A NOTE ON THE SIZE RAMSEY NUMBERS FOR MATCHINGS VERSUS CYCLES

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Received December 30, 2018. Published online June 2, 2020.
Communicated by Riste Škrekovski

Abstract. For graphs G, F_1, F_2, we write $G \to (F_1, F_2)$ if for every red-blue colouring of the edge set of G we have a red copy of F_1 or a blue copy of F_2 in G. The size Ramsey number $\hat{r}(F_1, F_2)$ is the minimum number of edges of a graph G such that $G \to (F_1, F_2)$. Erdős and Faudree proved that for the cycle C_n of length n and for $t \geqslant 2$ matchings tK_2, the size Ramsey number $\hat{r}(tK_2, C_n) < n + (4t+3)\sqrt{n}$. We improve their upper bound for $t = 2$ and $t = 3$ by showing that $\hat{r}(2K_2, C_n) \leqslant n + 2\sqrt{3n} + 9$ for $n \geqslant 12$ and $\hat{r}(3K_2, C_n) < n + 6\sqrt{n} + 9$ for $n \geqslant 25$.

Keywords: size Ramsey number; matching; cycle

MSC 2020: 05C55, 05C35

1. Introduction

Ramsey theory studies problems which can be grouped under the common theme that every large system contains a highly organized subsystem. Ramsey-type theorems have roots in various branches of mathematics and the theory developed from them has influenced areas such as set theory, number theory, ergodic theory, geometry and theoretical computer science.

The size Ramsey number was introduced by Erdős et al. (see [3]) who investigated the size Ramsey number for various graphs. Size Ramsey numbers for all pairs of connected graphs having at most four vertices were found by Faudree and Sheehan (see [4]). Bounds on the size Ramsey number for trees were presented by Ke in [5], paths and stars were investigated by Lortz and Mengersen in [6]. Modifications of

The work of E. T. Baskoro was supported by the WCU-ITB Research Incentive in House Collaboration, Institut Teknologi Bandung, Ministry of Research, Technology and Higher Education, Indonesia.

DOI: 10.21136/MB.2020.0174-18

the size Ramsey number have been studied extensively, too (see [1] and [7] for results on the size multipartite Ramsey numbers).

We denote the edge set of a graph G by $E(G)$ and the number of edges in G by $|E(G)|$. For an edge set $E_1 \subseteq E(G)$, the edge-induced subgraph of G consists of the edges in E_1 and the vertices incident to edges in E_1.

A *cycle* of length n (an n-cycle) $C_n = v_1v_2\dots v_nv_1$ is a graph with n vertices v_1, $v_2, \dots, v_n$ and n edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$ and v_nv_1. Similarly, a *path* $v_1v_2\dots v_n$ of length $n-1$ contains $n-1$ edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$. For $t \geqslant 1$, the graph which consists of t *independent edges* (*matchings*) is denoted by tK_2 (it is a 1-regular graph having $2t$ vertices).

For simple graphs G, F_1, F_2, we write $G \to (F_1, F_2)$ if for each 2-*colouring* (say red and blue) of $E(G)$ we necessarily get a red copy of F_1 or a blue copy of F_2 in G. The *size Ramsey number* $\hat{r}(F_1, F_2)$ is the minimum number of edges in a graph G such that $G \to (F_1, F_2)$.

Erdős and Faudree (see [2]) mentioned that the difficulty in calculating $\hat{r}(tK_2, C_n)$ is surprising. They proved that for a fixed $t \geqslant 2$, there exist positive constants c_1, c_2, such that $n + c_1\sqrt{n} < \hat{r}(tK_2, C_n) < n + c_2\sqrt{n}$. Their upper bound depends on t and it has the form

$$\hat{r}(tK_2, C_n) < n + (4t+3)\sqrt{n}.$$

We considerably improve this bound for $t = 2$ and $t = 3$ by showing that $\hat{r}(2K_2, C_n) \leqslant n + 2\sqrt{3n} + 9$ for $n \geqslant 12$ and $\hat{r}(3K_2, C_n) < n + 6\sqrt{n} + 9$ for $n \geqslant 25$.

2. Results

Let us present upper bounds on the size Ramsey numbers $\hat{r}(2K_2, C_n)$ and $\hat{r}(3K_2, C_n)$.

