NONOSCILLATORY SOLUTIONS OF DISCRETE FRACTIONAL ORDER EQUATIONS WITH POSITIVE AND NEGATIVE TERMS

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Abstract. This paper aims at discussing asymptotic behaviour of nonoscillatory solutions of the forced fractional difference equations of the form

$$\begin{aligned}&\Delta^{\gamma}u(\kappa)+\Theta[\kappa+\gamma,w(\kappa+\gamma)]\\&\qquad=\Phi(\kappa+\gamma)+\Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma)+\Psi[\kappa+\gamma,w(\kappa+\gamma)],\quad \kappa\in\mathbb{N}_{1-\gamma},\\&u_0=c_0,\end{aligned}$$

where $\mathbb{N}_{1-\gamma}=\{1-\gamma,2-\gamma,3-\gamma,\ldots\}$, $0<\gamma\leqslant 1$, Δ^{γ} is a Caputo-like fractional difference operator. Three cases are investigated by using some salient features of discrete fractional calculus and mathematical inequalities. Examples are presented to illustrate the validity of the theoretical results.

Keywords: fractional difference equation; nonoscillatory; Caputo fractional difference; forcing term

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1. Introduction

Mathematical theory regarding fractional calculus was put forward before the commencement of 20th century. The development of the theory of fractional calculus is credited to the direct and indirect contributions of many eminent mathematicians. Non-local property of fractional derivatives makes it unique and thus opening new directions and avenues for exploration and applications. It has emerged as one of the

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significant interdisciplinary subjects in physical, biological sciences and engineering. Specifically, in the last three decades, the subject witnessed exponential growth regarding its applicability in engineering, natural, physical and social sciences. Some bibliographic metrics of this evolution are seen in [4], [15], [16], [21].

In the recent decade, theory of fractional difference equations is evolving as a mathematical means to analyze problems arising in interdisciplinary applications. Discrete fractional calculus is a nascent discipline and is analogous to its continuous counter part. Fractional calculus is evolving as a notable platform to model real world phenomena. The theory of delta fractional calculus is enriched by the contributions of mathematicians like Atici, Eloe, Abdeljawad, Holm, Goodrich, Peterson and Cheng, to note a few (see [5], [6], [13]). Delta fractional calculus has come to the forefront during the recent decade due its inherent complexity and non-local property. Development of its theory is still in the progressing stage and thus opening new opportunities and scope to explore in this area.

The oscillation theory provides valuable insights into the dynamics of solutions to problems modeled with equations in various areas of engineering and science. In recent years, the study of the oscillation theory of fractional order difference equation has been remarkably constructive, advancing rapidly and has been the focus of research for many scientists, see [1], [2], [3], [7], [17], [19], [20]. For the nonoscillatory solutions of fractional differential equation, its asymptotic behaviour (see [11]) and boundedness (see [10]) are discussed. The existence of nonoscillatory solutions of nonlinear neutral delay difference equation of fractional order are dealt in [18].

In the recent times, the authors in [12] established the asymptotic behaviour of nonoscillatory solutions of certain fractional differential equations with positive and negative terms. Inspired by the above literature, the following forced discrete form of fractional equations is considered

$$\begin{aligned}(1.1)\qquad &\Delta^{\gamma}u(\kappa)+\Theta[\kappa+\gamma,w(\kappa+\gamma)]\\ &\qquad=\Phi(\kappa+\gamma)+\Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma)+\Psi[\kappa+\gamma,w(\kappa+\gamma)],\quad \kappa\in\mathbb{N}_{1-\gamma},\\ &u_0=c_0,\end{aligned}$$

where $\mathbb{N}_{1-\gamma}=\{1-\gamma,2-\gamma,3-\gamma,\ldots\}$, $0<\gamma\leqslant 1$, Δ^{γ} is a Caputo-like discrete fractional difference operator. The asymptotic behaviour of the nonoscillatory solution is discussed in the below mentioned three cases:

$$u(\kappa)=\Delta[\varrho(\kappa)[\Delta w(\kappa)]^{\nu}], \tag{1.2}$$

$$u(\kappa)=\varrho(\kappa)[\Delta w(\kappa)]^{\nu}, \tag{1.3}$$

$$u(\kappa)=w(\kappa). \tag{1.4}$$

The following assumptions are made:

($\mathcal{H}_1$) ν is the ratio of odd positive integers.

($\mathcal{H}_2$) $\varrho, \Upsilon\colon [\kappa_1, \infty) \to (0, \infty)$ are continuous functions and Φ is a positive sequence.

($\mathcal{H}_3$) $\Theta, \Psi\colon (\kappa_1, \infty) \times \mathbb{R} \to \mathbb{R}$ are real valued continuous functions and there exist continuous functions $\varphi, \sigma\colon [\kappa_1, \infty) \to (0, \infty)$ and positive real numbers λ and α with $\lambda > \alpha$ such that

$$\begin{cases} w\Theta(\kappa, w) \geqslant \varphi(\kappa)|w|^{\lambda+1}, \\ 0 < w\Psi(\kappa, w) \leqslant \sigma(\kappa)|w|^{\alpha+1} \end{cases} \quad \text{for } w \neq 0,\ \kappa \geqslant 0.$$

The remaining part of this paper is arranged as follows: The basic definitions and lemmas are presented in Section 2 for further discussion on important results. Results on nonoscillation are established in Section 3 by using features of discrete fractional calculus and mathematical inequalities. Suitable examples are demonstrated in Section 4 for the theoretical findings. The paper ends with a brief conclusion in the last section.

2. Preliminaries

The essential definitions and lemmas are included in this section for the proofs of further results.

Definition 2.1 ([9]). A solution $\{w(\kappa)\}$ is said to be *oscillatory* if the terms of the solution are neither eventually positive nor eventually negative. Otherwise, the solution is called *nonoscillatory.*

Definition 2.2 ([5]). Let $\gamma > 0$. Then the γth fractional sum $\Delta^{-\gamma}\colon \mathbb{N}_a \mapsto \mathbb{N}_{a+\gamma}$ of w is given by

$$\Delta^{-\gamma} w(\kappa) = \frac{1}{\Gamma(\gamma)} \sum_{\beta=a}^{\kappa-\gamma} (\kappa - \beta - 1)^{\gamma-1} w(\beta), \quad \kappa \in \mathbb{N}_{a+\gamma}.$$

Here w is defined for $\beta \equiv a \pmod 1$ and $\Delta^{-\gamma} w$ is defined for $\kappa \equiv (a+\gamma) \pmod 1$.

