MULTIPlicity OF POSITIVE SOLUTIONS FOR SECOND ORDER QUASILINEAR EQUATIONS

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Abstract. We discuss the existence and multiplicity of positive solutions for a class of second order quasilinear equations. To obtain our results we will use the Ekeland variational principle and the Mountain Pass Theorem.

Keywords: critical point; Ekeland variational principle; Mountain Pass Theorem; Palais-Smale condition; positive solution

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1. INTRODUCTION

Our aim in this paper is to obtain at least two positive solutions for the problem

\[
\begin{cases}
-u'' + u = \lambda h(x)|u|^\beta - 2 u + q(x)f(u), & x \in (0, \infty), \\
u(0) = u(\infty) = 0,
\end{cases}
\]

where \( f \in C(\mathbb{R}, \mathbb{R}) \), \( \beta \) and \( \lambda \) are real parameters with \( 1 < \beta < 2 \) and \( \lambda > 0 \).

Throughout this paper we assume the following hypotheses are satisfied:

(H0) \( h \) and \( q \): \([0, \infty) \rightarrow (0, \infty) \) belong to \( L^1(0, \infty) \cap L^\infty(0, \infty) \);

(H1) there is a continuously differentiable and bounded function \( p \): \([0, \infty) \rightarrow (0, \infty) \) belonging to \( L^1(0, \infty) \cap L^\infty(0, \infty) \) such that the functions \( q/p \), \( q/p^2 \), \( q/p^\beta \), \( q/p^{\beta+1} \), \( h/p^{\beta-1} \) and \( h/p^\beta \) all belong to \( L^1(0, \infty) \);

(H2) \( M = \max(\|p\|_{L^2}, \|p'\|_{L^2}) < \infty \),

\[ M_{r,g} = \|p\|_{L^\infty}^{1/2} \left( \int_0^\infty g(x) \left( \int_0^x \frac{ds}{p(s)} \right)^{r/2} dx \right)^{1/r} < \infty \]

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for all \( r \in \{ \beta, 2, \beta + 1 \} \) and all \( g \in \{ q, h \} \) and
\[
M_{2,q} = \| p \|_\infty^{1/2} \left( \int_0^\infty q(x) \left( \int_0^x \frac{ds}{p(s)} \right) dx \right)^{1/2} < \frac{1}{\sqrt{A}},
\]
where the constant \( A \) satisfies

\( \text{(H}_3) \lim_{u \to 0^+} f(u)/\|u\| = A \in (0, \lambda_{2,q}^2) \) and \( \lim_{u \to \infty} f(u)/\|u\|^\beta = B \in (\lambda_{2,q}^2, \infty) \), where \( \lambda_{2,q} \)

is the first eigenvalue of problem (2) which is defined in Lemma 1.3;

\( \text{(H}_4) \) there exists \( \mu > \beta + 1 \) such that
\[
F(s) \leq \frac{1}{\mu} sf(s) \quad \forall |s| > 0, \text{ where } F(s) = \int_0^s f(t) \, dt.
\]

Now we introduce the Hilbert space \( H^1_0(0, \infty) \) which is suitable for the study of our

problem. Let
\[
H^1_0(0, \infty) = \{ u \text{ measurable: } u, u' \in L^2(0, \infty), u(0) = u(\infty) = 0 \}
\]
equipped with the norm
\[
\| u \| = \left( \int_0^\infty |u'(x)|^2 \, dx + \int_0^\infty |u(x)|^2 \, dx \right)^{1/2}
\]
and endowed with the inner product
\[
(u, v) = \int_0^\infty u'(x) \cdot v'(x) \, dx + \int_0^\infty u(x) \cdot v(x) \, dx.
\]

We consider the spaces \( L^r_{g}(0, \infty) \) which are defined by
\[
L^r_{g}(0, \infty) = \left\{ u: (0, \infty) \to \mathbb{R} \text{ measurable such that } \int_0^\infty g(x)|u(x)|^r \, dx < \infty \right\}
\]
for all \( r \in \{ \beta, 2, \beta + 1 \} \) and all \( g \in \{ h, q \} \) equipped, respectively, with the norms
\[
\| u \|_{r,g} = \left( \int_0^\infty g(x)|u(x)|^r \, dx \right)^{1/r}.
\]

Let the space \( C_{l,p}[0, \infty) \) be defined by
\[
C_{l,p}[0, \infty) = \left\{ u \in C([0, \infty), \mathbb{R}): \lim_{x \to \infty} p(x) u(x) \text{ exists} \right\}.
\]
The corresponding norm is defined by
\[ \| u \|_{\infty,p} = \sup_{x \in [0,\infty)} p(x)|u(x)|. \]

Now we give some necessary lemmas and corollaries, which are used below.

**Lemma 1.1** ([5]). \( H^1_0(0,\infty) \) embeds continuously and compactly in \( C_{l,p}[0,\infty) \), i.e.
\[ \| u \|_{\infty,p} \leq \sqrt{2} M \| u \| \quad \forall u \in H^1_0(0,\infty). \]

**Lemma 1.2** ([2]). \( C_{l,p}[0,\infty) \) is continuously embedded in \( L^{r,g}(0,\infty) \) for all \( r \in \{ \beta, 2, \beta + 1 \} \) and all \( g \in \{ h, q \} \).

**Corollary 1.1** ([2]). \( H^1_0(0,\infty) \) embeds continuously and compactly in \( L^{r,g}(0,\infty) \) with the embedding constant \( M_{r,g} \).

Let \( \lambda_{r,g} \) be the first eigenvalue of the problem
\[ \begin{cases} -u''(x) + u(x) = \lambda g(x)|u(x)|^{r-2}u(x), & x > 0, \\ u(0) = u(\infty) = 0, \end{cases} \tag{2} \]
and note
\[ \lambda_{r,g} = \inf_{u \in H^1_0(0,\infty) \setminus \{0\}} \frac{\| u \|}{\| u \|_{r,g}}. \]

**Lemma 1.3** ([2]). The first eigenvalue \( \lambda_{r,g} \) is positive and is achieved for some positive function \( \psi_{r,g} \in H^1_0(0,\infty) \setminus \{0\} \), i.e.
\[ \lambda_{r,g} := \inf_{u \in H^1_0(0,\infty) \setminus \{0\}} \frac{\| u \|}{\| u \|_{r,g}} = \frac{\| \psi_{r,g} \|}{\| \psi_{r,g} \|_{r,g}}. \]

**Theorem 1.1** ([4], Weak Ekeland variational principle). Let \( (E,d) \) be a complete metric space and let \( J : E \to \mathbb{R} \) be a functional that is lower semi-continuous and bounded from below. Then for each \( \varepsilon > 0 \) there exists \( u_\varepsilon \in E \) with
\[ J(u_\varepsilon) \leq \inf_{E} J + \varepsilon, \]
and whenever \( w \in E \) with \( w \neq u_\varepsilon \), then
\[ J(u_\varepsilon) < J(w) + \varepsilon d(u_\varepsilon, w). \]
**Definition 1.1** ([6]). Let $E$ be a Banach space and $J: E \to \mathbb{R}$ a $C^1$-functional and $c \in \mathbb{R}$. The functional $J$ is said to satisfy the (local) Palais-Smale condition at the level $c$, denoted by (P.S)$_c$, if any sequence $(u_n)$ in $E$ such that

\[ J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0, \]

admits a convergent subsequence.

