumar [6]. In this study, we extend the theorem by Gajić and Rakočević to apply for two pairs of non-self mappings in convex partial metric spaces.

We now introduce the results which will be of use in this paper.

Definition 1.1 ([8])**.** A partial metric on a nonempty set X is a mapping $p\colon X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

(P0) $0 \leqslant p(x, x) \leqslant p(x, y)$,
(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(P2) $p(x, y) = p(y, x)$ and
(P3) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$.

A pair (X, p) is said to be a partial metric space.

From Definition 1.1, we deduce that for all x, y, z in a partial metric space (X, p), we have:

(1.1)
$$p(x, y) = 0 \Rightarrow x = y,$$

(1.2)
$$p(x, y) \leqslant p(x, z) + p(z, y).$$

P r o o f. If $p(x, y) = 0$, then $p(x, x) = 0$ because $0 \leqslant p(x, x) \leqslant p(x, y)$ from (P0). Similarly, $p(x, y) = 0$ implies $p(y, y) = 0$ because $0 \leqslant p(y, y) \leqslant p(x, y)$. Hence $p(x, y) = 0$ implies $p(x, x) = p(x, y) = p(y, y) = 0$. From (P1) this means that $x = y$.
From (P3), we infer that

$$p(x, y) \leqslant p(x, z) + p(z, y).$$

$\square$

As an example, let $X = \mathbb{R}^+$ and let $p \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x, \varepsilon) \colon x \in X,\ \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X \colon p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to $x \in X$ if and only if

$$p(x, x) = \lim_{n \to \infty} p(x, x_n).$$

Definition 1.2 ([8]). Let (X, p) be a partial metric space and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim\limits_{n \to \infty} p(x, x_n)$,

(ii) $\{x_n\}$ is called a *Cauchy sequence* if $\lim\limits_{n, m \to \infty} p(x_n, x_m)$ exists and is finite,

(iii) a partial metric space (X, p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m).$$

Lemma 1.3 ([8]). *If p is a partial metric on X, then the mapping $p^s \colon X \times X \to [0, \infty)$ given by*
$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$
is a metric.

In this paper we will denote by p^s the metric derived from the partial metric p.

Lemma 1.4 ([8]). *Let (X, p) be a partial metric space and $\{x_n\}$ a sequence in X. Then*

(i) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s),*

(ii) *(X, p) is complete if and only if (X, p^s) is complete. Furthermore, $\lim\limits_{n \to \infty} p(x_n, x)$ is zero if and only if $p(x, x) = \lim\limits_{n \to \infty} p(x_n, x) = \lim\limits_{n,m \to \infty} p(x_n, x_m)$.*

Definition 1.5 ([8]). Let (X, p) be a partial metric space and $\{x_n\}$ a sequence in X. Then

(i) the sequence $\{x_n\}$ is called 0-*Cauchy* if $\lim\limits_{n,m \to \infty} p(x_n, x_m) = 0$,

(ii) (X, p) is said to be 0-*complete* if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Definition 1.6 ([9]). Let (X, p) be a partial metric space and $I = [0, 1]$ the closed unit interval. A mapping $W \colon X \times X \times I \to X$ is said to be a *convex* structure on X if for all $(x, y, t) \in X \times X \times I$,

$$p(u, W(x, y, t)) \leqslant tp(u, x) + (1 - t)p(u, y)$$

for every $u \in X$. A partial metric space (X, p), together with the convex structure W, is called a *convex partial metric space.*

If (X, p) is a convex partial metric space, then for every $x, y \in X$, we define

(1.3) $$\operatorname{seg}[x, y] := \{W(x, y, t) \colon t \in [0, 1]\}.$$

In this study, we will use the following properties of a convex partial metric space with convex structure W.

Lemma 1.7. *Let $x, y \in X$ where (X, p) is a convex partial metric space with convex structure W. Let $w \in \operatorname{seg}[x, y]$. Then for all $u \in X$, we have*

(i) *$p(u, w) \leqslant \max\{p(u, x), p(u, y)\}$,*

(ii) *$p(x, w) \leqslant p(x, y)$.*

P r o o f. Suppose $\Gamma = \max\{p(u, x), p(u, y)\}$. Applying Definition 1.6, we have

$$p(u, w) \leqslant tp(u, x) + (1 - t)p(u, y) \leqslant t\Gamma + (1 - t)\Gamma = \Gamma = \max\{p(u, x), p(u, y)\}.$$

We have proved Lemma 1.7 (i). Now let us set $x = u$ in Lemma 1.7 (i). We get

$$p(x, w) \leqslant \max\{p(x, x), p(x, y)\} = p(x, y),$$

from (P0) of Definition 1.1, so we have proved Lemma 1.7 (ii). $\qquad \square$

Definition 1.8 ([1]). Let (X, p) be a partial metric space and $B \subseteq X$. Then

(i) B is said to be *bounded* if there is a positive number M such that $p(x, y) \leqslant M$ for all $x, y \in B$,

(ii) if B is a bounded set, the *diameter* of B is defined as

$$\text{diam}(B) = \sup_{u,v \in B} \{p(u, v)\}.$$

Let $f \colon C \to X$ be a mapping, where $C \subseteq X$. We say that f is a *self mapping* if $C = X$, otherwise f is called a *non-self mapping*. If there is an element $x \in C$ such that $fx = x$, we say that x is a *fixed point* of f in X.

Suppose we have two mappings $f, g \colon C \to X$, with $C \subseteq X$. Let there be $x \in C$ such that $fx = gx = w$. We say that x is a *coincidence point* of f and g in X. If $x = w$, then we call x a *common fixed point* of f and g in X.

Suppose we have two mappings $f, g \colon C \to X$ with $C \subseteq X$. We say f and g are *coincidentally commuting* if for all $x \in C$ we have

$$fx = gx \Rightarrow fgx = gfx.$$

In this paper, we aim to extend the following theorem by Gajić and Rakočević (see [5]) which proves the existence of a common fixed point for non-self mappings in context of metric spaces under specified conditions.