Definition 2.3 ([5]). For $\gamma > 0$, f defined on $\mathbb{N}_a$ and $N \in \mathbb{N}$ such that $0 \leqslant N-1 < \gamma \leqslant N$ let us define:

(a) The γth order Riemann-Liouville ($\mathcal{RL}$) fractional difference of w is

$${}_{\mathcal{RL}}\Delta^{\gamma} w(\kappa) = \Delta^N \Delta_a^{-(N-\gamma)} w(\kappa) = \frac{\Delta^N}{\Gamma(N-\gamma)} \sum_{\beta=a}^{\kappa-(N-\gamma)} (\kappa - \beta - 1)^{N-\gamma-1} w(\beta)$$

for $\kappa \in \mathbb{N}_{a+N-\gamma}$.

(b) The γth order Caputo-like ($\mathcal{CA}$) fractional difference of w is

$$ {}_{\mathcal{CA}}\Delta^{\gamma} w(\kappa) = \Delta_a^{-(N-\gamma)} \Delta^N w(\kappa) = \frac{1}{\Gamma(N-\gamma)} \sum_{\beta=a}^{\kappa-(N-\gamma)} (\kappa-\beta-1)^{\underline{N-\gamma-1}} \Delta^N w(\beta) $$

for $\kappa \in \mathbb{N}_{a+N-\gamma}$.

Lemma 2.4 (Young's inequality, see [12]). *If $\mathcal{X}$ and $\mathcal{Y}$ are non-negative, $\delta > 1$ and $1/\delta + 1/\xi = 1$, then $\mathcal{X}\mathcal{Y} \leqslant \mathcal{X}^{\delta}/\delta + \mathcal{Y}^{\xi}/\xi$, where equality holds if and only if $\mathcal{Y} = \mathcal{X}^{\delta-1}$.*

Lemma 2.5 ([8]). *Assume that $\xi > 1$ and $\delta > 0$, then*

$$ [\kappa^{-\delta}]^{\xi} = \frac{\Gamma(1+\delta\xi)}{\Gamma^{\xi}(1+\delta)} \kappa^{-\delta\xi} \quad \textit{for } \kappa \in \mathbb{N}. $$

Lemma 2.6 ([8]). *Initial value problem (IVP)* (1.1) *is equivalent to the expression*

$$ u(\kappa) = c_0 + \frac{1}{\Gamma(\gamma)} \sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\underline{\gamma-1}} (\Phi(\beta+\gamma) + \Upsilon(\beta+\gamma) w^{\nu}(\beta+\gamma) $$
$$ - \Theta[(\beta+\gamma), w(\beta+\gamma)] + \Psi[(\beta+\gamma), w(\beta+\gamma)]), \quad \kappa \in \mathbb{N}_1. $$

Lemma 2.7 (Discrete Gronwall's inequality, see [14]). *Let w and ε be non-negative sequences and c be a non-negative constant. If*

$$ w(\kappa) \leqslant c + \sum_{\beta=0}^{\kappa} \varepsilon(\beta) w(\beta) \quad \textit{for } \kappa \geqslant 0, $$

then

$$ x(\kappa) \leqslant c \exp\left(\sum_{\beta=0}^{\kappa} \varepsilon(\beta) \right) \quad \textit{for } \kappa \geqslant 0. $$

Lemma 2.8 ([8]).

$$ \sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\underline{\gamma-1}} = \frac{\Gamma(\kappa+\gamma)}{\gamma\Gamma(\kappa)} = \frac{1}{\gamma(\kappa-1)^{\underline{-\gamma}}}. $$

3. Main results

This section is devoted to the nonoscillatory results. The results are stated and proved in two separate subsections. The results are established by using some features of discrete fractional calculus and mathematical inequalities.

3.1. Nonoscillatory solutions of equation (1.1) **with** (1.2)**.** Consider the equation

$$\tag{3.1}\begin{aligned}&\Delta^{\gamma+1}[\varrho(\kappa)[\Delta w(\kappa)]^{\nu}]+\Theta[\kappa+\gamma,w(\kappa+\gamma)]\\&\qquad=\Phi(\kappa+\gamma)+\Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma)+\Psi[\kappa+\gamma,w(\kappa+\gamma)],\\&\Delta[\varrho_0[\Delta w(0)]^{\nu}]=c_0.\end{aligned}$$

For the sake of convenience, let

$$\tag{3.2}\mathcal{R}(\kappa)=\sum_{\beta=1}^{\kappa-1}\varrho^{-1/\nu}(\beta)$$

and assume that

$$\tag{3.3}\lim_{\kappa\to\infty}\mathcal{R}(\kappa)=\infty,$$

$$\tag{3.4}G(\kappa)=\frac{\lambda-\alpha}{\alpha}\Big(\frac{\alpha}{\lambda}\sigma(\kappa)\Big)^{\lambda/(\lambda-\alpha)}(\varphi(\kappa))^{\alpha/(\alpha-\lambda)}.$$

The sufficient conditions under which any nonoscillatory solution $w(\kappa)$ of (1.1) with (1.2) satisfies the following condition:

$$|w(\kappa)|=O(\kappa^{1/\nu}\mathcal{R}(\kappa))\quad\text{as }\kappa\to\infty.$$

Theorem 3.1. *Let conditions* $(\mathcal{H}_1)$*–*$(\mathcal{H}_3)$ *and* (3.3) *hold and assume that there exist real numbers* $l>1$ *and* $0<\gamma<1$ *such that* $l(\gamma-1)+1>0$*. If*

$$\tag{3.5}\sum_{\beta=1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)(\beta+\gamma)^{m}\mathcal{R}^{\nu m}(\beta+\gamma)<\infty,\quad m=\frac{l}{l-1},$$

$$\tag{3.6}\lim_{\kappa\to\infty}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Phi(\beta+\gamma)<\infty$$

and

$$\tag{3.7}\lim_{\kappa\to\infty}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}G(\beta+\gamma)<\infty,$$

then every solution $w(\kappa)$ *of* (1.1) *with* (1.2) *satisfies*

$$\tag{3.8}\limsup_{\kappa\to\infty}\frac{|w(\kappa)|}{\kappa^{1/\nu}\mathcal{R}(\kappa)}<\infty.$$

P r o o f. Let $w(\kappa)$ be a nonoscillatory solution of (1.1) with (1.2), say $w(\kappa) \geqslant 0$ for $\kappa \geqslant \kappa_1$ for some $\kappa_1 \geqslant 0$. Set $\mathcal{F}(\kappa+\gamma) = \Psi[\kappa+\gamma, w(\kappa+\gamma)] - \Theta[\kappa+\gamma, w(\kappa+\gamma)]$. It follows from (1.1) and (1.2) that

$$\begin{aligned}\Delta^{\gamma+1}[\varrho(\kappa)[\Delta(w(\kappa))]^{\nu}] &= \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma) \\ &\quad + \Psi[\kappa+\gamma, w(\kappa+\gamma)] - \Theta[\kappa+\gamma, w(\kappa+\gamma)] \\ &= \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma) + \mathcal{F}(\kappa+\gamma).\end{aligned}$$