Theorem 2.1. *Let $n \geqslant 12$. Then $\hat{r}(2K_2, C_n) \leqslant n + 2\sqrt{3n} + 9$.*

Proof. Let k be an integer where $k\varphi + 2 \leqslant n < (k+1)\varphi + 2$, and $\varphi = \left\lfloor \sqrt{\frac{1}{3}n} \right\rfloor$. Then we can write

$$n = k\varphi + 2 + p,$$

where $0 \leqslant p \leqslant \varphi - 1$. Let $t_1, t_2 \in \mathbb{Z}$, where $1 \leqslant t_1 + 1 < t_2 \leqslant k+2$ and $(t_1, t_2) \neq (0, k+2)$. Let G be a graph having $n' = (k+3)\varphi$ vertices $v_0, v_1, v_2, \dots, v_{n'-1}$ and

$$\begin{aligned} E(G) = \{&v_iv_{i+1}\colon\ i = 0, 1, 2, \dots, n'-1\} \\ &\cup \{v_jv_{j+2\varphi}\colon\ j = 0, \varphi, 2\varphi, \dots, (k+2)\varphi\} \\ &\cup \{v_{t_1\varphi}v_{(t_1+1)\varphi-p}, v_{t_2\varphi}v_{(t_2+1)\varphi-p}\}, \end{aligned}$$

the indices are taken modulo n'. Since $n \geqslant 12$, we have $\varphi \geqslant 2$ and $v_{t_1\varphi}v_{(t_1+1)\varphi-p}$, $v_{t_2\varphi}v_{(t_2+1)\varphi-p} \notin \{v_iv_{i+1}\colon\ i = 0, 1, 2, \ldots, n'-1\}$. Thus

$$|E(G)| = (k+3)\varphi + (k+3) + 2 = n - p + 3\varphi + k + 3.$$

It can be checked that $k \leqslant 3\varphi + 6$ (where $k = 3\varphi + 6$ only if $\frac{1}{3}(n+1)$ is a square). Therefore

$$|E(G)| \leqslant n + 6\varphi + 9 \leqslant n + 2\sqrt{3n} + 9.$$

It remains to show that G contains a red $2K_2$ or a blue C_n. Assume that G does not contain a red $2K_2$. We show that G contains a blue C_n. Let us consider two cases.

Case 1. A graph induced by red edges is a subgraph of a star. Let v_i be the center of this star. Without loss of generality we can suppose that $1 \leqslant i \leqslant \varphi$. We have a blue path $v_{2\varphi}v_{2\varphi+1}\ldots v_{(k+3)\varphi-1}v_0$ of length $(k+1)\varphi$ and a blue cycle $C' = v_0v_{2\varphi}v_{2\varphi+1}\ldots v_{(k+3)\varphi-1}v_0$ of length $(k+1)\varphi + 1$. Then we can replace the path $v_{t_2\varphi}v_{t_2\varphi}\ldots v_{(t_2+1)\varphi}$ having length φ in C' by the blue path $v_{t_2\varphi}v_{(t_2+1)\varphi-p}v_{(t_2+1)\varphi-p+1}\ldots v_{(t_2+1)\varphi}$ of length $p+1$ to obtain a blue cycle C of length $k\varphi + p + 2 = n$.

Case 2. A graph induced by red edges is a 3-cycle. The graph G contains 3-cycles only if $p = \varphi - 2$ (cycles $v_{t_i\varphi}v_{t_i\varphi+1}v_{t_i\varphi+2}v_{t_i\varphi}$ for $i = 1, 2$). Without loss of generality we can suppose that $t_1 = 0$ and the 3-cycle $v_0v_1v_2v_0$ is red. Then G contains blue cycles C' and C described in the previous case. The proof is complete. $\square$

Theorem 2.2. *Let $n \geqslant 25$. Then $\hat{r}(3K_2, C_n) < n + 6\sqrt{n} + 9$.*

Proof. Let k be an integer such that $k\omega + 3 \leqslant n < (k+1)\omega + 3$, where $\omega = \lfloor\sqrt{n}\rfloor$. Then we can write

$$n = k\omega + 3 + p, \tag{2.1}$$

where $0 \leqslant p \leqslant \omega - 1$. Let $t_1, t_2, t_3 \in \mathbb{Z}$, where $1 \leqslant t_1 + 1 < t_2 < t_3 - 1 \leqslant k + 1$ and $(t_1, t_3) \neq (0, k+2)$. Let G be a graph having $n' = (k+3)\omega$ vertices $v_0, v_1, v_2, \ldots, v_{n'-1}$ and