**Lemma 1.4** (Mountain Pass Theorem). Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$ with $J(0) = 0$. Suppose $J(u)$ satisfies (P.S)$_c$ condition and

(a) there are $\varrho, \alpha > 0$ such that $J(u) \geq \alpha$ when $\|u\|_E = \varrho$,

(b) there is an $e \in E$, $\|e\|_E > \varrho$ such that $J(e) < 0$.

Define

\[ \Gamma = \{ \gamma \in C^1([0, 1], E) : \gamma(0) = 0, \gamma(1) = e \}. \]

Then

\[ c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) \geq \alpha \]

is a critical value of $J(u)$.

## 2. Main existence results

Now we define the Euler-Lagrange functional associated to problem (1). Let $J_\lambda: H^1_0(0, \infty) \to \mathbb{R}$ be defined by

\[ J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta(x) \, dx - \int_0^\infty q(x)F(u) \, dx. \]

**Proposition 2.1.** Suppose that the conditions (H$_0$)–(H$_3$) hold. Then the functional $J_\lambda$ is continuously differentiable. The Fréchet derivative of $J_\lambda$ has the form

\[ \langle J'_\lambda(u), v \rangle = \int_0^\infty u'(x)v'(x) \, dx + \int_0^\infty u(x)v(x) \, dx - \lambda \int_0^\infty h(x)|u|^\beta-2(x)u(x)v(x) \, dx - \int_0^\infty q(x)f(u)v(x) \, dx \]

for all $v \in H^1_0(0, \infty)$.  

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Proof. The proof of the proposition will be done consecutively.

Claim 2.1. $J_\lambda$ is Gâteaux-differentiable.

For all $v \in H^1_0(0, \infty)$ and for any $t > 0$ we have

$$J_\lambda(u + tv) - J_\lambda(u)$$

$$= \frac{1}{2} \int_0^\infty |(u + tv)'|^2 \, dx + \frac{1}{2} \int_0^\infty |u + tv|^2 \, dx - \frac{\lambda}{\beta} \int_0^\infty h(x)|u + tv|^\beta \, dx$$

$$- \int_0^\infty q(x)F(u + tv) \, dx - \frac{1}{2} \int_0^\infty |u'|^2 \, dx - \frac{1}{2} \int_0^\infty |u|^2 \, dx$$

$$+ \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta \, dx + \int_0^\infty q(x)F(u) \, dx$$

$$= \frac{t^2}{2} \int_0^\infty |v'|^2 \, dx + t \int_0^\infty u'v' \, dx + \frac{t^2}{2} \int_0^\infty |v|^2 \, dx + t \int_0^\infty uv \, dx$$

$$- \frac{\lambda}{\beta} \int_0^\infty h(x)(|u + tv|^\beta - |u|^\beta) \, dx - \int_0^\infty q(x)(F(u + tv) - F(u)) \, dx$$

$$= \frac{t^2}{2} \int_0^\infty |v'|^2 \, dx + t \int_0^\infty u'v' \, dx + \frac{t^2}{2} \int_0^\infty |v|^2 \, dx + t \int_0^\infty uv \, dx$$

$$- t\lambda \int_0^\infty h(x)|u + t\theta v|^\beta - 2(u + t\theta v)v \, dx - t \int_0^\infty q(x)f(u + t\theta v)v \, dx,$$

where $0 < \theta < 1$, and then

$$\frac{J_\lambda(u + tv) - J_\lambda(u)}{t} = \frac{t}{2} \int_0^\infty |v'|^2 \, dx + \int_0^\infty u'v' \, dx + \frac{t}{2} \int_0^\infty |v|^2 \, dx$$

$$+ \int_0^\infty uv \, dx - \lambda \int_0^\infty h(x)|u + t\theta v|^\beta - 2(u + t\theta v)v \, dx$$

$$- \int_0^\infty q(x)f(u + t\theta v)v \, dx.$$

Let $t \to 0$ and we have

$$\langle J'_\lambda(u), v \rangle = \int_0^\infty u'v' \, dx + \int_0^\infty uv \, dx - \lambda \int_0^\infty h(x)|u|^\beta - 2uv \, dx - \int_0^\infty q(x)f(u)v \, dx$$

for all $v \in H^1_0(0, \infty)$.

Claim 2.2. $J'_\lambda$ is continuous.

Let $(u_n) \subset H^1_0(0, \infty)$ with $u_n \to u$ when $n \to \infty$, so there exists $R > 0$ such that $\|u_n\| \leq R$ for all $n \in \mathbb{N}$. 

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From \((H_3)\), given \(\varepsilon\) small enough, there exists \(\delta_2 > \delta_1 > 0\) such that

\[
(A - \varepsilon)|s| < f(s) < (A + \varepsilon)|s| \quad \forall 0 < s < \delta_1
\]

and

\[
(B - \varepsilon)|s|^{\beta} < f(s) < (B + \varepsilon)|s|^{\beta} \quad \forall s > \delta_2,
\]

so from (8) and (9) and since (9) and since (10) we

\[
-D_1 + (A - \varepsilon)|s| + (B - \varepsilon)|s|^{\beta} < f(s) < D_1 + (A + \varepsilon)|s| + (B + \varepsilon)|s|^{\beta}
\]

for all \(s \in (0, \infty)\). This yields

\[
F(s) \leq D_2 + \frac{A + \varepsilon}{2} s^2 + \frac{B + \varepsilon}{\beta} |s|^{\beta+1} \quad \forall s \in (0, \infty)
\]

and

\[
F(s) \geq -D_2 + \frac{A - \varepsilon}{2} s^2 + \frac{B - \varepsilon}{\beta} |s|^{\beta+1} \quad \forall s \in (0, \infty),
\]

where \(D_2 = D_1(\delta_2 - \delta_1)\). Furthermore, from Lemma 1.1, \((H_0)-(H_1)\) and (10) we obtain

\[
q(x)|f(u_n(x))| \leq (A + \varepsilon)q(x)|u_n(x)| + (B + \varepsilon)q(x)|u_n(x)|^{\beta} + D_1q(x)
\]