Using Lemma 2.6 yields

$$\begin{aligned}&\Delta[\varrho(\kappa)[\Delta(w(\kappa))]^{\nu}] \\ &\quad\leqslant c_0 + \frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}(\Phi(\beta+\gamma) + \Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma) + \mathcal{F}(\beta+\gamma)) \\ &\quad\leqslant c_0 + \frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa_1-1-\gamma}(\kappa-\beta-1)^{\gamma-1}|\mathcal{F}(\beta+\gamma)| \\ &\qquad + \frac{1}{\Gamma(\gamma)}\sum_{\beta=\kappa-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}(\Psi[\beta+\gamma, w(\beta+\gamma)] - \Theta[\beta+\gamma, w(\beta+\gamma)]) \\ &\qquad + \frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa_1-1-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma) \\ &\qquad + \frac{1}{\Gamma(\gamma)}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma) \\ &\qquad + \frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}|\Phi(\beta+\gamma)|.\end{aligned}$$

Applying $(\mathcal{H}_3)$, we get

$$\begin{aligned}(3.9)\quad &\Delta[\varrho(\kappa)[\Delta(w(\kappa))]^{\nu}] \\ &\quad\leqslant c_0 + \frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}|\Phi(\beta+\gamma)| \\ &\qquad + \frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa_1-1-\gamma}(\kappa-\beta-1)^{\gamma-1}|\mathcal{F}(\beta+\gamma)| \\ &\qquad + \frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa_1-1-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)\end{aligned}$$

$$+\frac{1}{\Gamma(\gamma)}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)$$

$$+\frac{1}{\Gamma(\gamma)}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}(\sigma(\beta+\gamma)w^{\alpha}(\beta+\gamma)-\varphi(\beta+\gamma)w^{\lambda}(\beta+\gamma)).$$

Lemma 2.4 yields $\delta=\lambda/\alpha$, $\mathcal{X}=w^{\nu}(\beta+\alpha)$, $\mathcal{Y}=(\alpha/\lambda)\sigma(\beta+\alpha)/\varphi(\beta+\alpha)$ and $\xi=\lambda/(\lambda-\alpha)$. Hence,

$$\begin{aligned}(3.10)\qquad &\sigma(\beta+\gamma)w^{\alpha}(\beta+\gamma)-\varphi(\beta+\gamma)w^{\lambda}(\beta+\gamma)\\ &=\frac{\lambda}{\alpha}\varphi(\beta+\gamma)\Big(w^{\alpha}(\beta+\gamma)\frac{\alpha}{\lambda}\frac{\sigma(\beta+\gamma)}{\varphi(\beta+\gamma)}-\frac{\alpha}{\lambda}(w^{\alpha}(\beta+\gamma))^{\lambda/\alpha}\Big)\\ &=\frac{\lambda}{\alpha}\varphi(\beta+\gamma)\Big(\mathcal{X}\mathcal{Y}-\frac{1}{\delta}\mathcal{X}^{\delta}\Big)\\ &\leqslant\frac{\lambda}{\alpha}\varphi(\beta+\gamma)\frac{1}{\xi}\mathcal{Y}^{\xi}\\ &\leqslant\frac{\lambda}{\alpha}\varphi(\beta+\gamma)\frac{\lambda-\alpha}{\lambda}\Big(\frac{\alpha}{\lambda}\frac{\sigma(\beta+\gamma)}{\varphi(\beta+\gamma)}\Big)^{\lambda/(\lambda-\alpha)}\\ &\leqslant\frac{\lambda-\alpha}{\alpha}\Big(\frac{\alpha}{\lambda}\sigma(\beta+\gamma)\Big)^{\lambda/(\lambda-\alpha)}(\varphi(\beta+\gamma))^{\alpha/(\alpha-\lambda)}\\ &=G(\beta+\gamma).\end{aligned}$$

Using (3.10) in (3.9) gives

$$\begin{aligned}\Delta[\varrho(\kappa)[\Delta w(\kappa)]^{\nu}]\leqslant c_0&+\frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}|\Phi(\beta+\gamma)|\\ &+\frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa_1-1-\gamma}(\kappa-\beta-1)^{\gamma-1}|\mathcal{F}(\beta+\gamma)|\\ &+\frac{1}{\Gamma(\gamma)}\sum_{\beta=1-\gamma}^{\kappa_1-1-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)\\ &+\frac{1}{\Gamma(\gamma)}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)\\ &+\frac{1}{\Gamma(\gamma)}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}G(s+\gamma).\end{aligned}$$

In view of (3.6) and (3.7)

$$(3.11)\qquad \Delta[\varrho(\kappa)[\Delta w(\kappa)]^{\nu}]\leqslant c_1+\frac{1}{\Gamma(\gamma)}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)$$

for a constant $c_1 > 0$. Taking summation from κ_1 to $\kappa - 1$ we get

$$\sum_{\varepsilon=\kappa_1}^{\kappa-1} \Delta[\varrho(\varepsilon)[\Delta w(\kappa)]^\nu] \leqslant \sum_{\varepsilon=\kappa_1}^{\kappa-1} \biggl(c_1 + \frac{1}{\Gamma(\gamma)} \sum_{\beta=\varepsilon_1-\gamma}^{\varepsilon-\gamma} (\varepsilon-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^\nu(\beta+\gamma)\biggr),$$

$$\begin{aligned} \varrho(\kappa)[\Delta w(\kappa)]^\nu - \varrho(\kappa_1)[\Delta w(\kappa_1)]^\nu &\leqslant c_1(\kappa-\kappa_1) \\ &\quad + \frac{1}{\Gamma(\gamma)} \sum_{\varepsilon=\kappa_1}^{\kappa-1} \sum_{\beta=\varepsilon_1-\gamma}^{\varepsilon-\gamma} (\varepsilon-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^\nu(\beta+\gamma), \end{aligned} \tag{3.12}$$

$$\varrho(\kappa)[\Delta w(\kappa)]^\nu := V(\kappa). \tag{3.13}$$

Now

$$[\Delta w(\kappa)]^\nu = \frac{V(\kappa)}{\varrho(\kappa)}, \quad \Delta w(\kappa) = \Bigl[\frac{V(\kappa)}{\varrho(\kappa)}\Bigr]^{1/\nu}, \quad \sum_{\beta=\kappa_1}^{\kappa-1} \Delta w(\beta) = \sum_{\beta=\kappa_1}^{\kappa-1} \Bigl[\frac{V(\beta)}{\varrho(\beta)}\Bigr]^{1/\nu},$$