$$\begin{aligned} E(G) = {} & \{v_iv_{i+1}\colon\ i = 0, 1, 2, \ldots, n'-1\} \cup \{v_jv_{j+\omega}\colon\ j = 0, \omega, 2\omega, \ldots, n'-\omega\} \\ & \cup \left\{v_rv_{r+\omega}\colon\ r = \left\lfloor\frac{\omega}{2}\right\rfloor, \left\lfloor\frac{3\omega}{2}\right\rfloor, \left\lfloor\frac{5\omega}{2}\right\rfloor, \ldots, n' - \left\lceil\frac{\omega}{2}\right\rceil\right\} \\ & \cup \{v_sv_{s+2\omega-1}\colon\ s = 0, \omega, 2\omega, \ldots, n'-\omega\} \cup \{v_{t_i\omega}v_{(t_i+1)\omega-p}\colon\ i = 1, 2, 3\}, \end{aligned}$$

the indices are taken modulo n'. Then

$$(2.2)\qquad |E(G)| = (k+3)\omega + 3(k+3) + 3 = (n-3-p) + 3\omega + 3(k+3) + 3 \leqslant n + 3\omega + 3k + 9.$$

It can be checked that $k \leqslant \lceil\sqrt{n}\rceil$. Note that $k = \lceil\sqrt{n}\rceil$ only if n has the form $n = b^2 - b + 3 + p$ (where $0 \leqslant p \leqslant b-4$). Then $\omega = b-1$ and $k = b$. We obtain

$$(2.3)\qquad \omega + k = 2\Big(b - \frac{1}{2}\Big) < 2\sqrt{n},$$

since $n = (b-\frac{1}{2})^2 + \frac{11}{4} + p$. From (2.2) and (2.3) we get $|E(G)| < n + 6\sqrt{n} + 9$.

If $k = \omega$, then from (2.1) we know that $\sqrt{n}$ is not an integer ($\omega < \sqrt{n}$) and by (2.2), we obtain $|E(G)| < n + 6\sqrt{n} + 9$. If $k < \omega$, again by (2.2), $|E(G)| < n + 6\sqrt{n} + 9$.

It remains to show that G has a red $3K_2$ or a blue C_n. Assume that G does not contain a red $3K_2$. We show that G contains a blue C_n. Let us consider a few cases.

Case 1. A graph induced by red edges is a subgraph of two stars. Let v_i and v_j be the centers of these stars. Without loss of generality we can suppose that $1 \leqslant i \leqslant \omega$ and $1 \leqslant j - i \leqslant \frac{1}{2}n'$. If $j \leqslant 2\omega - 2$, then G has a blue path $v_0 v_{2\omega-1} v_{2\omega}$ of length 2 and also a blue cycle $C' = v_0 v_{2\omega-1} v_{2\omega} \dots v_{(k+3)\omega-1} v_0$ of length $(k+1)\omega + 2$. We can replace the path $v_{t_3\omega} v_{t_3\omega+1} \dots v_{(t_3+1)\omega}$ of length ω in C' by the blue path

$$v_{t_3\omega} v_{(t_3+1)\omega-p} v_{(t_3+1)\omega-p+1} \dots v_{(t_3+1)\omega}$$

of length $p+1$ to obtain a blue cycle of length $k\omega + p + 3 = n$.

Let $j \geqslant 2\omega - 1$. If $i \neq \omega$, then G contains a blue path $v_0 v_\omega v_{\omega+1} \dots v_{\lfloor 3\omega/2\rfloor}$ of length $\lfloor\frac{1}{2}\omega\rfloor + 1$, and if $i = \omega$, then G contains a blue path $v_0 v_1 \dots v_{\lfloor\omega/2\rfloor} v_{\lfloor 3\omega/2\rfloor}$ of length $\lfloor\frac{1}{2}\omega\rfloor + 1$.

Note that $\lfloor\frac{1}{2}(c-1)\omega\rfloor < j \leqslant \lfloor\frac{1}{2}(c+1)\omega\rfloor$ for some even $c \geqslant 4$. If $j \neq \lfloor\frac{1}{2}(c+1)\omega\rfloor$, then $v_{\lfloor(c-1)\omega/2\rfloor} v_{\lfloor(c+1)\omega/2\rfloor} v_{\lfloor(c+1)\omega/2\rfloor+1} \dots v_{(c/2+1)\omega}$ is a blue path having length $\lceil\frac{1}{2}\omega\rceil + 1$, and if $j = \lfloor\frac{1}{2}(c+1)\omega\rfloor$, then $v_{\lfloor(c-1)\omega/2\rfloor} v_{\lfloor(c-1)\omega/2\rfloor+1} \dots v_{(c/2)\omega} v_{(c/2+1)\omega}$ is a blue path of length $\lceil\frac{1}{2}\omega\rceil + 1$.