\[
\leq (A + \varepsilon) \sup_{x \in [0, \infty)} \frac{|(pu_n)(x)|^{\beta} q(x)}{p(x)} + (B + \varepsilon)\|u_n\|_{\infty,p}^{\beta} \frac{q(x)}{p^{\beta}(x)} + D_1q(x)
\]

\[
= (A + \varepsilon)\|u_n\|_{\infty,p}^{\beta} \frac{q(x)}{p^{\beta}(x)} + (B + \varepsilon)(\sqrt{2MR})^{\beta} \frac{q(x)}{p^{\beta}(x)} + D_1q(x) \in L^1(0, \infty)
\]

and

\[
h(x)|u_n(x)|^{\beta-2}|u_n(x)| \leq h(x)|u_n(x)|^{\beta-1} = p^{\beta-1}(x)|u_n(x)|^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)}
\]

\[
\leq \sup_{x \in [0, \infty)} \frac{|(pu_n)(x)|^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)}}{\|u_n\|_{\infty,p}^{\beta-1}} = \|u_n\|_{\infty,p}^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)}
\]

\[
\leq (\sqrt{2MR})^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)} \in L^1(0, \infty).
\]

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Then from the Lebesgue dominated convergence theorem we obtain

\[
\lim_{n \to \infty} \int_0^\infty q(x)f(u_n(x)) \, dx = \int_0^\infty q(x)f(u(x)) \, dx
\]

and also

\[
\lim_{n \to \infty} \int_0^\infty h(x)|u_n|^{\beta-2}(x)u_n(x) \, dx = \int_0^\infty h(x)|u|^{\beta-2}(x)u(x) \, dx.
\]

Thus we have

\[
\langle J'_\lambda(u_n) - J'_\lambda(u), v \rangle = \int_0^\infty u'_n v' \, dx + \int_0^\infty u_n v \, dx - \lambda \int_0^\infty h(x)|u_n|^{\beta-2}u_n v \, dx
\]

\[
- \int_0^\infty q(x)f(u_n)v \, dx - \int_0^\infty u'v' \, dx - \int_0^\infty uv \, dx
\]

\[
+ \lambda \int_0^\infty h(x)|u|^{\beta-2}uv \, dx + \int_0^\infty q(x)f(u)v \, dx
\]

\[
= \int_0^\infty (u'_n - u')v' \, dx + \int_0^\infty (u_n - u)v \, dx
\]

\[
- \lambda \int_0^\infty h(x)(|u_n|^{\beta-2}u_n - |u|^{\beta-2}u)v \, dx
\]

\[
- \int_0^\infty q(x)(f(u_n) - f(u))v \, dx,
\]

and from (13), (14) and the continuity of \( f \), passing to the limit in \( \langle J'_\lambda(u_n) - J'_\lambda(u), v \rangle \) when \( n \to \infty \), we obtain that \( J'_\lambda(u_n) \to J'_\lambda(u) \) as \( n \to \infty \).

**Definition 2.1.** We say that \( u \in H_0^1(0, \infty) \) is a weak solution of problem (1) if for any \( v \in H_0^1(0, \infty) \) we have

\[
\langle J'_\lambda(u), v \rangle = \int_0^\infty u'v' \, dx + \int_0^\infty uv \, dx - \lambda \int_0^\infty h(x)|u|^{\beta-2}uv \, dx
\]

\[
- \int_0^\infty q(x)f(u)v \, dx = 0.
\]

**Remark 2.1.** Since the nonlinear term \( f \) is continuous, then a weak solution of problem (1) is a classical solution.

In our next two sections we will prove the main result of this paper.

**Theorem 2.1.** Suppose that (H\(_0\))–(H\(_4\)) hold. Then there exists \( \xi > 0 \) such that for \( 0 < \lambda < \xi \), problem (1) has at least two positive solutions.
2.1. Existence of a first solution.

**Lemma 2.1.** Suppose that the hypotheses (H\(_0\))–(H\(_4\)) hold. Then there exists \(\xi_1 > 0\) such that for \(0 < \lambda \leq \xi_1\), the functional \(J_\lambda\) satisfies the geometric conditions (a) and (b) in Lemma 1.4, i.e.

(a) there are \(\varrho, \alpha > 0\) such that \(J_\lambda(u) \geq \alpha\) when \(\|u\| = \varrho\),

(b) there is \(e \in H_0^1(0, \infty), \|e\| > \varrho\) such that \(J_\lambda(e) < 0\).

**Proof.** (a) From (H\(_0\))–(H\(_3\)), (11) and using Corollary 1.1, we have

\[
J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^{\beta}(x) \, dx - \int_0^\infty q(x) F(u) \, dx
\]

\[
\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^{\beta}(x) \, dx - D_2 \int_0^\infty q(x) \, dx
\]

\[
- A + \frac{\varepsilon}{2} \int_0^\infty q(x)|u|^2 \, dx - \frac{B + \varepsilon}{\beta + 1} \int_0^\infty q(x)|u|^{\beta+1} \, dx
\]

\[
\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} M_{\beta,h} \|u\|^{\beta} - \frac{A + \varepsilon}{2} M_{\beta,q}^2 \|u\|^2
\]

\[
- \frac{B + \varepsilon}{\beta + 1} M_{\beta+1,h} \|u\|^{\beta+1} - D_2 \|q\|_{L^1}
\]

\[
\geq \left( \frac{1}{2} - \frac{A + \varepsilon}{2} M_{\beta,q}^2 \right) \|u\|^2 - \frac{\lambda}{\beta} M_{\beta,h} \|u\|^{\beta}
\]

\[
- \frac{B + \varepsilon}{\beta + 1} M_{\beta+1,h} \|u\|^{\beta+1} - D_2 \|q\|_{L^1}
\]

\[
\geq \|u\|^2 \left( \frac{1}{2} (1 - (A + \varepsilon) M_{\beta,q}^2) - \frac{\lambda}{\beta} M_{\beta,h} \|u\|^{\beta-1} - \frac{B + \varepsilon}{\beta + 1} M_{\beta+1,h} \|u\|^{\beta} \right)
\]

\[
- D_2 \|q\|_{L^1}
\]

\[
\geq \|u\|^2 \left( \frac{1}{2} (1 - (A + \varepsilon) M_{\beta,q}^2) - \lambda K_1 \|u\|^{\beta-1} - K_2 \|u\|^{\beta} \right) - K_3,
\]

where \(K_1 = \beta^{-1} M_{\beta,h} \), \(K_2 = ((B + \varepsilon)/(\beta + 1)) M_{\beta+1,h}^{\beta+1} \), and \(K_3 = D_2 \|q\|_{L^1}\); here \(\varepsilon\) and \(D_2\) are given in the proof of Proposition 2.1. Let

\[
g(t) = \lambda K_1 t^{\beta-2} + K_2 t^{\beta-1} \quad \text{for} \ t \geq 0.
\]