$$w(\kappa) - w(\kappa_1) \leqslant V^{1/\nu}(\kappa) \sum_{\beta=\kappa_1}^{\kappa-1} \Bigl[\frac{1}{\varrho(\beta)}\Bigr]^{1/\nu}, \quad w(\kappa) \leqslant w(\kappa_1) + V^{1/\nu}(\kappa) \sum_{\beta=\kappa_1}^{\kappa-1} \varrho^{-1/\nu}(\beta),$$

$$w(\kappa) \leqslant w(\kappa_1) + V^{1/\nu}(\kappa)\mathcal{R}(\kappa) = \Bigl[\frac{w(\kappa_1)}{\mathcal{R}(\kappa)} + V^{1/\nu}(\kappa)\Bigr]\mathcal{R}(\kappa), \quad \frac{w(\kappa)}{\mathcal{R}(\kappa)} \leqslant c_2 + V^{1/\nu}(\kappa)$$

for some $\kappa \geqslant \kappa_2$, where $c_2 = w(\kappa_1)/\mathcal{R}(\kappa_2)$.

We shall apply the inequality

$$(\mathcal{A}+\mathcal{B})^\mu \leqslant 2^{\mu-1}(\mathcal{A}^\mu + \mathcal{B}^\mu), \quad \mathcal{A},\mathcal{B} \geqslant 0,\ \mu \geqslant 1. \tag{3.14}$$

Also,

$$\begin{aligned} \Bigl[\frac{w(\kappa)}{\mathcal{R}(\kappa)}\Bigr]^\nu &\leqslant [c_2 + V^{1/\nu}(\kappa)]^\nu \leqslant 2^{\nu-1}c_2^\nu + 2^{\nu-1}[V^{1/\nu}(\kappa)]^\nu \leqslant 2^{\nu-1}c_2^\nu + 2^{\nu-1}V(\kappa) \\ &\leqslant 2^{\nu-1}c_2^\nu + 2^{\nu-1}\varrho(\kappa_1)[\Delta w(\kappa_1)]^\nu + 2^{\nu-1}c_1(\kappa-\kappa_1) \\ &\quad + \frac{2^{\nu-1}}{\Gamma(\gamma)} \sum_{\varepsilon=\kappa_1}^{\kappa-1} \sum_{\beta=\varepsilon_1-\gamma}^{\varepsilon-\gamma} (\varepsilon-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^\nu(\beta+\gamma) \\ &\leqslant c_3\kappa + \frac{2^{\nu-1}}{u(\gamma+1)}(\kappa) \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^\nu(\beta+\gamma), \end{aligned}$$

and

$$\Bigl[\frac{w(\kappa)}{\kappa^{1/\nu}}\mathcal{R}(\kappa)\Bigr]^\nu \leqslant c_3 + \frac{2^{\nu-1}}{u(\gamma+1)} \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^\nu(\beta+\gamma) \tag{3.15}$$

for a constant $c_3 > 0$. Applying Holder's inequality with Lemma 2.5 and Lemma 2.6 we have

$$\begin{aligned}
&\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)\\
&\quad=\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)\\
&\quad=\frac{\Gamma(\kappa+\gamma)}{\gamma\Gamma(\kappa)}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)\\
&\quad=\Big(\Big(\frac{1}{\gamma(\kappa-1)^{-\gamma}}\Big)^{l}\Big)^{1/l}\Big(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma)\Big)^{1/m}
\end{aligned}$$

for $\kappa > \kappa_1$.

$$\begin{aligned}
(3.16)\quad&\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)\\
&=\Big(\frac{1}{\gamma l}\frac{1}{\kappa_1^{-\gamma l}\Gamma(1+\gamma l)/\Gamma^{l}(1+\gamma)}\Big)^{1/l}\Big(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma)\Big)^{1/m}\\
&=\Big(\frac{\Gamma^{l}(1+\gamma)\Gamma(\kappa_1+1+\gamma l)}{\gamma^{l}\Gamma(1+\gamma l)\Gamma(\kappa_1+1)}\Big)^{1/l}\Big(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma)\Big)^{1/m}\\
&=\mathcal{Q}^{1/l}\Big(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma)\Big)^{1/m},
\end{aligned}$$

where $\mathcal{Q} = \Gamma^l(1+\gamma)\Gamma(\kappa_1+1+\gamma l)/(\gamma^l\Gamma(1+\gamma l)\Gamma(\kappa_1+1))$. Using (3.16) in (3.15),

$$\begin{aligned}
Z(\kappa) := \Big[\frac{w(\kappa)}{\kappa^{1/\nu}\mathcal{R}(\kappa)}\Big]^{\nu} &\leqslant c_3+\frac{2^{\nu-1}}{\Gamma(\gamma+1)}\mathcal{Q}^{1/l}\Big(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma)\Big)^{1/m}\\
&\leqslant c_3+D\Big(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma)\Big)^{1/m},\\
Z^{m}(\kappa) &\leqslant \Big(c_3+D\Big(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma)\Big)^{1/m}\Big)^{m}\\
&\leqslant 2^{m-1}c_3^{m}+2^{m-1}D^{m}\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma}\Upsilon^{m}(\beta+\gamma)w^{\nu m}(\beta+\gamma).
\end{aligned}$$

By setting $\mathcal{P}_1 = 2^{m-1}c_3^m$, $\mathcal{Q}_1 = 2^{m-1}D^m$, $V^{1/m}(\kappa) = Z(\kappa)$ and $w^{\nu m}(\beta+\gamma) = (\beta+\gamma)^m\mathcal{R}^{\nu m}(\beta+\gamma)\Theta^m(\beta+\gamma)$ in the above inequality, we obtain

$$V(\kappa) \leqslant \mathcal{P}_1 + \mathcal{Q}_1 \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)(\beta+\gamma)^m\mathcal{R}^{\nu m}(\beta+\gamma)\Psi(\beta+\gamma).$$

Using Lemma 2.7,

$$V(\kappa) \leqslant \mathcal{P}_1 \exp\Bigg(\mathcal{Q}_1 \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)(\beta+\gamma)^m\mathcal{R}^{\nu m}(\beta+\gamma)\Bigg)$$

Using (3.5),

$$\limsup_{\kappa\to\infty} \frac{w(\kappa)}{\kappa^{1/\nu}\mathcal{R}(\kappa)} < \infty.$$