Theorem 1.9 ([5]). *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let C be a nonempty closed subset of X and ∂C the boundary of C. Let $f, g \colon C \to X$ and suppose $\partial C \neq \emptyset$. Let us assume that f and g satisfy the following conditions:*

(i) *For every $x, y \in C$, $d(gx, gy) \leqslant M_\omega(x, y)$ where*

$$M_\omega(x, y) = \max\{\omega_1[d(fx, fy)], \omega_2[d(fx, gx)], \omega_3[d(fy, gy)],$$
$$\omega_4[d(fx, gy)], \omega_5[d(gx, fy)]\},$$

$\omega_i \colon [0, \infty) \to [0, \infty)$, $i \in \{1, 2, 3, 4, 5\}$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $\lim_{r \to \infty} (r - \omega_i(r)) = \infty$,*

(ii) $\partial C \subseteq f(C)$,

(iii) $g(C) \cap C \subset f(C)$,

(iv) $fx \in \partial C \Rightarrow gx \in C$ *and*

(v) $f(C)$ *is closed in X.*

Then there exists a coincidence point v in C. Moreover, if $\{f, g\}$ are coincidentally commuting, then v remains a unique common fixed point of f and g.

We now proceed to the main results.

2. Main results

In this section, we extend Theorem 1.9 to two pairs of non-self mappings. We prove the following theorem:

Theorem 2.1. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a closed subset of X with a nonempty boundary ∂C. Let $S, T, A, B\colon C \to X$. Let us assume that S, T, A and B satisfy the following conditions:*

(i) *For every $x, y \in C$, $p(Ax, By) \leqslant M_\omega(x, y)$ where*

$$M_\omega(x, y) = \max\{\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(By, Ty)],$$
$$\omega_4[p(Ax, Ty)], \omega_5[p(Sx, By)]\},$$

$\omega_i\colon [0, \infty) \to [0, \infty)$, $i = 1, 2, 3, 4, 5$, is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $\lim\limits_{r \to \infty}(r - 2\omega_i(r)) = \infty$.

(ii) *$\partial C \subseteq TC$, $\partial C \subseteq SC$,*

(iii) *$Sx \in \partial C \Rightarrow Ax \in C$; $Tx \in \partial C \Rightarrow Bx \in C$,*

(iv) *$AC \cap C \subset TC$, $BC \cap C \subset SC$ and*

(v) *SC, TC are closed in C.*

Then there exists a coincidence point $z \in C$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

P r o o f. Commencing with an arbitrary point $w \in \partial C$, we construct a sequence $\{x_n\}$ of points in C as follows:

By assumption (ii), there is a point $x_0 \in C$ such that $Sx_0 = w$. We find Ax_0. Then we proceed inductively as follows.

If $Ax_{2n} \in C$, then, by (iv), we choose $x_{2n+1} \in C$ such that $Tx_{2n+1} = Ax_{2n}$.

If however $Ax_{2n} \notin C$, because W is continuous in the third variable, it means that, by (iii), there is $\lambda_{2n,2n} \in (0, 1)$ such that

$$W(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}) \in \partial C.$$

By (ii), this means we can choose $x_{n+1} \in C$ such that

$$Tx_{n+1} = W(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}) \in \partial C.$$

We then determine Bx_{2n+1}.

If $Bx_{2n+1} \in C$, then, by (iv), we choose $x_{2n+2} \in C$ such that $Sx_{2n+2} = Bx_{2n+1}$.

However if $Bx_{2n+1} \notin C$, because W is continuous in the third variable, this means there is $\lambda_{2n+1,2n+1} \in (0,1)$ such that

$$W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \partial C.$$

By (ii), this means we can choose $x_{n+2} \in C$ such that

$$Sx_{n+2} = W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \partial C.$$

We then determine Ax_{2n+2}.

We show that, for $n \geqslant 1$, we have

$$(2.1) \qquad\qquad Ax_{2n} \neq Tx_{2n+1} \Rightarrow Bx_{2n-1} = Sx_{2n}.$$

Suppose we have $Bx_{2n-1} \neq Sx_{2n}$. Then we have $Sx_{2n} \in \partial C$, which by (iii) means $Ax_{2n} \in C$. This, by (iv), implies that $Ax_{2n} = Tx_{2n+1}$, which is a contradiction. Using a similar argument we have

$$(2.2) \qquad\qquad Bx_{2n+1} \neq Sx_{2n+2} \Rightarrow Ax_{2n} = Tx_{2n+1}.$$

We now prove that the sequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ are bounded. For each $n \geqslant 1$ let

$$D_n = \bigcup_{i=0}^{n-1}\{Ax_{2i}\} \cup \bigcup_{i=0}^{n-1}\{Bx_{2i+1}\} \cup \bigcup_{i=0}^{n-1}\{Sx_{2i}\} \cup \bigcup_{i=0}^{n-1}\{Tx_{2i+1}\}, \quad n \geqslant 1.$$

Let $\alpha_n = \operatorname{diam}(D_n)$. We show that

$$(2.3) \qquad \alpha_n \leqslant \max\{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1})\}, \quad 0 \leqslant j \leqslant n-1.$$

Let us consider the case where $\alpha_n = 0$, $n \geqslant 1$.