Clearly,

\[
g'(t) = \lambda K_1 (\beta - 2) t^{\beta-3} + K_2 (\beta - 1) t^{\beta-2} \quad \text{for} \ t \geq 0.
\]

From \(g'(t_0) = 0\) we have

\[
t_0 = \frac{\lambda K_1 (2 - \beta)}{K_2 (\beta - 1)}.
\]
Then
\[ g(t_0) = \frac{2\lambda^{\beta-1}K_1^{\beta-1}}{(\beta-1)K_2^{\beta-2}}. \]

Thus, there exists
\[ 0 < \xi_1 < \left(\frac{(\beta-1)K_2}{4K_1}(1 - (A + \varepsilon)M_{2,q}^2)\right)^{1/\beta-1} \]
such that
\[ g(t_0) < \frac{1}{2}(1 - (A + \varepsilon)M_{2,q}^2) \quad \forall 0 < \lambda \leq \xi_1. \]

Consequently, taking \( \rho = t_0 \) and choosing \( \lambda \in (0, \xi_1) \) such that
\[ m_0 = \rho^2\left(\frac{1}{2}(1 - (A + \varepsilon)M_{2,q}^2) - \lambda K_1 \rho^{\beta-2} - K_2 \rho^{\beta-1}\right) > K_3, \]
from (16) we have
\[ J(\lambda) \geq \alpha > 0 \text{ when } \|u\| = \rho, \]
where \( \alpha = m_0 - K_3. \) Thus (a) is proved.

(b) For \( t > 0 \) large enough, from (12) and Lemma 1.3 we have
\[
J(\lambda(t\overline{\psi}_{\beta+1,q})) \\
= \frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^{\beta} \int_0^\infty h(x)|\overline{\psi}_{\beta+1,q}|^\beta \, dx - \int_0^\infty q(x)F(t\overline{\psi}_{\beta+1,q}) \, dx \\
\leq \frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^{\beta} \int_0^\infty h(x)|\overline{\psi}_{\beta+1,q}|^\beta \, dx - \frac{A - \varepsilon}{2}t^2 \int_0^\infty q(x)|\overline{\psi}_{\beta+1,q}|^2 \, dx \\
- \frac{B - \varepsilon}{\beta + 1}\int_0^\infty q(x)|\overline{\psi}_{\beta+1,q}|^{\beta+1} \, dx + D_2 \int_0^\infty q(x) \, dx \\
\leq \frac{1}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta}t^{\beta}\|\overline{\psi}_{\beta+1,q}\|_{\beta,h} - \frac{A - \varepsilon}{2}t^2\|\overline{\psi}_{\beta+1,q}\|^2_{2,q} \\
- \frac{B - \varepsilon}{\beta + 1}\|\overline{\psi}_{\beta+1,q}\|_{\beta+1,q} + D_2\|q\|_{L^1} \\
\leq \frac{1}{2}(\|\overline{\psi}_{\beta+1,q}\|^2 - (A - \varepsilon)\|\overline{\psi}_{\beta+1,q}\|^2_{2,q})t^2 - \frac{\lambda}{\beta}\|\overline{\psi}_{\beta+1,q}\|_{\beta,h}^{\beta}t^{\beta} \\
- \frac{B - \varepsilon}{\beta + 1}\|\overline{\psi}_{\beta+1,q}\|_{\beta+1,q}^{\beta+1} + K_3.
\]

Therefore \( J(\lambda(t\overline{\psi}_{\beta+1,q})) \to -\infty \) as \( t \to \infty. \) Choose \( t_1 > 0 \) large enough and \( e = t_1\overline{\psi}_{\beta+1,q}. \) Hence, we conclude that
\[ J(\lambda(e)) < 0 \text{ when } \|e\| > \rho. \]

Thus (b) is proved. \( \square \)

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From a version of the Mountain Pass Theorem without the Palais-Smale condition (see [7]), there exists a (P.S)\textsubscript{c} sequence \((u_n) \subset H^1_0(0, \infty)\) for \(J_\lambda\) which satisfies (3), i.e.

\[
J_\lambda(u_n) \to c \quad \text{and} \quad J'_\lambda(u_n) \to 0,
\]

where

\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t))
\]

with

\[
\Gamma = \{ \gamma \in C([0,1], H^1_0(0, \infty)) : \gamma(0) = 0, \, \gamma(1) = e \},
\]

where \(e\) is given in Lemma 2.1.

R e m a r k 2.2. Since the sequence \((u_n^+)\) also satisfies (3) (see [1], Lemma 1), we assume, without of loss generality, that \(u_n \geq 0\) for all \(n \in \mathbb{N}\).

**Lemma 2.2.** Suppose that the hypotheses \((H_0)-(H_4)\) hold. Then the mountain level \(c\) satisfies the following inequality:

\[
c < \left( \frac{\lambda^{\beta+1}}{B - \varepsilon} \right)^{2/(\beta-1)} \left( \frac{1}{2} \frac{1}{\mu} \right) + K_3;
\]

here \(K_3\) is given in the proof of Lemma 2.1.

**Proof.** From the proof of Lemma 2.1 we can consider \(\gamma(t) = tt_1 \overline{\psi}_{\beta+1, q}\), where \(t_1 > 0\) is sufficiently large such that \(e = t_1 \overline{\psi}_{\beta+1, q}\). Thus, from the definition of \(c\),

\[
c \leq \max_{t \geq 0} J_\lambda(t \overline{\psi}_{\beta+1, q}),
\]

that is,

\[
c \leq \max_{t \geq 0} \left\{ \frac{1}{2} t^2 \| \overline{\psi}_{\beta+1, q} \|^2 - \frac{\lambda}{\beta} t^\beta \| \overline{\psi}_{\beta+1, q} \|_{\beta,h}^\beta - \int_0^\infty q(x) F(t \overline{\psi}_{\beta+1, q}) \, dx \right\}.
\]

From (12),

\[
c \leq \max_{t \geq 0} \left\{ \frac{1}{2} t^2 \| \overline{\psi}_{\beta+1, q} \|^2 - \frac{\lambda}{\beta} t^\beta \| \overline{\psi}_{\beta+1, q} \|_{\beta,h}^\beta - \frac{A - \varepsilon}{2} t^2 \| \overline{\psi}_{\beta+1, q} \|_{2,q}^2
\]

\[
- \frac{B - \varepsilon}{\beta + 1} t^{\beta+1} \| \overline{\psi}_{\beta+1, q} \|_{\beta+1,q}^{\beta+1} + K_3 \right\}
\]

\[
\leq \max_{t \geq 0} \left\{ \frac{1}{2} t^2 \| \overline{\psi}_{\beta+1, q} \|^2 - \frac{B - \varepsilon}{\beta + 1} t^{\beta+1} \| \overline{\psi}_{\beta+1, q} \|_{\beta+1,q}^{\beta+1} \right\} + K_3,
\]