This completes the proof. □

3.2. Nonoscillatory solutions of equation (1.1) **with criteria** (1.3)**.** Consider the equation

(3.17)

$$\begin{aligned}\Delta^\gamma[\varrho(\kappa)[\Delta w(\kappa)]^\nu] + \Theta[\kappa+\gamma, w(\kappa+\gamma)]\\ = \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^\nu(\kappa+\gamma) + \Psi[\kappa+\gamma, w(\kappa+\gamma)], \quad \kappa \in \mathbb{N}_{1-\gamma},\\ \varrho_0[\Delta w(0)]^\nu = c_0.\end{aligned}$$

The sufficient conditions under which any nonoscillatory solution $w(\kappa)$ of (1.1) with (1.3) satisfies the following condition:

$$|w(\kappa)| = O(\mathcal{R}(\kappa)) \quad \text{as } \kappa \to \infty.$$

Theorem 3.2. *Let conditions* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *and* (3.3) *hold. Assume that there exist real numbers* $l > 1$ *and* $0 < \gamma < 1$ *such that* $l(\gamma-1)+1 > 0$. *If*

$$\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)\mathcal{R}^{\nu m}(\beta+\gamma) < \infty, \quad m = \frac{l}{l-1} \tag{3.18}$$

and conditions (3.6) *and* (3.7) *hold, then every solution* $w(\kappa)$ *of equation* (1.1) *with* (1.3) *satisfies*

$$\limsup_{\kappa\to\infty} \frac{|w(\kappa)|}{\mathcal{R}(\kappa)} < \infty. \tag{3.19}$$

P r o o f. Let $w(\kappa)$ be a nonoscillatory solution of equation (1.1) with (1.3), say $w(\kappa) > 0$ for $\kappa > \kappa_1$ for some $\kappa_1 > 0$. Set $\mathcal{F}[\kappa+\gamma] = \Psi[\kappa+\gamma, w(\kappa+\gamma)] - \Theta[\kappa+\gamma, w(\kappa+\gamma)]$. It follows from (1.1) and (1.3) that

$$\begin{aligned}\Delta^{\gamma}[\varrho(\kappa)[\Delta w(\kappa)]^{\nu}] &= \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma) \\ &\quad + \Psi[\kappa+\gamma, w(\kappa+\gamma)] - \Theta[\kappa+\gamma, w(\kappa+\gamma)] \\ &= \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma) + \mathcal{F}(\kappa+\gamma).\end{aligned}$$

Using Lemma 2.6, we get

$$\begin{aligned}\varrho(\kappa)[\Delta w(\kappa)]^{\nu} \leqslant c_0 + \frac{1}{u(\gamma)} \sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-l)^{\gamma-1} \\ \times (\Phi(\beta+\gamma) + \Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma) + \mathcal{F}(\beta+\gamma)).\end{aligned}$$

Proceeding as in the proof of Theorem 3.1, equation (3.11) is obtained in the following form:

$$\varrho(\kappa)[\Delta w(\kappa)]^{\nu} \leqslant c_4 + \frac{1}{\Gamma(\gamma)} \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma) := T(\kappa),$$

$$[\Delta w(\kappa)] = \frac{T^{1/\nu}(\kappa)}{\varrho^{1/\nu}(\kappa)}.$$

Now

$$\sum_{\beta=\kappa_1}^{\kappa-1} \Delta w(\beta) = \sum_{\beta=\kappa_1}^{\kappa-1} \Big[\frac{T(\beta)}{\varrho(\beta)}\Big]^{1/\nu}, \quad w(\kappa) - w(\kappa_1) \leqslant T^{1/\nu}(\kappa) \sum_{\beta=\kappa_1}^{\kappa-1} \Big(\frac{1}{\varrho(\beta)}\Big)^{1/\nu},$$

$$\begin{aligned}w(\kappa) \leqslant w(\kappa_1) + T^{1/\nu}(\kappa) \sum_{\beta=\kappa_1}^{\kappa-1} \varrho^{-1/\nu}(\beta) &\leqslant w(\kappa_1) + T^{1/\nu}(\kappa)\mathcal{R}(\kappa) \\ &\leqslant \Big[\frac{w(\kappa_1)}{\mathcal{R}(\kappa)} + T^{1/\nu}(\kappa)\Big]\mathcal{R}(\kappa),\end{aligned}$$

$$\frac{w(\kappa)}{\mathcal{R}(\kappa)} \leqslant c_5 + T^{1/\nu}(\kappa)$$

for some $\kappa \geqslant \kappa_2$, where $c_5 = w(\kappa_1)/\mathcal{R}(\kappa_2)$. Applying inequality (3.14), we have

(3.20)
$$\begin{aligned}\Big[\frac{w(\kappa)}{\mathcal{R}(\kappa)}\Big]^{\nu} &\leqslant [c_5 + T^{1/\nu}(\kappa)]^{\nu} \leqslant 2^{\nu-1}c_5^{\nu} + 2^{\nu-1}[T^{1/\nu}(\kappa)]^{\nu} \leqslant 2^{\nu-1}c_5^{\nu} + 2^{\nu-1}T(\kappa) \\ &= 2^{\nu-1}c_5^{\nu} + 2^{\nu-1}c_4 + \frac{2^{\nu-1}}{\Gamma(\gamma)} \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma)(\kappa-\beta-1)^{\gamma-1} \\ &\leqslant c_6 + \frac{2^{\nu-1}}{\Gamma(\gamma)} \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\gamma-1}\Upsilon(\beta+\gamma)w^{\nu}(\beta+\gamma).\end{aligned}$$

Repeating the procedure of the proof of Theorem 3.1 from equation (3.15) to (3.16), we obtain

$$\mathcal{M}(\kappa) = \Big[\frac{w(\kappa)}{\mathcal{R}(\kappa)}\Big]^{\nu} \leqslant c_6 + \frac{2^{\nu-1}}{\Gamma(\gamma)}\mathcal{Q}^{1/l}\bigg(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)w^{\nu m}(s+\gamma)\bigg)^{1/m}$$

$$\leqslant c_6 + \mathcal{E}\bigg(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)w^{\nu m}(\beta+\gamma)\bigg)^{1/m},$$

$$\mathcal{M}^m(\kappa) \leqslant \bigg(c_6 + \mathcal{E}\bigg(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)w^{\nu m}(\beta+\gamma)\bigg)^{1/m}\bigg)^m$$

$$\leqslant 2^{m-1}c_6^m + 2^{m-1}\mathcal{E}^m \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)w^{\nu m}(\beta+\gamma).$$