G also contains a blue path $v_{\lfloor 3\omega/2\rfloor} v_{\lfloor 3\omega/2\rfloor+1} \dots v_{\lfloor(c-1)\omega/2\rfloor}$ having length $(\frac{1}{2}c-2)\omega$ and a blue path $v_{(c/2+1)\omega} v_{(c/2+1)\omega+1} \dots v_{(k+3)\omega-1} v_0$ having length $(k - \frac{1}{2}c + 2)\omega$.

Thus G contains a blue cycle C'' having length

$$\Big(\Big\lfloor\frac{\omega}{2}\Big\rfloor + 1\Big) + \Big(\Big\lceil\frac{\omega}{2}\Big\rceil + 1\Big) + \Big(\frac{c}{2} - 2\Big)\omega + \Big(k - \frac{c}{2} + 2\Big)\omega = k\omega + \omega + 2.$$

From the definition of the edges $v_{t_i\omega} v_{(t_i+1)\omega-p}$, it follows that the cycle C'' contains a path $v_{t_i\omega} v_{t_i\omega+1} \dots v_{(t_i+1)\omega}$ of length ω for some $i \in \{1,2,3\}$. That path can be

replaced by the blue path $v_{t_i\omega}v_{(t_i+1)\omega-p}v_{(t_i+1)\omega-p+1}\dots v_{(t_i+1)\omega}$ of length $p+1$, which implies that G has a blue cycle of length $k\omega+p+3=n$.

Case 2. A graph induced by red edges contains a 3-cycle. G contains 3-cycles only if $p=1$ (cycles $v_{t_i\omega}v_{(t_i+1)\omega-1}v_{(t_i+1)\omega}v_{t_i\omega}$ for $i=1,2,3$) or $p=\omega-2$ (cycles $v_{t_i\omega}v_{t_i\omega+1}v_{t_i\omega+2}v_{t_i\omega}$ for $i=1,2,3$).

Without loss of generality we can suppose that $t_1=0$ and the 3-cycle $v_0\ v_{\omega-1}v_\omega v_0$ is red for $p=1$, and the 3-cycle $v_0v_1v_2v_0$ is red for $p=\omega-2$.

Case 2.1. A graph induced by red edges is a 3-cycle and a subgraph of a star. Let v_j be the center of this star. If $j\leqslant 2\omega-2$, then G has a blue cycle $C'=v_0v_{2\omega-1}v_{2\omega}\dots v_{(k+3)\omega-1}v_0$ of length $(k+1)\omega+2$. Let us replace the path $v_{t_3\omega}v_{t_3\omega+1}\dots v_{(t_3+1)\omega}$ of length ω in C' by the blue path

$$v_{t_3\omega}v_{(t_3+1)\omega-p}v_{(t_3+1)\omega-p+1}\dots v_{(t_3+1)\omega}$$

of length $p+1$ to obtain a blue cycle of length $k\omega+p+3=n$.

If $j\geqslant 2\omega-1$, then G contains a blue path $v_0v_1\dots v_{\lfloor\omega/2\rfloor}v_{\lfloor 3\omega/2\rfloor}$ of length $\lfloor\frac{1}{2}\omega\rfloor+1$ for $p=1$, and a blue path $v_0v_\omega v_{\omega+1}\dots v_{\lfloor 3\omega/2\rfloor}$ of length $\lfloor\frac{1}{2}\omega\rfloor+1$ for $p=\omega-2$. It can be shown as in the last three paragraphs of Case 1 that G has a blue cycle of length n.

Case 2.2. A graph induced by red edges consists of two 3-cycles. Without loss of generality we can suppose that $v_{z\omega}v_{z\omega+1}v_{z\omega+2}v_{z\omega}$ or $v_{z\omega}v_{(z-1)\omega}v_{z\omega-1}v_{z\omega}$ for some $z\geqslant 2$ is the other red 3-cycle. Thus G contains a blue path

$$v_{\lfloor 3\omega/2\rfloor}v_{\lfloor 3\omega/2\rfloor+1}\dots v_{\lfloor(z-1/2)\omega\rfloor}v_{\lfloor(z+1/2)\omega\rfloor}v_{\lfloor(z+1/2)\omega\rfloor+1}\dots v_{(k+3)\omega-1}v_0$$

of length $k\omega+\lceil\frac{1}{2}\omega\rceil+1$. Note that we also have a blue path of length $\lfloor\frac{1}{2}\omega\rfloor+1$ between v_0 and $v_{\lfloor 3\omega/2\rfloor}$, which means that G contains a blue cycle C'' of length $k\omega+\omega+2$. This cycle contains a path $v_{t_i\omega}v_{t_i\omega+1}\dots v_{(t_i+1)\omega}$ for some $i\in\{1,2,3\}$. That path can be replaced by the blue path $v_{t_i\omega}v_{(t_i+1)\omega-p}v_{(t_i+1)\omega-p+1}\dots v_{(t_i+1)\omega}$, so G has a blue cycle of length $k\omega+p+3=n$.