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and then
\[
\frac{c}{\|\psi_{\beta+1,q}\|_{\beta+1,q}^2} \leq \max_{t \geq 0} \left\{ \frac{\lambda_{\beta+1,q}^2 t^2}{2} - \frac{B - \varepsilon}{\beta + 1} \|\psi_{\beta+1,q}\|_{\beta+1,q}^{-1} t^{\beta+1} \right\} + \frac{K_3}{\|\psi_{\beta+1,q}\|_{\beta+1,q}^2}.
\]

Let
\[
Z(t) = \frac{\lambda_{\beta+1,q}^2 t^2}{2} - \frac{B - \varepsilon}{\beta + 1} \|\psi_{\beta+1,q}\|_{\beta+1,q}^{-1} t^{\beta+1}.
\]
Clearly,
\[
Z'(t) = \frac{\lambda_{\beta+1,q}^2}{B - \varepsilon} - (B - \varepsilon) \|\psi_{\beta+1,q}\|_{\beta+1,q}^{-1} t^{\beta+1}.
\]
Since the function $Z$ attains its maximum at
\[
t = \left( \frac{\lambda_{\beta+1,q}^2}{(B - \varepsilon) \|\psi_{\beta+1,q}\|_{\beta+1,q}^{-1}} \right)^{1/(\beta-1)},
\]
\[
c < \left( \frac{\lambda_{\beta+1,q}^2}{B - \varepsilon} \right)^{2/(\beta-1)} \left( \frac{1}{2} - \frac{1}{\beta + 1} \right) + K_3,
\]
and therefore we have
\[
c < \left( \frac{\lambda_{\beta+1,q}^2}{B - \varepsilon} \right)^{2/(\beta-1)} \left( \frac{1}{2} - \frac{1}{\mu} \right) + K_3.
\]

\[\square\]

**Lemma 2.3.** There exists $\xi_2 > 0$ such that for $0 < \lambda < \xi_2$, the Palais-Smale sequence $(u_n)$ associated with the functional $J_\lambda$ satisfies
\[
\limsup_{n \to \infty} \|u_n\|^2 < 2 \left( \frac{\lambda_{\beta+1,q}}{B - \varepsilon} \right)^{2/(\beta-1)} \left( \frac{2\beta/2 - \beta}{\mu} \right) + \frac{4K_3\mu}{\mu - 2}.
\]

**Proof.** First, observe that $(u_n)$ is bounded in $H^1_0(0, \infty)$. In fact, from (3)
\[
J_\lambda(u_n) \to c \quad \text{and} \quad \langle J'_\lambda(u_n), u_n \rangle \to 0 \quad \text{as} \quad n \to \infty.
\]
Notice that from (7) we have
\[
\int_0^\infty q(x)f(u_n)u_n \, dx = \|u_n\|^2 - \lambda\|u_n\|_{\beta,h}^\beta - \langle J'_\lambda(u_n), u_n \rangle.
\]

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Using Corollary 1.1 and (H₄), it follows from (3) that

\begin{align*}
(18) \quad c + \varepsilon > J_{\lambda}(u_n) &= \frac{1}{2} \|u_n\|^2 - \frac{\lambda}{\beta} \|u_n\|_{\beta,h}^\beta - \int_0^\infty q(x) F(u_n) \, dx \\
&\geq \frac{1}{2} \|u_n\|^2 - \frac{\lambda}{\beta} \|u_n\|_{\beta,h}^\beta - \frac{1}{\mu_n} \int_0^\infty q(x) f(u_n) u_n \, dx \\
&\geq \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 - \lambda \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|_{\beta,h}^\beta + \frac{1}{\mu_n} (J'_{\lambda}(u_n), u_n) \\
&\geq \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 - \lambda M_{\beta,h}^\beta \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^\beta + \frac{1}{\mu_n} (J'_{\lambda}(u_n), u_n) \\
&\geq \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 - \lambda M_{\beta,h}^\beta \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^\beta - \frac{1}{\mu_n} \|J'_{\lambda}(u_n)\| \|u_n\|.
\end{align*}

Since \( J_{\lambda}(u_n) \to 0 \), there exists \( N_0 \in \mathbb{N} \) large enough such that

\begin{align*}
(19) \quad c + \varepsilon > \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 - \lambda M_{\beta,h}^\beta \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^\beta - o_n(1) \|u_n\| \quad \forall \, n > N_0.
\end{align*}

This implies that \( (u_n) \subset H_{\lambda}^1(0, \infty) \) is bounded.

Now we can write (19) as

\begin{align*}
(20) \quad \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 \leq \lambda M_{\beta,h}^\beta \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^\beta + o_n(1) \|u_n\| + c + \varepsilon.
\end{align*}

Using Young’s inequality in (20), we get

\begin{align*}
\left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 &\leq \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \left( \frac{2 - \beta}{2} M_{\beta,h}^{2\beta/(2-\beta)} + \frac{\beta}{2} \lambda^{2/\beta} \|u_n\|^2 \right) + o_n(1) \|u_n\| + c + \varepsilon \\
&\leq \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \frac{2 - \beta}{2} M_{\beta,h}^{2\beta/(2-\beta)} + \frac{\beta}{2} \lambda^{2/\beta} \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 + o_n(1) \|u_n\| + c + \varepsilon,
\end{align*}

and then we have

\begin{align*}
\left( \frac{1}{2} - \frac{1}{\mu_n} \right) - \frac{\beta}{2} \lambda^{2/\beta} \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \|u_n\|^2 \leq \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \frac{2 - \beta}{2} M_{\beta,h}^{2\beta/(2-\beta)} + o_n(1) \|u_n\| + c + \varepsilon.
\end{align*}

Choosing

\begin{align*}
0 < \lambda \leq \xi_2 = \left( \frac{\frac{1}{2} - \frac{1}{\mu_n}}{\beta(\beta^{-1} - \mu^{-1})} \right)^{\beta/2},
\end{align*}

then using Lemma 2.2, we conclude that

\begin{align*}
\|u_n\|^2 &\leq \left( \frac{1}{2} - \frac{1}{\mu_n} \right)^{-1} \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \left( \frac{\lambda^{\beta+1}}{\lambda_{\beta+1,q}} \right)^{2/(\beta-1)} \\
&\quad + K_3 + \left( \frac{1}{2} - \frac{1}{\mu_n} \right) \frac{2 - \beta}{2} M_{\beta,h}^{2\beta/(2-\beta)} + o_n(1) \|u_n\| + \varepsilon,
\end{align*}