This means $p(Sx_0, Ax_0) = p(Ax_0, Bx_1) = p(Bx_1, Tx_1) = 0$. Applying (1.2) this means

$$Sx_0 = Ax_0 = Bx_1 = Tx_1.$$

We shall show that Sx_0 is a common fixed point of S and A. As the mappings S and A are coincidentally commuting at the coincidence point x_0, we have

$$(2.4) \qquad\qquad Sx_0 = Ax_0 \Rightarrow SSx_0 = SAx_0 = ASx_0.$$

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From (i), we have

$$
\begin{aligned}
p(SSx_0, Sx_0) = p(ASx_0, Bx_1) &\leqslant M_\omega(Sx_0, x_1) \\
&= \max\{\omega_1[p(SSx_0, Tx_1)], \omega_2[p(ASx_0, SSx_0)], \omega_3[p(Bx_1, Tx_1)], \\
&\qquad \omega_4[p(ASx_0, Tx_1)], \omega_5[p(SSx_0, Bx_1)]\} \\
&= \max\{\omega_1[p(SSx_0, Sx_0)], \omega_2[p(SSx_0, SSx_0)], \omega_3[p(Sx_0, Sx_0)], \\
&\qquad \omega_4[p(SSx_0, Sx_0)], \omega_5[p(SSx_0, Sx_0)]\} \\
&\leqslant \omega_t[p(SSx_0, Sx_0)] \quad \text{for some } t \in \{1, 2, 3, 4, 5\}, \\
&< \tfrac{1}{2} p(SSx_0, Sx_0) \quad \text{for } p(SSx_0, Sx_0) > 0 \\
&\Rightarrow p(SSx_0, Sx_0) = 0.
\end{aligned}
$$

By (1.2), this implies

$$
(2.5) \qquad\qquad\qquad\qquad SSx_0 = Sx_0.
$$

Hence Sx_0 is a fixed point of S. From (2.4) we have $SSx_0 = ASx_0$. Thus (2.5) implies $ASx_0 = Sx_0$, making Sx_0 a fixed point of A too.

Using a similar argument we have $Tx_1 = Sx_0$ being a common fixed point of T and B. Hence, $z = Sx_0$ is a common fixed point of all four mappings S, T, A and B.

To show the uniqueness of the fixed point, let z' be also a fixed point of S, T, A and B. Then we have

$$
\begin{aligned}
p(z, z') = p(Az, Bz') &\leqslant \max\{\omega_1[p(Sz, Tz')], \omega_2[p(Az, Sz)], \omega_3[p(Bz', Tz')], \\
&\qquad \omega_4[p(Az, Tz')], \omega_5[p(Sz, Bz')]\} \\
&= \max\{\omega_1[p(z, z')], \omega_2[p(z, z)], \omega_3[p(z', z')], \\
&\qquad \omega_4[p(z, z')], \omega_5[p(z, z')]\} \\
&\leqslant \omega_i[p(z, z')] \quad \text{for some } i \in \{1, 2, 3, 4, 5\} \\
&< \tfrac{1}{2} p(z, z') \quad \text{for } p(z, z') > 0, \\
&\Rightarrow p(z, z') = 0 \\
&\Rightarrow z = z, \quad \text{by (1.1)}.
\end{aligned}
$$

Hence when $\alpha_n = 0$, then $z = Sx_0$ is the unique common fixed point of S, T, A and B.

We now consider the cases when $\alpha_n > 0$.

Case 1: Consider the case where $\alpha_n = p(Sx_{2i}, Ax_{2j})$ for some $0 \leqslant i, j \leqslant n - 1$.

Subcase 1.1: If $i \geqslant 1$ and $Sx_{2i} = Bx_{2i-1}$ we have

$$
\begin{aligned}
\alpha_n = p(Sx_{2i}, Ax_{2j}) &= p(Ax_{2j}, Bx_{2i-1}) \leqslant M_\omega(x_{2j}, x_{2i-1}) \\
&\leqslant \omega_s(\alpha_n) \quad \text{for some } s \in \{1, 2, 3, 4, 5\} \\
&< \tfrac{1}{2}\alpha_n,
\end{aligned}
$$

which is a contradiction. Hence $i = 0$.

Subcase 1.2: If however $i \geqslant 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it follows that $Sx_{2i} \in$ seg$[Ax_{2i-2}, Bx_{2i-1}]$ and hence, by Lemma 1.7 (i), we have

$$
\alpha_n = p(Sx_{2i}, Ax_{2j}) \leqslant \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i-2}, Ax_{2j})\}.
$$

Subcase 1.2.1: If $p(Ax_{2j}, Bx_{2i-1}) \geqslant p(Ax_{2i-2}, Ax_{2j})$, we have

$$
\alpha_n = p(Sx_{2i}, Ax_{2j}) \leqslant p(Ax_{2j}, Bx_{2i-1}),
$$

which leads to the contradiction in Case 1.1, meaning that $i = 0$.

Subcase 1.2.2: Otherwise, if $p(Ax_{2j}, Bx_{2i-1}) < p(Ax_{2i-2}, Ax_{2j})$, then for some k such that $2i - 2 < 2k + 1 < 2j$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have

$$
\begin{aligned}
\alpha_n = p(Sx_{2i}, Ax_{2j}) &\leqslant p(Ax_{2i-2}, Ax_{2j}) \\
&\leqslant p(Ax_{2i-2}, Bx_{2k+1}) + p(Ax_{2j}, Bx_{2k+1}), \text{ by } (1.2) \\
&\leqslant M_\omega(x_{2i-2}, x_{2i-1}) + M_\omega(x_{2j}, x_{2i-1}) \\
&\leqslant \omega_s(\alpha_n) + \omega_t(\alpha_n) \\
&< \tfrac{1}{2}\alpha_n + \tfrac{1}{2}\alpha_n \Rightarrow \alpha_n < \alpha_n,
\end{aligned}
$$

which is a contradiction. Hence $i = 0$.

Case 2: The case where $\alpha_n = p(Ax_{2i}, Bx_{2j+1})$ leads to contradiction by Case 1.1.

Case 3: The case where $\alpha_n = p(Ax_{2i}, Ax_{2j})$ leads to contradiction by Case 1.2.2.