By setting $\mathcal{P}_2 = 2^{m-1}c_6^m$, $\mathcal{Q}_2 = 2^{m-1}\mathcal{E}^m$, $T(\kappa) = \mathcal{M}^m(\kappa) \Rightarrow T^{1/m}(\kappa) = \mathcal{M}(\kappa)$,

$$\mathcal{M}(\kappa) = \Big[\frac{w(\kappa)}{\mathcal{R}(\kappa)}\Big]^{\nu}, \quad \mathcal{M}^m(\kappa) = \frac{w^{\nu m}(\kappa)}{\mathcal{R}^{\nu m}(\kappa)},$$
$$w^{\nu m}(\kappa) = \mathcal{M}^m(\kappa)\mathcal{R}^{\nu m}(\kappa) = \mathcal{R}^{\nu m}(\kappa)T(\kappa), \quad w^{\nu m}(\beta+\gamma) = \mathcal{R}^{\nu m}(\beta+\gamma)T(\beta+\gamma),$$

we obtain

$$T(\kappa) \leqslant \mathcal{P}_2 + \mathcal{Q}_2 \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)\mathcal{R}^{\nu m}(\beta+\gamma)T(\beta+\gamma).$$

Using Lemma 2.7,

$$T(\kappa) \leqslant \mathcal{P}_2 \exp\bigg(\mathcal{Q}_2 \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)\mathcal{R}^{\nu m}(\beta+\gamma)\bigg).$$

Using (3.18), we finally arrive at

$$\limsup_{\kappa\to\infty} \frac{w(\kappa)}{\mathcal{R}(\kappa)} < \infty.$$

This completes the proof. □

3.3. Nonoscillatory solutions of equation (1.1) **with criteria** (1.4)**.** Consider

$$\begin{aligned}(3.21)\quad &\Delta^{\gamma}w(\kappa) + \Theta[\kappa+\gamma, w(\kappa+\gamma)]\\ &\quad = \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^{\nu}(\kappa+\gamma) + \Psi[\kappa+\gamma, w(\kappa+\gamma)], \quad \kappa\in\mathbb{N}_{1-\gamma},\\ &w(0) = c_0.\end{aligned}$$

Theorem 3.3. *Let conditions* $(\mathcal{H}_1)$–$(\mathcal{H}_3)$ *hold. Assume that there exist real numbers* $l > 1$ *and* $0 < \gamma < 1$ *such that* $l(\gamma - 1) + 1 > 0$. *If*

$$\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma) < \infty, \quad m = \frac{l}{l-1}, \tag{3.22}$$

then every solution $w(\kappa)$ *of equation* (1.1) *with* (1.4) *satisfies*

$$\limsup_{\kappa\to\infty} |w(\kappa)| < \infty. \tag{3.23}$$

P r o o f. Let $w(\kappa)$ be a nonoscillatory solution of equation (1.1) with (1.4), say $w(\kappa) > 0$ for $\kappa > \kappa_1$ for some $\kappa_1 > 0$. By setting $\mathcal{F}[\kappa+\gamma] = \Psi[\kappa+\gamma, w(\kappa+\gamma)] - \Theta[\kappa+\gamma, w(\kappa+\gamma)]$, it follows from (1.1) and (1.4) that

$$\begin{aligned}\Delta^\gamma w(\kappa) &= \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^\nu(\kappa+\gamma)\\ &\quad + \Psi[\kappa+\gamma, w(\kappa+\gamma)] - \Theta[\kappa+\gamma, w(\kappa+\gamma)]\\ &= \Phi(\kappa+\gamma) + \Upsilon(\kappa+\gamma)w^\nu(\kappa+\gamma) + \mathcal{F}(\kappa+\gamma).\end{aligned}$$

Using Lemma 2.6, we get

$$w(\kappa) \leqslant c_0 + \frac{1}{\Gamma(\gamma)} \sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-l)^{\gamma-1}(\Phi(\beta+\gamma) + \Upsilon(\beta+\gamma)w^\nu(\beta+\gamma) + \mathcal{F}(\beta+\gamma)).$$

Proceeding as in the proof of Theorem 3.1, equation (3.11) is obtained in the form

$$w(\kappa) \leqslant c_7 + \frac{1}{\Gamma(\gamma)} \sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-l)^{\gamma-1}\Upsilon(\beta+\gamma)w^\nu(\beta+\gamma) \tag{3.24}$$

for a constant $c_7 > 0$. Applying Holder's inequality with Lemma 2.5 and Lemma 2.6, we obtain

$$\sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-l)^{\gamma-1}\Upsilon(\beta+\gamma)w^\nu(\beta+\gamma) = \mathcal{Q}^{1/l}\bigg(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)w^{\nu m}(\beta+\gamma)\bigg)^{1/m}, \tag{3.25}$$

where $\mathcal{Q} = \Gamma^l(1+\gamma)\Gamma(\kappa_1+1+\gamma l)/(\gamma^l\Gamma(1+\gamma l)\Gamma(\kappa_1+1))$. Using (3.27) in (3.26),

$$\begin{aligned}w(\kappa) &\leqslant c_7 + \frac{1}{\Gamma(\gamma)}\mathcal{Q}^{1/l}\bigg(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)w^{\nu m}(\beta+\gamma)\bigg)^{1/m}\\ &\leqslant c_7 + J\bigg(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)w^{\nu m}(\beta+\gamma)\bigg)^{1/m},\end{aligned}$$

$$w^m(\kappa) \leqslant \Bigg(c_7 + J\Bigg(\sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma) w^{\nu m}(\beta+\gamma)\Bigg)^{1/m}\Bigg)^m$$

$$\leqslant 2^{m-1}c_7^m + 2^{m-1}J^m \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma) w^{\nu m}(\beta+\gamma).$$

By setting $\mathcal{P}_3 = 2^{m-1}c_7^m$, $\mathcal{Q}_3 = 2^{m-1}\mathcal{J}^m$, the above inequality yields

$$w(\kappa) \leqslant \mathcal{P}_3 + \mathcal{Q}_3 \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma) w^{\nu m}(\beta+\gamma).$$

Applying Lemma 2.7 we have

$$w(\kappa) \leqslant \mathcal{P}_3 \exp\Bigg(\mathcal{Q}_3 \sum_{\beta=\kappa_1-\gamma}^{\kappa-\gamma} \Upsilon^m(\beta+\gamma)\Bigg).$$