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Thus

\[
\limsup_{n \to \infty} \|u_n\|^2 \leq \left(\frac{1}{2} - \frac{1}{\mu} \right)^{-1} \left(\frac{1}{2} - \frac{1}{\mu} \right)^{2/(\beta-1)} \left(\frac{L^{\beta+1,q}}{B - \varepsilon} \right)^{2/(\beta-1)} + K_3 + \left(\frac{1}{\beta} - \frac{1}{\mu} \right) M^{\beta/(2-\beta)} + 4K_3 \mu^2 \mu - 2.
\]

□

Since \((u_n)\) satisfying (3) is bounded in \(H^0_0(0, \infty)\) (see Lemma 2.3), there exists \(u_1 \in H^1_0(0, \infty)\) such that for a subsequence we have

\[
\begin{align*}
(21) & \quad u_n \rightharpoonup u_1 \quad \text{in} \quad H^0_0(0, \infty), \\
(22) & \quad u_n \to u_1 \quad \text{in} \quad L^r_g(0, \infty)
\end{align*}
\]

for all \(r \in \{\beta, 2, \beta + 1\}\) and all \(g \in \{h, q\}\) and

\[
(23) \quad u_n(x) \to u_1(x) \quad \text{a.e. in} \quad (0, \infty).
\]

In the next lemma we obtain some convergences results involving the sequence \((u_n)\) and its weak limit \(u_1\).

**Lemma 2.4.** The following limits are satisfied:

(c) \(\int_0^\infty q(x) |f(u_n) - f(u_1)| |u_n - u_1| \, dx = o_n(1)\),

(d) \(\int_0^\infty h(x) |u_n|^{\beta-2} u_n - |u_1|^{\beta-2} u_1| |u_n - u_1| \, dx = o_n(1)\).

**Proof.** (c) From (10) and using Corollary 1.1 and Lemmas 2.2 and 2.3, we obtain

\[
\begin{align*}
\int_0^\infty q(x) |f(u_n) - f(u_1)| |u_n - u_1| \, dx & \leq \int_0^\infty q(x) |f(u_n)||u_n - u_1| \, dx + \int_0^\infty q(x) |f(u_1)||u_n - u_1| \, dx \\
& \leq 2D_1 \int_0^\infty q(x) |u_n - u_1| \, dx \\
& \quad + (A + \varepsilon) \int_0^\infty q(x) |u_n| |u_n - u_1| \, dx + (A + \varepsilon) \int_0^\infty q(x) |u_1| |u_n - u_1| \, dx \\
& \quad + (B + \varepsilon) \int_0^\infty q(x) |u_n|^{\beta} |u_n - u_1| \, dx + (B + \varepsilon) \int_0^\infty q(x) |u_1|^{\beta} |u_n - u_1| \, dx
\end{align*}
\]
and using the Cauchy-Schwarz inequality, we have

\[
\int_0^\infty q(x)|f(u_n) - f(u_1)||u_n - u_1| \, dx
\]

\[
\leq 2D_1 \left( \int_0^\infty q(x) \, dx \right)^{1/2} \left( \int_0^\infty q(x)|u_n - u_1|^2 \, dx \right)^{1/2}
\]

\[
+ (A + \varepsilon) \left( \int_0^\infty q(x)|u_n|^2 \, dx \right)^{1/2} \left( \int_0^\infty q(x)|u_n - u_1|^2 \, dx \right)^{1/2}
\]

\[
+ (A + \varepsilon) \left( \int_0^\infty q(x)|u_1|^2 \, dx \right)^{1/2} \left( \int_0^\infty q(x)|u_n - u_1|^2 \, dx \right)^{1/2}
\]

\[
+ (B + \varepsilon) \left( \int_0^\infty q(x)|u_n|^{\beta} \, dx \right)^{(\beta-1)/\beta} \left( \int_0^\infty q(x)|u_n - u_1|^{\beta} \, dx \right)^{1/\beta}
\]

Thus,

\[
\int_0^\infty q(x)|f(u_n) - f(u_1)||u_n - u_1| \, dx
\]

\[
\leq 2D_1\|q\|_{L^1}\|u_n - u_1\|_{2,q} + (A + \varepsilon)\|u_n\|_{2,q}\|u_n - u_1\|_{2,q}
\]

\[
+ (A + \varepsilon)\|u_1\|_{2,q}\|u_n - u_1\|_{2,q} + (B + \varepsilon)\|u_n\|_{\beta,q}^{\beta-1}\|u_n - u_1\|_{\beta,q}
\]

\[
+ (B + \varepsilon)\|u_1\|_{\beta,q}^{\beta-1}\|u_n - u_1\|_{\beta,q}
\]

\[
\leq C_{1,\varepsilon}\|u_n - u_1\|_{2,q} + C_{2,\varepsilon}\|u_n - u_1\|_{\beta,q},
\]

where

\[
C_{1,\varepsilon} = 2(A + \varepsilon)M_{2,q}\overline{C}^{1/2} + 2D_1\|q\|_{L^1}, \quad C_{2,\varepsilon} = 2(B + \varepsilon)(M_{2,h}\overline{C}^{1/2})^{\beta-1},
\]

\[
\overline{C} = 2\left(\frac{\lambda_{\beta+1,q}}{B - \varepsilon}\right)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/2-\beta} + \frac{4K_3\mu}{\mu - 2}.
\]

Then according to (22) we have

\[
\int_0^\infty (q(x)|f(u_n) - f(u_1)||u_n - u_1|) \, dx = o_n(1).
\]

(d) From Corollary 1.1 and Lemmas 2.2 and 2.3, we have

\[
\int_0^\infty h(x)|u_n|^{\beta-2}u_n - |u_1|^{\beta-2}u_1| |u_n - u_1| \, dx
\]

\[
\leq \int_0^\infty h(x)|u_n|^{\beta-1}|u_n - u_1| \, dx + \int_0^\infty h(x)|u_1|^{\beta-1}|u_n - u_1| \, dx
\]

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Therefore, from Lemma 2.4 above and taking into account that
\[ J_\lambda(u_1) \]
Proposition 2.1), we have
\[
\exists \xi > 0 \text{ such that for } 0 < \lambda < \xi, \text{ problem } (1) \text{ has a positive solution } u_1 \text{ satisfying } J_\lambda(u_1) > 0.
\]

Proposition 2.2. Suppose that \( f \) is a function satisfying (H0)–(H1). Then there exists a constant \( \overline{\xi} > 0 \) such that for \( 0 < \lambda < \overline{\xi} \), problem (1) has a positive solution \( u_1 \) satisfying \( J_\lambda(u_1) > 0 \).