Case 4: If $\alpha_n = p(Bx_{2i+1}, Bx_{2j+1})$ then for k such that $2i + 1 < 2k < 2j + 1$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have

$$
\begin{aligned}
\alpha_n = p(Bx_{2i+1}, Bx_{2j+1}) & \\
&\leqslant p(Ax_{2k}, Bx_{2i+1}) + p(Ax_{2k}, Bx_{2j+1}) \\
&\leqslant M_\omega(x_{2k}, x_{2i+1}) + M_\omega(x_{2k}, x_{2j+1}) \\
&\leqslant \omega_s(\alpha_n) + \omega_t(\alpha_n) \\
&< \tfrac{1}{2}\alpha_n + \tfrac{1}{2}\alpha_n \Rightarrow \alpha_n < \alpha_n,
\end{aligned}
$$

which is a contradiction.

Case 5: If $\alpha_n = p(Tx_{2i+1}, Bx_{2j+1})$ for some $0 \leqslant i, j \leqslant n - 1$, then:

Subcase 5.1: If $Tx_{2i+1} = Ax_{2i}$, then we have

$$\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) = p(Ax_{2i}, Bx_{2j+1}),$$

which is a contradiction by Case 1.1.

Subcase 5.2: Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence by Lemma 1.7 (i) we have

$$\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) \leqslant \max\{p(Bx_{2i-1}, Bx_{2j+1}), p(Ax_{2i}, Bx_{2j+1})\}.$$

This means we have either $p(Tx_{2i+1}, Bx_{2j+1}) \leqslant p(Bx_{2i-1}, Bx_{2j+1})$, which is a contradiction by Case 4 or $p(Tx_{2i+1}, Bx_{2j+1}) \leqslant p(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by Case 1.1.

Case 6: If $\alpha_n = p(Tx_{2i+1}, Ax_{2j})$ for some $0 \leqslant i, j \leqslant n - 1$, then:

Subcase 6.1: If $Tx_{2i+1} = Ax_{2i}$, then we have

$$\alpha_n = p(Tx_{2i+1}, Ax_{2j}) = p(Ax_{2i}, Ax_{2j})$$

which is not possible by Case 1.2.2.

Subcase 6.2: Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$, then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence $\alpha_n = p(Tx_{2i+1}, Ax_{2j}) \leqslant \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i}, Ax_{2j})\}$. This implies we have either $p(Tx_{2i+1}, Ax_{2j}) \leqslant p(Ax_{2j}, Bx_{2i-1})$, which is a contradiction by Case 1.1 or else we have $p(Tx_{2i+1}, Ax_{2j}) \leqslant p(Ax_{2i}, Ax_{2j})$, which is a contradiction by Case 1.2.2.

Case 7: Suppose $\alpha_n = p(Tx_{2i+1}, Tx_{2j+1})$ for some $0 \leqslant i, j \leqslant n - 1$.

Subcase 7.1: If $Tx_{2j+1} = Ax_{2j}$, we have

$$\alpha_n = p(Tx_{2i+1}, Tx_{2j+1}) = p(Tx_{2i+1}, Ax_{2j}),$$

which is a contradiction by Case 6.

Subcase 7.2: Otherwise, if $Tx_{2j+1} \neq Ax_{2j}$, then $Tx_{2j+1} \in \text{seg}[Bx_{2j-1}, Ax_{2j}]$ and hence $\alpha_n = p(Tx_{2i+1}, Tx_{j+1}) \leqslant \max\{p(Tx_{2i+1}, Bx_{2j-1}), p(Tx_{2i+1}, Ax_{2j})\}$.

This implies we have either $p(Tx_{2i+1}, Tx_{2j+1}) \leqslant p(Tx_{2i+1}, Bx_{2j-1})$, which results in a contradiction by Case 5, or else we have $p(Tx_{2i+1}, Tx_{2j+1}) \leqslant p(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

Case 8: Let $\alpha_n = p(Sx_{2i}, Bx_{2j+1})$ for some $0 \leqslant i, j \leqslant n - 1$.

Subcase 8.1: If $i \geqslant 1$ and $Sx_{2i} = Bx_{2i-1}$, then $\alpha_n = p(Sx_{2i}, Bx_{2j+1}) = p(Bx_{2i-1}, Bx_{2j+1})$, which is not possible as per Case 4. Hence $i = 0$.

Subcase 8.2: If however $i \geqslant 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it means that $Sx_{2i} \in$ seg$[Ax_{2i-2}, Bx_{2i-1}]$. This implies that $\alpha_n = p(Sx_{2i}, Bx_{2j+1}) \leqslant \max\{p(Ax_{2i-2}, Bx_{2j+1}), p(Bx_{2i-1}, Bx_{2j+1})\}$. This leads to a contradiction by Case 1.1 when $p(Sx_{2i}, Bx_{2j+1}) \leqslant p(Ax_{2i-2}, Bx_{2j+1})$ and a contradiction by Case 4 when it happens that $p(Sx_{2i}, Bx_{2j+1}) \leqslant p(Bx_{2i-1}, Bx_{2j+1})$. Hence $i = 0$.

Case 9: Let us consider the case when $\alpha_n = p(Sx_{2i}, Sx_{2j})$ for some $0 \leqslant i < j \leqslant n - 1$.

Subcase 9.1: If $i \geqslant 1$ and $Sx_{2j} = Bx_{2j-1}$, then we have $\alpha_n = p(Sx_{2i}, Sx_{2j}) = p(Sx_{2i}, Bx_{2j-1})$, which leads to a contradiction according to Case 8. Hence $i = 0$.

Subcase 9.2: If $i \geqslant 1$ and $Sx_{2j} \neq Bx_{2j-1}$, it follows that $Sx_{2j} \in$ seg$[Ax_{2j-2}, Bx_{2j-1}]$ and $p(Sx_{2i}, Sx_{2j}) \leqslant \max\{p(Sx_{2i}, Ax_{2j-2}), p(Sx_{2i}, Bx_{2j-1})\}$.

If it happens $p(Sx_{2i}, Sx_{2j}) \leqslant p(Sx_{2i}, Ax_{2j-2})$, we get a contradiction by Case 1. However if it happens that $p(Sx_{2i}, Sx_{2j}) \leqslant p(Sx_{2i}, Bx_{2j-1})$, then we get a contradiction by Case 8. Hence $i = 0$.