From 3.24 we arrive at

$$\limsup_{\kappa\to\infty} w(\kappa) < \infty.$$

This completes the proof. □

4. Examples

Example 4.1. Consider the fractional difference equation

$$\Delta^{1/2}u(\kappa) = \mathrm{e}^{-2(\kappa+1/2)} + \frac{1}{1+(\kappa+\frac{1}{2})^2} w^3\Big(\kappa+\frac{1}{2}\Big) + \frac{\Gamma(\kappa+\frac{3}{2})}{\Gamma(\kappa+2)}\mathrm{e}^{-(\kappa+1/2)} w^{1/2}\Big(\kappa+\frac{1}{2}\Big) - \frac{\Gamma(\kappa+\frac{3}{2})}{\Gamma(\kappa+3)}\mathrm{e}^{-(\kappa+1/2)} w^{1/3}\Big(\kappa+\frac{1}{2}\Big), \quad \kappa\in\mathbb{N}_{1/2}. \tag{4.1}$$

In equation (4.1), $\gamma = \frac{1}{2}$, $\Phi(\kappa) = \mathrm{e}^{-2\kappa}$, $\Upsilon(\kappa) = 1/(1+\kappa^2)$, $\nu = 3$, $\Theta(\kappa, w) = \varphi(\kappa)w^\lambda = (\Gamma(\kappa+1)/\Gamma(\kappa+\frac{1}{2}))\mathrm{e}^{-\kappa}w^{1/2}$, $\Psi(\kappa, w) = \sigma(\kappa)w^\alpha = (\Gamma(\kappa+1)/\Gamma(\kappa+\frac{5}{2}))\times \mathrm{e}^{-\kappa}w^{1/3}$, $\varphi(\kappa) = \sigma(\kappa) = \mathrm{e}^{-\kappa}$, $\lambda = \frac{1}{2}$, $\alpha = \frac{1}{3}$, $\lambda > \alpha$. Also, set $l = \frac{1}{2}$, $m = l/(l-1) = -1$, $l(\gamma-1)+1 = \frac{3}{4} > 0$ and $\varrho(\kappa) = \kappa$. Now

$$\mathcal{R}(\kappa) = \sum_{\beta=1}^{\kappa-1}(\varrho(\beta))^{-1/\nu} = \sum_{\beta=1}^{\kappa-1}\beta^{-1/3} = \sum_{\beta=1}^{\kappa-1}\frac{1}{\beta^{1/3}}.$$

Also,

$$\lim_{\kappa\to\infty}\mathcal{R}(\kappa) = \lim_{\kappa\to\infty}\sum_{\beta=1}^{\kappa-1}\frac{1}{\beta^{1/3}} = \infty.$$

Moreover,

$$G(\kappa)=\frac{\lambda-\alpha}{\alpha}\Big(\frac{\alpha}{\lambda}\sigma(\kappa)\Big)^{\lambda/(\lambda-\alpha)}(\varphi(\kappa))^{\alpha/(\alpha-\lambda)}=\frac{\mathrm{e}^{-\kappa}}{436},$$

$$\sum_{\beta=1-\gamma}^{\kappa-\gamma}\Upsilon^m(\beta+\gamma)(\beta+\gamma)^m\mathcal{R}^{\nu m}(\beta+\gamma)$$
$$=\sum_{\beta=0}^{\kappa-1/2}\frac{1}{(\beta+\frac{1}{2})}\Big(1+\Big(\beta+\frac{1}{2}\Big)^2\Big)\sum_{\varepsilon=1}^{\beta-1}\Big(\varepsilon+\frac{1}{2}\Big)^{1/3}<\infty.$$

Furthermore,

$$\lim_{\kappa\to\infty}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Phi(\beta+\gamma)$$
$$=\sum_{\beta=0}^{\infty}(\kappa-\beta-1)^{-1/2}\mathrm{e}^{-2(\kappa+1/2)}=\sum_{\beta=0}^{\infty}\frac{\Gamma(\kappa-\beta)}{\Gamma(\kappa-\beta+\frac{1}{2})}\mathrm{e}^{-2\kappa-1}<\infty,$$

$$\lim_{\kappa\to\infty}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}G(\beta+\gamma)$$
$$=\sum_{\beta=0}^{\infty}(\kappa-\beta-1)^{-1/2}\frac{\mathrm{e}^{-(\beta+1/2)}}{432}=\frac{\mathrm{e}^{-1/2}}{432}\sum_{\beta=0}^{\infty}\frac{\Gamma(\kappa-\beta)}{\Gamma(\kappa-\beta+\frac{1}{2})}\mathrm{e}^{-\beta}<\infty.$$

Hence,

$$\limsup_{\kappa\to\infty}\frac{|w(\kappa)|}{\kappa^{1/\nu}\mathcal{R}(\kappa)}=\limsup_{\kappa\to\infty}\frac{|w(\kappa)|}{\kappa^{1/3}\sum_{\beta=1}^{\kappa-1}\beta^{-1/3}}<\infty.$$

Example 4.2. Consider the fractional difference equation

$$\begin{aligned}\Delta^{3/4}u(\kappa)&=\mathrm{e}^{-(\kappa+3/4)}+\frac{1}{1+(\kappa+\frac{3}{4})^3}w^3\Big(\kappa+\frac{3}{4}\Big)\\&+\frac{\Gamma(\kappa+\frac{7}{4})}{\Gamma(\kappa+2)}\mathrm{e}^{-(\kappa+3/4)}w^{1/4}\Big(\kappa+\frac{3}{4}\Big)\\&-\frac{\Gamma(\kappa+\frac{7}{4})}{\Gamma(\kappa+1)}\mathrm{e}^{-(\kappa+3/4)}w^{1/2}\Big(\kappa+\frac{3}{4}\Big),\quad \kappa\in\mathbb{N}_{1/4}.\end{aligned}\tag{4.2}$$

In equation (4.2), $\gamma=\frac{3}{4}$, $\Phi(\kappa)=\mathrm{e}^{-\kappa}$, $\Upsilon(\kappa)=1/(1+\kappa^3)$, $\nu=3$, $\Theta(\kappa,w)=\varphi(\kappa)w^\lambda=(\Gamma(\kappa+1)/\Gamma(\kappa+\frac{1}{4}))\mathrm{e}^{-\kappa}w^{1/2}$, $\Psi(\kappa,w)=\sigma(\kappa)w^\alpha=(\Gamma(\kappa+1)/\Gamma(\kappa+\frac{3}{4}))\times\mathrm{e}^{-\kappa}w^{1/4}$, $\varphi(\kappa)=\sigma(\kappa)=\mathrm{e}^{-\kappa}$, $\lambda=\frac{1}{2}$, $\alpha=\frac{1}{4}$, $\lambda>\alpha$. Also, set $l=2$, $m=2$, with $\varrho(\kappa)=\kappa^3$. Now

$$\mathcal{R}(\kappa)=\sum_{\beta=1}^{\kappa-1}\Big(\frac{1}{\beta^3}\Big)^{1/3}=\sum_{\beta=1}^{\infty}\frac{1}{\beta},\quad\lim_{\kappa\to\infty}\mathcal{R}(\kappa)=\sum_{\beta=1}^{\infty}\frac{1}{\beta}=\infty.$$