Case 3. A graph induced by red edges consists of a 5-cycle. Since $n\geqslant 25$, every 5-cycle of G contains an edge $v_{t_i\omega}v_{(t_i+1)\omega-p}$ for some i, where $1\leqslant i\leqslant 3$. Without loss of generality we can suppose that $t_i=0$ and the red 5-cycle contains the edge $v_0v_{\omega-p}$.

All 5-cycles (except for two 5-cycles) containing the edge $v_0v_{\omega-p}$ consist only of edges connecting some of the vertices $v_0,v_1,\dots,v_{\lfloor 3\omega/2\rfloor}$ or some of the vertices $v_{(k+2)\omega},v_{(k+2)\omega+1},\dots,v_{\lfloor\omega/2\rfloor}$. Then we have a blue cycle $v_0v_{2\omega-1}v_{2\omega}\dots v_{(k+3)\omega-1}v_0$ or a blue cycle $v_{(k+2)\omega}v_{\omega-1}v_\omega\dots v_{(k+2)\omega}$ (of length $(k+1)\omega+2$). These cycles contain the path $v_{t_3\omega}v_{t_3\omega+1}\dots v_{(t_3+1)\omega}$. We replace this path by the blue path $v_{t_3\omega}v_{(t_3+1)\omega-p}v_{(t_3+1)\omega-p+1}\dots v_{(t_3+1)\omega}$ to obtain a blue cycle of length $k\omega+p+3=n$.

Those two exceptions are the 5-cycles

$$v_0 v_{(k+2)\omega} v_{\omega-1} v_{\omega-2} v_{\omega-3} v_0 \quad \text{and} \quad v_0 v_{2\omega-1} v_{2\omega} v_\omega v_{\omega-1} v_0$$

(note that these cycles exist only for particular values of p).

If the 5-cycle $v_0 v_{(k+2)\omega} v_{\omega-1} v_{\omega-2} v_{\omega-3} v_0$ is red, then the cycle

$$v_0 v_{2\omega-1} v_{2\omega} \dots v_{(k+3)\omega-1} v_0$$

is blue, and if the 5-cycle $v_0 v_{2\omega-1} v_{2\omega} v_\omega v_{\omega-1} v_0$ is red, then we have a blue cycle $v_{\lfloor \omega/2 \rfloor} v_{\lfloor 3\omega/2 \rfloor} v_{\lfloor 5\omega/2 \rfloor} v_{\lfloor 5\omega/2 \rfloor+1} \dots v_{\lfloor \omega/2 \rfloor}$. These blue cycles have length $(k+1)\omega + 2$ and it is easy to obtain a blue cycle having length $k\omega + p + 3 = n$. $\square$

References

[1] *Effendi, B. Ginting, Surahmat, Dafik, S. Sy*: The size multipartite Ramsey numbers of large paths versus small graph. Far East J. Math. Sci. (FJMS) *102* (2017), 507–514. zbl doi

[2] *P. Erdős, R. J. Faudree*: Size Ramsey numbers involving matchings. Finite and Infinite Sets. Vol. I, II (Eger, 1981) (A. Hajnal et al., eds.). Colloquia Mathematica Societatis János Bolyai 37. János Bolyai Mathematical Society, Budapest; North-Holland, Amsterdam, 1984, pp. 247–264. zbl MR doi

[3] *P. Erdős, R. J. Faudree, C. C. Rousseau, R. H. Schelp*: The size Ramsey number. Period. Math. Hung. *9* (1978), 145–161. zbl MR doi

[4] *R. J. Faudree, J. Sheehan*: Size Ramsey numbers for small-order graphs. J. Graph Theory *7* (1983), 53–55. zbl MR doi

[5] *X. Ke*: The size Ramsey number of trees with bounded degree. Random Struct. Algorithms *4* (1993), 85–97. zbl MR doi

[6] *R. Lortz, I. Mengersen*: Size Ramsey results for paths versus stars. Australas. J. Comb. *18* (1998), 3–12. zbl MR

[7] *P. H. Perondi, E. L. Monte Carmelo*: Set and size multipartite Ramsey numbers for stars. Discrete Appl. Math. *250* (2018), 368–372. zbl MR doi

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