Case 10: Suppose $\alpha_n = p(Sx_{2i}, Tx_{2j+1})$ for some $0 \leqslant i, j \leqslant n - 1$.

Subcase 10.1: If $i \geqslant 1$ and $Sx_{2i} = Bx_{2i-1}$, then we have $\alpha_n = p(Sx_{2i}, Tx_{2j+1}) = p(Tx_{2j+1}, Bx_{2i-1})$, which is not possible as per Case 8. Hence $i = 0$.

Subcase 10.2: If however $i \geqslant 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it follows that $Sx_{2i} \in$ seg$[Ax_{2i-2}, Bx_{2i-1}]$ and

$$p(Sx_{2i}, Tx_{2j+1}) \leqslant \max\{p(Tx_{2j+1}, Ax_{2i-2}), p(Tx_{2j+1}, Bx_{2i-1})\}.$$

This leads to contradictions by Case 6 and Case 5. Hence $i = 0$.

We have considered ten possible cases for α_n and conclude from Cases 1, 8, 9 and 10 that for some $0 \leqslant j \leqslant n - 1$ we have

$$(2.6) \qquad \alpha_n \in \{p(Sx_0, Sx_{2j}), p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1}), p(Sx_0, Tx_{2j+1})\}.$$

Note that, from the construction of the sequence, $Sx_{2j} \in C$ implies $Sx_{2j} = Bx_{2j-1}$. This leads to

$$(2.7) \qquad\qquad p(Sx_0, Sx_{2j}) = p(Sx_0, Bx_{2j-1}).$$

For $Sx_{2j} \notin C$, we have $Sx_{2j} \in$ seg$[Ax_{2j-2}, Bx_{2j-1}]$. By Lemma 1.7 (i), this implies

$$(2.8) \qquad p(Sx_0, Sx_{2j}) \leqslant \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}.$$

From (2.7) and (2.8) we conclude that

$$(2.9) \qquad p(Sx_0, Sx_{2j}) \leqslant \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}.$$

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Using a similar argument we also have

(2.10) $\qquad p(Sx_0, Tx_{2j+1}) \leqslant \max\{p(Sx_0, Bx_{2j-1}), p(Sx_0, Ax_{2j})\}.$

Applying (2.9) and (2.10) to (2.6) we get

$$\alpha_n \leqslant \max\{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1})\}, \quad 0 \leqslant j \leqslant n - 1.$$

We have proved (2.3).

Consider the case where $\max\{p(Sx_0, Ax_{2j})\} \leqslant \max\{p(Sx_0, Bx_{2j+1})\}$, $0 \leqslant j \leqslant n - 1$. Then, for some $u \in \{1, 2, \ldots, 5\}$, (2.3) implies

$$\begin{aligned}
\alpha_n &\leqslant p(Sx_0, Bx_{2j+1}) \\
&\leqslant p(Sx_0, Ax_0) + p(Ax_0, Bx_{2j+1}), \quad \text{by (1.2)} \\
&\leqslant p(Sx_0, Ax_0) + \omega_u[\alpha_n] \\
&\leqslant p(Sx_0, Ax_0) + 2\omega_u[\alpha_n] \\
&\Rightarrow \alpha_n - 2\omega_u[\alpha_n] \leqslant p(Sx_0, Ax_0).
\end{aligned}$$

Consider now the case when $\max\{p(Sx_0, Ax_{2j})\} > \max\{p(Sx_0, Bx_{2j+1})\}$, $0 \leqslant j \leqslant n - 1$. Then for some $v \in \{1, 2, \ldots, 5\}$, (2.3) implies

$$\begin{aligned}
\alpha_n &\leqslant p(Sx_0, Ax_{2j}) \\
&\leqslant p(Sx_0, Ax_0) + p(Ax_0, Ax_{2j}), \text{ by (1.2)} \\
&\leqslant p(Sx_0, Ax_0) + 2\omega_v[\alpha_n], \quad \text{by Case 1.2.2} \\
&\Rightarrow \alpha_n - 2\omega_v[\alpha_n] \leqslant p(Sx_0, Ax_0).
\end{aligned}$$

Thus in both cases, we have for some $s \in \{1, 2, \ldots, 5\}$

(2.11) $\qquad\qquad \alpha_n - 2\omega_s[\alpha_n] \leqslant p(Sx_0, Ax_0).$

By assumption (i) there is an $r_0 \in [0, \infty)$ such that for each $s \in \{1, 2, \ldots, 5\}$ we have $r - 2\omega_s(r) > p(Sx_0, Ax_0)$ for $r > r_0$.

There is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ and $s \in \{1, 2, \ldots, 5\}$ such that for each n, we have

$$a_n - 2\omega_s[a_n] \leqslant p(Sx_0, Ax_0).$$

Thus by (2.11), $a_n \leqslant r_0$. Thus we have

$$a = \lim_{n \to \infty} a_n = \operatorname{diam}(D) \leqslant r_0.$$

We have hence proved that $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ are bounded sequences.

To prove that $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge in C, we consider the set

$$(2.12) \qquad E_n = \bigcup_{i=n}^{\infty}\{Ax_{2i}\} \cup \bigcup_{i=n}^{\infty}\{Bx_{2i+1}\} \cup \bigcup_{i=n}^{\infty}\{Sx_{2i}\} \cup \bigcup_{i=n}^{\infty}\{Tx_{2i+1}\},$$

$n = 2, 3, \ldots$

By (2.3) we have for $n = 2, 3, \ldots$

$$(2.13) \qquad e_n := \mathrm{diam}(E_n) \leqslant \sup_{j \geqslant n}\{p(Sx_{2n}, Ax_{2j}), p(Sx_{2n}, Bx_{2j+1})\}.$$