Proof. Let \( u_1 \) be the weak limit of the sequence \( (u_n) \) that satisfies (3). Consider \( \overline{\xi} = \min\{\xi_1, \xi_2\} \), where \( \xi_1 \) and \( \xi_2 \) are given in Lemmas 2.1 and 2.3, respectively. We will prove that \( u_n \to u_1 \) in \( H_0^1(0, \infty) \).

From (15) we have
\[
\langle J'_\lambda(u_n) - J'_\lambda(u_1), u_n - u_1 \rangle \\
= \int_0^\infty \langle u'_n - u'_1, u'_n - u'_1 \rangle \, dx + \int_0^\infty (u_n - u_1)(u_n - u_1) \, dx \\
- \lambda \int_0^\infty h(x)(|u_n|^{\beta-2}u_n - |u_1|^{\beta-2}u_1)(u_n - u_1) \, dx \\
- \int_0^\infty q(x)(f(u_n) - f(u_1))(u_n - u_1) \, dx.
\]

Thus,
\[
\|u_n - u_1\|^2 \leq |\langle J'_\lambda(u_n) - J'_\lambda(u_1), u_n - u_1 \rangle| \\
+ \lambda \int_0^\infty h(x)(|u_n|^{\beta-2}u_n - |u_1|^{\beta-2}u_1)|u_n - u_1| \, dx \\
+ \int_0^\infty q(x)|f(u_n) - f(u_1)||u_n - u_1| \, dx.
\]

Therefore, from Lemma 2.4 above and taking into account that \( J'_\lambda \) is continuous (see Proposition 2.1), we have
\[
\|u_n - u_1\|^2 = \int_0^\infty |u'_n - u'_1|^2 \, dx + \int_0^\infty |u_n - u_1|^2 \, dx = o_n(1).
\]
Consequently,
\[ \lim_{n \to \infty} \left( \int_0^\infty |u'_n - u'_1|^2 \, dx + \int_0^\infty |u_n - u_1|^2 \, dx \right) = 0. \]

That is, \( u_n \to u_1 \) as \( n \to \infty \) in \( H_0^1(0, \infty) \), i.e. \( (u_n) \) satisfies the Palais-Smale condition. Now by applying the Mountain Pass Theorem, we obtain
\[ J'_\lambda(u_1) = 0 \quad \text{and} \quad J_\lambda(u_1) = c > 0. \]

\[ \square \]

2.2. Existence of a second solution. Now we apply the Ekeland variational principle to prove the existence of a weak solution \( u_2 \) which is different from the solution \( u_1 \).

**Lemma 2.5.** Suppose that \((H_0)-(H_4)\) hold. Then there exists a constant \( \xi_3 > 0 \) such that for \( 0 < \lambda < \xi_3 \), the functional \( J_\lambda \) satisfies \((P.S)_d\) condition with \( d < 0 \).

**Proof.** Fix \( d < 0 \) and suppose that \( (u_n) \subset H_0^1(0, \infty) \) satisfies
\[ J_\lambda(u_n) \to d \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty. \]

We need to show that \((u_n)\) admits a subsequence converging strongly in \( H_0^1(0, \infty) \).

Proceeding as in (20) we get
\[ \left( \frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|^2 \leq \lambda M_{\beta,h}^\beta \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \|u_n\|^\beta + o_n(1)\|u_n\| + d + \varepsilon. \]

Thus, for a subsequence we have
\[ \left( \frac{1}{2} - \frac{1}{\mu} \right) \limsup_{n \to \infty} \|u_n\|^{2-\beta} - \lambda M_{\beta,h}^\beta \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \limsup_{n \to \infty} \|u_n\|^\beta \leq d < 0. \]

Hence,
\[ \limsup_{n \to \infty} \|u_n\|^2 \leq \left( \frac{\lambda M_{\beta,h}^\beta (\beta^{-1} - \mu^{-1})}{\left( \frac{1}{2} - \mu^{-1} \right)^{2/(2-\beta)}} \right)^{2/(2-\beta)} < (\lambda M_{\beta,h}^\beta)^{2/(2-\beta)}. \]

Choosing
\[ \xi_3 = M_{\beta,h}^{-\beta} \left( 2 \left( \frac{\lambda_{\beta+1,h}^{\beta+1} + M_{\beta,h}^2}{B - \varepsilon} \right)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/(2-\beta)} + \frac{4K_3\mu}{\mu - 2} \right)^{(2-\beta)/2}, \]

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we have that for $\lambda < \xi_3$,

$$
\limsup_{n \to \infty} \|u_n\|^2 < 2 \left( \frac{\lambda^\beta + 1}{B - \varepsilon} \right)^{2/(\beta - 1)} + M^{2\beta/(2 - \beta)} + \frac{4K\mu}{\mu - 2}.
$$

From (26) we have that $(u_n)$ is bounded in $H^1_0(0, \infty)$ and there exists $u \in H^1_0(0, \infty)$ such that $u_n \to u$ in $H^1_0(0, \infty)$. Now, we can repeat the same arguments employed in the proofs of Lemma 2.3 and Proposition 2.2 to conclude that $u_n \to u$ in $H^1_0(0, \infty)$.

\textbf{Proposition 2.3.} Suppose that $f$ is a function satisfying (H$_0$)–(H$_4$). Then there exists a constant $\hat{\xi} > 0$ such that for $0 < \lambda < \hat{\xi}$, problem (1) has a positive solution $u_2$ satisfying $J_\lambda(u_2) < 0$.

\textbf{Proof.} Consider the complete metric space

$$
\overline{B}_\rho(0) := \{ u \in H^1_0(0, \infty) : \|u\| \leq \rho \}
$$

with a metric given by $d(u, w) = \|u - w\|$. The functional $J_\lambda$ is bounded from below on $\overline{B}_\rho(0)$ for $\lambda < \xi_1$ (see Lemma 1.4). Note that

$$
\forall t < \min \left\{ \frac{\delta_1}{\|\psi_{\beta,h}\|}, \frac{1}{\|\psi_{\beta,h}\|^{\beta,h}} \left( \frac{2\lambda}{\beta \lambda_{\beta,h}^2} \right)^{1/2 - \beta} \right\}
$$