If $Sx_{2n} = Bx_{2n-1}$ we have as in Case 1 and Case 8, for each $j \geqslant n$ and for some $u \in \{1, 2, \ldots, 5\}$

$$(2.14) \qquad \begin{aligned} e_n &\leqslant \sup_{j \geqslant n}\{p(Ax_{2j}, Bx_{2n-1}), p(Bx_{2j+1}, Bx_{2n-1})\}, \quad n = 2, 3 \ldots \\ &\leqslant 2\omega_u[e_{n-1}]. \end{aligned}$$

If however $Sx_{2n} \neq Bx_{2n-1}$, it follows that $Sx_{2n} \in \mathrm{seg}[Ax_{2n-2}, Bx_{2n-1}]$. Hence, as in Case 1 and Case 8, for each $j \geqslant n$ and for some $v \in \{1, 2, \ldots, 5\}$, we have

$$(2.15) \qquad \begin{aligned} e_n &\leqslant \sup_{j \geqslant n}\{p(Ax_{2n-2}, Ax_{2j}), p(Bx_{2n-1}, Ax_{2j}), \\ &\qquad\qquad p(Ax_{2n-2}, Bx_{2j+1}), p(Bx_{2n-1}, Bx_{2j+1})\}, \quad n = 2, 3, \ldots \\ &\leqslant 2\omega_v[e_{n-2}]. \end{aligned}$$

By (2.14) and (2.15), there is a subsequence $\{\varepsilon_n\}$ of $\{e_n\}$ and $s \in \{1, 2, \ldots, 5\}$ such that for each n

$$(2.16) \qquad \begin{aligned} \varepsilon_n &\leqslant 2\omega_s[\varepsilon_{n-2}], \quad n = 2, 3, \ldots \\ &< \varepsilon_{n-2}. \end{aligned}$$

We note that $e_n \geqslant e_{n+1}$ for every n. Let $\lim_{n \to \infty} e_n = \lim_{n \to \infty} \varepsilon_n = e$. We claim that $e = 0$.

If $e > 0$, then by (2.16) and assumption (i) of Theorem 2.1 we have

$$\lim_{n \to \infty} \varepsilon_n < \lim_{n \to \infty} \varepsilon_{n-2} \Rightarrow e < e,$$

which is a contradiction. Hence $e = 0$.

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We recall from (2.13) that $e_n = \text{Diam}(E_n)$. Taking $n, m \to \infty$ in (2.12), we get

$$(2.17) \qquad \lim_{n,m\to\infty} p(Ax_{2n}, Ax_{2m}) = \lim_{n,m\to\infty} p(Bx_{2n+1}, Bx_{2m+1}) = 0.$$

This means both $\{A_{2n}\}$ and $\{B_{2n+1}\}$ are Cauchy sequences.

Because X is a complete partial metric space, this means there is $z \in X$ such that

$$(2.18) \qquad \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = z \quad \text{and} \quad p(z, z) = 0.$$

Consider the subsequence Sx_{2n_k} of Sx_{2n} such that $Sx_{2n_k} = Bx_{2n_k-1}$. Taking $n_k \to \infty$ we have

$$(2.19) \qquad \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Sx_{2n_k} = \lim_{n\to\infty} Bx_{2n_k-1} = z, \quad \text{with } p(z, z) = 0.$$

Using a similar argument we have

$$(2.20) \qquad \lim_{n\to\infty} Tx_{2n+1} = z, \quad \text{with } p(z, z) = 0.$$

But both SC, TC are 0-complete. This implies $z \in SC$ and $z \in TC$.

As $z \in SC$, there is a point $u \in C$ such that $Su = z$. We show that u is a coincidence point of A, B and S:

$$p(Au, Bx_{2n+1}) \leqslant \max\{\omega_1[p(Su, Tx_{2n+1})], \omega_2[p(Au, Su)], \omega_3[p(Bx_{2n+1}, Tx_{2n+1})],$$
$$\omega_4[p(Au, Tx_{2n+1})], \omega_5[p(Su, Bx_{2n+1})]\}$$
$$= \max\{\omega_1[p(z, Tx_{2n+1})], \omega_2[p(Au, z)], \omega_3[p(Bx_{2n+1}, Tx_{2n+1})],$$
$$\omega_4[p(Au, Tx_{2n+1})], \omega_5[p(z, Bx_{2n+1})]\}.$$

Taking $n \to \infty$ and applying (2.19) and (2.20), we get

$$(2.21) \; p(Au, z) \leqslant \max\{\omega_1[p(z, z)], \omega_2[p(Au, z)], \omega_3[p(z, z], \omega_4[p(Au, z)], \omega_5[p(z, z)]\}$$
$$\leqslant \omega_i[p(Au, z)] \quad \text{for some } i \in \{1, 2, \ldots, 5\}$$
$$< p(Au, z) \quad \text{for } p(Au, z) > 0$$
$$\Rightarrow p(Au, z) = 0$$
$$\Rightarrow Au = z, \quad \text{from (1.1)}.$$

Using a similar procedure, when we expand $p(Ax_{2n}, Bu)$, we get $Bu = z$, making u a coincidence point of A, B and S. By the coincidental commutativity of S and A, we have $SAu = ASu \Rightarrow Sz = Az$.

In the same vein, $z \in TC$ means there is $v \in C$ such that $Tv = z$. We show that $Bv = z$:

$$
\begin{aligned}
p(z, Bv) &= p(Au, Bv) \\
&\leqslant \max\{\omega_1[p(Su, Tv)], \omega_2[p(Su, Au)], \omega_3[p(Tv, Bv)], \\
&\qquad\quad \omega_4[p(Au, Tv)], \omega_5[p(Su, Bv)]\} \\
&= \max\{\omega_1[p(z, z)], \omega_2[p(z, z)], \omega_3[p(z, Bv)], \\
&\qquad\quad \omega_4[p(z, z)], \omega_5[p(z, Bv)]\} \\
&\leqslant \omega_j[p(z, Bv)] \quad \text{for some } j \in \{1, 2, \ldots, 5\} \\
&< p(z, Bv) \quad \text{for } p(z, Bv) > 0 \\
&\Rightarrow p(z, Bv) = 0 \\
&\Rightarrow Bv = z, \quad \text{from (1.1)}.
\end{aligned}
$$

Thus v is a coincidence point of B and T. By the coincidental commutativity property, we have

$$
BTv = TBv \Rightarrow Bz = Tz.
$$

Now the following holds:

$$
\begin{aligned}
p(Az, Bz) &\leqslant \max\{\omega_1[p(Sz, Tz)], \omega_2[p(Sz, Az)], \omega_3[p(Tz, Bz)], \\
&\qquad\quad \omega_4[p(Az, Tz)], \omega_5[p(Sz, Bz)]\} \\
&= \max\{\omega_1[p(Az, Bz)], \omega_2[p(Az, Az)], \omega_3[p(Bz, Bz)], \\
&\qquad\quad \omega_4[p(Az, Bz)], \omega_5[p(Az, Bz)]\} \\
&\leqslant \omega_i[p(Az, Bz)] \quad \text{for } i \in \{1, 2, \ldots, 5\} \\
&< p(Az, Bz) \quad \text{for } p(Az, Bz) > 0 \\
&\Rightarrow p(Az, Bz) = 0 \\
&\Rightarrow Az = Bz.
\end{aligned}
$$

Hence we have

(2.22)
$$
Az = Bz = Sz = Tz.
$$

Now the following holds:

$$
\begin{aligned}
p(z, Bz) &= p(Au, Bz), \quad \text{from (2.21)} \\
&\leqslant \max\{\omega_1[p(Su, Tz)], \omega_2[p(Au, Su)], \omega_3[p(Bz, Tz)], \\
&\qquad\quad \omega_4[p(Au, Tz)], \omega_5[p(Su, Bz)]\} \\
&= \max\{\omega_1[p(z, Bz)], \omega_2[p(z.z)], \omega_3[p(Bz, Bz)], \omega_4[p(z, Bz)], \omega_5[p(z, Bz)]\}.
\end{aligned}
$$

This implies

$$p(z, Bz) \leqslant \omega_j[p(z, Bz)] \quad \text{for some } j \in \{1, 2, \ldots, 5\}$$
$$< p(z, Bz) \quad \text{for } p(z, Bz) > 0$$
$$\Rightarrow p(z, Bz) = 0$$
$$\Rightarrow Bz = z, \quad \text{by (1.1)}.$$

From (2.22) we conclude that $Az = Bz = Sz = Tz = z$, meaning that z is a common fixed point of A, B, S and T.

We now show that z is unique. Suppose z' is also a common fixed point of A, B, S and T. We get

$$p(z, z') = p(Az, Bz')$$
$$\leqslant \max\{\omega_1[p(Sz, Tz')], \omega_2[p(Az, Sz)], \omega_3[p(Bz', Tz')],$$
$$\omega_4[p(Az, Tz')], \omega_5[p(Sz, Bz')]\}$$
$$= \max\{\omega_1[p(z, z')], \omega_2[p(z, z)], \omega_3[p(z', z')],$$
$$\omega_4[p(z, z')], \omega_5[p(z, z')]\}.$$

This implies

$$p(z, z') \leqslant \omega_k[p(z, z')] \quad \text{for } k \in \{1, 2, \ldots, 5\}$$
$$< p(z, z') \quad \text{for } p(z, z') > 0$$
$$\Rightarrow p(z, z') = 0$$
$$\Rightarrow z = z'.$$

This proves that the common fixed point of A, B, S and T is unique. $\quad\square$

R e m a r k 2.2. Theorem 2.1 leads to corollaries if we consider the following cases:

(i) $A = B$;

(ii) $A = B$, $S = T$, we get Theorem 1.9;

(iii) $A = B$, $S = T = I$, we get an extension of a theorem proved by Ćirić (see [2]) into partial metric spaces;

(iv) $A = I$;

(v) $S = T$;

(vi) $x = y$.

The proof given above works even when we define C as closed in (X, p^s). This leads to the following theorem.

Theorem 2.3. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a closed subset*

of X, the closure being taken with respect to (X, p^s). Let ∂C, the boundary of C in (X, p^s), be nonempty. Let $A, B, S, T \colon C \to X$. Let us assume that A, B, S and T satisfy the following conditions:

(i) For every $x, y \in C, p(Ax, By) \leqslant M_\omega(x, y)$ where $M_\omega(x, y) = \max\{\omega_1[p(Sx, Ty)],$ $\omega_2[p(Ax, Sx)], \omega_3[p(By, Ty)], \omega_4[p(Ax, Ty)], \omega_5[p(Sx, By)]\}$, $\omega_i \colon \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2, 3, 4, 5$, is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $\lim\limits_{r \to \infty}(r - 2\omega_i(r)) = \infty$.

(ii) $\partial C \subseteq SC$, $\partial C \subseteq TC$,

(iii) $Sx \in \partial C \Rightarrow Ax \in C$; $Tx \in \partial C \Rightarrow Bx \in C$,

(iv) $AC \cap C \subset TC$, $BC \cap C \subset SC$ and

(v) SC, TC are closed in C.

Then there exists a coincidence point $z \in C$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

Here we give an example on the use of Theorem 2.3, as it is better suited for the partial metric that we will use.

E x a m p l e 2.4. Consider the partial metric space $(\mathbb{R}_+, p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$. Let $C = [0, 2]$. We note that C is closed in the derived metric $p^s(x, y) = |x - y|$ and $\partial C = \{0, 2\}$.

We define the mappings $A, B, S, T \colon C \to \mathbb{R}_+$ as follows:

$$Ax = \begin{cases} 3^x - 1, & x \in [0, 1], \\ 1, & x \in (1, 2], \end{cases} \qquad Bx = \begin{cases} 4^x - 1, & x \in [0, 1], \\ 2, & x \in (1, 2], \end{cases}$$

$$Sx = \begin{cases} 27^x - 1, & x \in [0, 1], \\ 6, & x \in (1, 2], \end{cases} \qquad Tx = \begin{cases} 64^x - 1, & x \in [0, 1], \\ 8, & x \in (1, 2]. \end{cases}$$

We have $AC = [0, 2]$ and $BC = [0, 3]$. We also have $TC = [0, 63]$ and $SC = [0, 26]$, both of which are closed in (X, p^s).