(t near 0) using (H$_3$) in (8), we get

$$
J_\lambda(t\bar{\psi}_{\beta,h}) = \frac{1}{2} t^2 \|\bar{\psi}_{\beta,h}\|^2 - \frac{\lambda}{\beta} t^\beta \|\bar{\psi}_{\beta,h}\|^{\beta,h} - \int_0^\infty q(x) F(t\bar{\psi}_{\beta,h}) \, dx
\leq \frac{1}{2} t^2 \|\bar{\psi}_{\beta,h}\|^2 - \frac{\lambda}{\beta} t^\beta \|\bar{\psi}_{\beta,h}\|^{\beta,h} - \frac{A - \varepsilon}{2} t^2 \|\bar{\psi}_{\beta,h}\|_{2,q}^2
= \frac{1}{2} t^2 \|\bar{\psi}_{\beta,h}\|^2 \left( 1 - \frac{2\lambda}{\beta \lambda_{\beta,h}^2} \|\bar{\psi}_{\beta,h}\|^{\beta,h} \right) - \frac{A - \varepsilon}{2} t^2 \|\bar{\psi}_{\beta,h}\|_{2,q}^2 < 0,
$$

by (17). Then, in view of (27), we see that

$$
\inf_{u \in \overline{B}_\rho(0)} J(u) < 0 < \inf_{u \in \partial \overline{B}_\rho(0)} J(u).
$$

Consequently, by applying Ekeland’s variational principle in $\overline{B}_\rho(0)$, there is a minimizing sequence $(u_n)_{n \geq 1} \subset \overline{B}_\rho(0)$ such that

$$
J_\lambda(u_n) \to d := \inf\{ J_\lambda(u) : u \in \overline{B}_\rho(0) \},
$$

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i.e.

\[ J(u_n) \leq \inf_{u \in \overline{B}_0(0)} J(u) + \frac{1}{n} \quad \forall n \geq 1, \]

and for every \( w \in \overline{B}_0(0) \) with \( w \neq u_n \),

\[ J_\lambda(w) - J_\lambda(u_n) + \frac{1}{n} \|u_n - w\| > 0. \tag{30} \]

Let \( v \in H^1_0(0, \infty) \). We consider the sequence \( w_n := u_n + tv \subset \overline{B}_0(0) \), \( t \) near 0 (small enough), and for all \( n \geq 1 \). From (30) we obtain

\[ \frac{1}{t}(J_\lambda(u_n + tv) - J_\lambda(u_n)) > -\frac{1}{n} \|v\|. \]

Thus, \( \langle J'_\lambda(u_n), v \rangle \geq -n^{-1}\|v\| \) and similarly, \( \langle J'_\lambda(u_n), (-v) \rangle \geq -n^{-1}\|v\| \). Therefore

\[ |\langle J'_\lambda(u_n), v \rangle| < \frac{1}{n} \|v\| \quad \forall v \in H^1_0(0, \infty). \]

Consequently,

\[ \|J'_\lambda(u_n)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{31} \]

Fix \( \widehat{\xi} := \min\{\xi_1, \xi_3\} \), where \( \xi_1 \) and \( \xi_3 \) are given by Lemmas 2.1 and 2.5, respectively. Then from (29) and (31) it follows that \( (u_n)_{n \geq 1} \) is a (P.S)\(_d\) sequence for the functional \( J_\lambda \) for all \( 0 < \lambda < \widehat{\xi} \).

Using Lemma 2.5 and Propositions 2.2, we obtain a subsequence, still denoted by \( (u_n)_{n \geq 1} \), which converges strongly to a function \( u_2 \in H^1_0(0, \infty) \). In this case

\[ J'_\lambda(u_2) = 0. \]

Now we will check \( J_\lambda(u_2) < 0 \) to complete the proof. Note that using (H\(_4\)) and (7) we obtain

\[ d + o_n(1) = J_\lambda(u_n) - \frac{1}{\mu}J'_\lambda(u_n)u_n = \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - \lambda\left(\frac{1}{\beta} - \frac{1}{\mu}\right)\int_0^\infty h(x)\|u_n\|^\beta \]

\[ - \int_0^\infty \left(F(u_n) - \frac{1}{\mu}f(u_n)u_n\right) + o_n(1). \]

From Fatou’s lemma (see [3], Lemma 4.1) we conclude that

\[ d = \liminf_{n \to \infty} \left(J_\lambda(u_n) - \frac{1}{\mu}J'_\lambda(u_n)u_n\right) \geq J_\lambda(u_2) - \frac{1}{\mu}J'_\lambda(u_2)u_2. \]

Thus

\[ J_\lambda(u_2) = d < 0. \]

\[ \square \]
Remark 2.3. If $u$ is a nontrivial solution for problem (1), by Remark 2.2, $u \geq 0$. Furthermore, as a consequence of (28) and $J_\lambda(0) = 0$, we have $u > 0$ in $(0, \infty)$.

Proof of Theorem 2.1. We take $\xi := \min\{\xi, \hat{\xi}\}$ and then the proof of Theorem 2.1 follows directly from Propositions 2.2, 2.3 and Remark 2.3. □

3. Example

In this section we give an example to illustrate our results.

Example 3.1. Consider the problem

\begin{equation}
\begin{cases}
-\ddot{u} + u = \lambda h(x)|u|^\beta - 2u + q(x)f(u), & x \in [0, \infty), \\
u(0) = u(\infty) = 0,
\end{cases}
\end{equation}

where

\[ f(u) = \begin{cases}
\frac{1}{2M_{2,q}^2} |u| + (\lambda_{2,q}^2 + 1)|u|^{\beta} & \text{if } |u| \leq 1, \\
(\lambda_{2,q}^2 + 1)|u|^{\beta} + \frac{1}{2M_{2,q}^2} & \text{if } |u| \geq 1,
\end{cases} \]

$q(x) = \frac{1}{4}D_2^{-1}e^{-3x/2}$ and $h(x) = e^{-4x/3}$. Choose $p(x) = e^{-x/4}$ and we see that

\[ \frac{q}{p}(x) = \frac{1}{4D_2}e^{-5x/4}, \quad \frac{q}{p^2}(x) = \frac{1}{4D_2}e^{-x}, \quad \frac{h}{p^{\beta-1}}(x) = e^{(3\beta-19)x/12}, \]

\[ \frac{q}{p^{\beta}}(x) = \frac{1}{4D_2}e^{(\beta-6)x/4}, \quad \frac{h}{p^{\beta}}(x) = e^{x(\beta/4-4/3)} \quad \text{and} \quad \frac{q}{p^{\beta+1}}(x) = \frac{1}{4D_2}e^{(\beta-5)x/4} \]

are in $L^1[0, \infty)$ for all $\beta \in (1, 2)$. Note that $\lambda_{2,q} > M_{2,q}^{-1}$, and we also obtain that

\[ M_{2,q} = \frac{\sqrt{2}}{\sqrt{15D_2}}, \quad A := \lim_{u \to 0^+} \frac{f(u)}{|u|} = \frac{1}{2M_{2,q}^2} \quad \text{and} \quad B := \lim_{u \to \infty} \frac{f(u)}{|u|^{\beta}} = \lambda_{2,q}^2 + 1. \]

It is easy to see that conditions (H$_0$)–(H$_4$) hold. Thus from Theorem 2.1, (32) has at least two positive solutions for each $\lambda \in (0, \xi)$. 

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References


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