$Sx \in \partial C$ implies $z \in \{0, \frac{1}{3}\} \subset C$. Similarly $Tx \in \partial C$ implies $x \in \{0, \frac{\ln 3}{\ln 64}\} \subset C$. We also have $\partial K = \{1, 3\} \subseteq SC, TC$.

We note that both $\{S, A\}$ and $\{T, B\}$ are coincidentally commuting at $x = 0$, that is, $SA(0) = AS(0)$ and $TB(0) = BT(0)$. We also note that all four mappings are discontinuous at 1.

Let us define functions $g, h \colon C \to \mathbb{R}_+$ as

$$g(x) = \frac{27^x - 1}{4^x - 1}, \qquad h(x) = \frac{64^x - 1}{4^x - 1}.$$

Both g and h are increasing functions. Using L'Hôpital rule, we can show that, as $x \to 0$, we have $h(x) \to 3$ and

(2.23)
$$g(x) \to \frac{\log 27}{\log 4} = \frac{1}{0.42062}.$$

Hence, for $x \in [0, 1]$, we have

(2.24)
$$4^x - 1 \leqslant \frac{1}{3}(64^x - 1) \leqslant 0.43(64^x - 1).$$

We also have, from (2.23)

(2.25)
$$4^x - 1 \leqslant 0.42062(27^x - 1) \Rightarrow 4^x - 1 \leqslant 0.43(27^x - 1).$$

When $x, y \in [0, 1]$ with $x \leqslant y$, we have

$$
\begin{aligned}
p(Ax, By) &= p(3^x - 1, 4^y - 1) \\
&= \max\{3^x - 1, 4^y - 1\} \\
&= 4^y - 1, \quad \text{because } x \leqslant y \\
&\leqslant 0.43(64^y - 1), \quad \text{by (2.24)} \\
&= 0.43Ty \\
&\leqslant 0.43 \max\{Sx, Ty\} \\
&= 0.43p(Sx, Ty).
\end{aligned}
$$

When $x, y \in [0, 1]$ with $x > y$, we have

$$
\begin{aligned}
p(Ax, By) &= p(3^x - 1, 4^y - 1) \\
&= \max\{4^x - 1, 4^y - 1\}, \quad \text{because } 3^x - 1 \leqslant 4^x - 1 \\
&= 4^x - 1, \text{ because } x > y \\
&\leqslant 0.43(27^x - 1), \quad \text{by (2.25)} \\
&= 0.43Sx \\
&\leqslant 0.43 \max\{Sx, Ty\} \\
&= 0.43p(Sx, Ty).
\end{aligned}
$$

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For $x, y \in (1, 2]$ we have

$$
\begin{aligned}
p(Ax, By) &= p(1, 2) \\
&= \max\{1, 2\} \\
&= 2 \\
&< 0.43 \times 6 \\
&\leqslant 0.43 \max\{6, Ty\} \\
&= 0.43 \max\{Sx, Ty\}, \quad \text{because } Sx = 6 \\
&= 0.43 p(Sx, Ty).
\end{aligned}
$$

Considering $x \in [0, 1]$, $y \in (1, 2]$ we get

$$
\begin{aligned}
p(Ax, By) &= p(3^x - 1, 2) \\
&= \max\{3^x - 1, 2\} \\
&= 2, \quad \text{because } 3^x - 1 \leqslant 2 \text{ for } x \in [0, 1] \\
&< 0.43 \times 8 \\
&= 0.43 Ty \quad \text{because } Ty = 8 \\
&\leqslant 0.43 \max\{Sx, Ty\} \\
&= 0.43 p(Sx, Ty).
\end{aligned}
$$

Considering $x \in (1, 2]$, $y \in (0, 1)$.

$$
p(Ax, By) = p(1, 4^y - 1) = \max\{1, 4^y - 1\} = \begin{cases} 1, \\ 6, & y \in (\tfrac{1}{2}, 1]. \end{cases}
$$

For $y \in [0, \tfrac{1}{2}]$ we have

$$
\begin{aligned}
p(Ax, By) &= p(1, 4^y - 1) \\
&= 1 \\
&< 0.43 \times 6 \\
&= 0.43 Sx \quad \text{because } Sx = 6 \\
&\leqslant 0.43 \max\{Sx, Ty\} \\
&= 0.43 p(Sx, Ty).
\end{aligned}
$$

For $y \in (\tfrac{1}{2}, 1]$ we have

$$
\begin{aligned}
p(Ax, By) &= p(1, 4^y - 1) \\
&= 4^y - 1
\end{aligned}
$$

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$$\leqslant 0.43(64^y - 1), \quad \text{by (2.24)}$$
$$= 0.43Ty$$
$$\leqslant 0.43 \max\{Sx, Ty\}$$
$$= 0.43p(Sx, Ty).$$

Thus in all cases, for every $x, y \in C$ we have

$$p(Ax, By) \leqslant 0.43p(Sx, Ty) \leqslant M_\omega(x, y),$$

where

$$M_\omega(x, y) = \max\{\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(Ay, Ty)],$$
$$\omega_4[p(Ax, Ty)], \omega_5[p(Sx, Ay)]\},$$

with $\omega_i(r) = 0.43r$. We note that $\omega_i(r) = 0.43r < \frac{1}{2}r$. As $r - 2\omega_i(r) = r - 2 \times 0.43r = 0.14r$, we also have $\lim_{r \to \infty} r - 2\omega_i(r) = \lim_{r \to \infty} 0.14r = \infty$.

Thus all the conditions of Theorem 2.3 are satisfied and 0 is the unique common fixed point of A, B, S and T.

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