Moreover,

$$G(\kappa) = \frac{\lambda-\alpha}{\alpha}\Big(\frac{\alpha}{\lambda}\sigma(\kappa)\Big)^{\lambda/(\lambda-\alpha)}(b(\kappa))^{\alpha/(\alpha-\lambda)} = \frac{1}{8},$$

$$\sum_{\beta=1-\gamma}^{\kappa-\gamma}\Upsilon^m(\beta+\gamma)\mathcal{R}^{\nu m}(\beta+\gamma) = \sum_{\beta=0}^{\kappa-3/4}\frac{1}{1+(\beta+\frac{1}{4})^2}\sum_{\varepsilon=1}^{\beta-1}\frac{1}{(\varepsilon+\frac{3}{4})} < \infty.$$

Furthermore,

$$\lim_{\kappa\to\infty}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}\Phi(\beta+\gamma)$$

$$= \sum_{\beta=0}^{\infty}(\kappa-\beta-1)^{-1/4}\mathrm{e}^{-(\kappa+3/4)} = \sum_{\beta=0}^{\infty}\frac{\Gamma(\kappa-\beta)}{\Gamma(\kappa-\beta+\frac{1}{4})}\mathrm{e}^{-(\kappa+3/4)} < \infty,$$

$$\lim_{\kappa\to\infty}\sum_{\beta=1-\gamma}^{\kappa-\gamma}(\kappa-\beta-1)^{\gamma-1}G(\beta+\gamma)$$

$$= \sum_{\beta=0}^{\infty}(\kappa-\beta-1)^{-1/4}\frac{1}{8} = \frac{1}{8}\sum_{\beta=0}^{\infty}\frac{\Gamma(\kappa-\beta)}{\Gamma(\kappa-\beta+\frac{1}{4})} < \infty.$$

Hence,

$$\limsup_{\kappa\to\infty}\frac{|w(\kappa)|}{\mathcal{R}(\kappa)} = \limsup_{\kappa\to\infty}\frac{|w(\kappa)|}{\sum_{\beta=1}^{\kappa-1}\beta^{-1}} < \infty.$$

Example 4.3. Ion traps, combined effect of electric and magnetic fields in capturing ions, have a wide range of applications in physics, mass spectrometry and in controlling quantum states. There are different types of traps with most commonly used being Penning and Paul trap. Dynamic electric field is employed by Paul trap also known as quadrupole ion trap to trap the charged particles. A relatively strong influence on quadrupole ion traps is obtained with higher field imperfections. These influence by the higher field imperfections are described by beat-envelope equation. The motion of ion equation with octopole-only imperfections can be obtained as

$$\Delta^\gamma[\Delta(w(\kappa))] + 2q\cos(2\kappa)w(\kappa+\gamma) = -4q\alpha_4\cos(2\kappa)w(\kappa+\gamma)^3, \tag{4.3}$$

where $w(\kappa)$ represents the motion of ion, α_4 is the 4th field harmonic in comparison to the quadrupole field, q is a real parameter. This is the special case of discrete fractional nonlinear equation (1.1).

Consider the fractional difference equation (1.1) with $\gamma=\frac{1}{2}$, $\Phi(\kappa)=\Upsilon(\kappa)=0$, $\nu=1$, $\Theta(\kappa,w)=\varphi(\kappa)w^\lambda=4q\alpha_4\cos(2\kappa)w(\kappa+\gamma)^3$, $\Psi(\kappa,w)=\sigma(\kappa)w^\alpha=2q\cos(2\kappa)w(\kappa+\gamma)$, $\varphi(\kappa)=2\mathrm{e}^{-\kappa}\cos(2\kappa), \sigma(\kappa)=\cos(2\kappa)$, $q=\frac{1}{2}$, $\alpha_4=\mathrm{e}^{-\kappa}$, $\lambda=3$,

$\alpha = 1$, $\lambda > \alpha$. Also, set $l = \frac{1}{2}$, $m = l/(l-1) = -1$, $l(\gamma - 1) + 1 = \frac{3}{4} > 0$ and $\varrho(\kappa) = 1$. Now

$$\mathcal{R}(\kappa) = \sum_{\beta=1}^{\kappa-1} (\varrho(\beta))^{-1/\nu} = \sum_{\beta=1}^{\kappa-1} 1^{-1} = \sum_{\beta=1}^{\kappa-1} 1.$$

Also,

$$\lim_{\kappa\to\infty} \mathcal{R}(\kappa) = \lim_{\kappa\to\infty} \sum_{\beta=1}^{\kappa-1} 1 = \infty.$$

Moreover,

$$G(\kappa) = \frac{\lambda-\alpha}{\alpha}\Big(\frac{\alpha}{\lambda}\sigma(\kappa)\Big)^{\lambda/\lambda-\alpha}(\varphi(\kappa))^{\alpha/\alpha-\lambda} = \frac{1}{3}\sqrt{\frac{2}{3}}\frac{\cos(2\kappa)}{\sqrt{\mathrm{e}^{\kappa}}},$$

$$\sum_{\beta=1-\gamma}^{\kappa-\gamma} \Upsilon^{m}(\beta+\gamma)(\beta+\gamma)^{m}\mathcal{R}^{\nu m}(\beta+\gamma) = 0 < \infty.$$

Furthermore,

$$\lim_{\kappa\to\infty} \sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\gamma-1}\Phi(\beta+\gamma) = 0 < \infty,$$

$$\begin{aligned}\lim_{\kappa\to\infty} \sum_{\beta=1-\gamma}^{\kappa-\gamma} (\kappa-\beta-1)^{\gamma-1}G(\beta+\gamma) &= \sum_{\beta=0}^{\infty}(\kappa-\beta-1)^{-1/2}\frac{1}{3}\sqrt{\frac{2}{3}}\frac{\cos(2\beta+1)}{\sqrt{\mathrm{e}^{\beta+1/2}}}\\ &= \frac{1}{3}\sqrt{\frac{2}{3}}\sum_{\beta=0}^{\infty}\frac{\Gamma(\kappa-\beta)}{\Gamma(\kappa-\beta+\frac{1}{2})}\frac{\cos(2\beta+1)}{\sqrt{\mathrm{e}^{\beta+1/2}}} < \infty.\end{aligned}$$

Hence,

$$\limsup_{\kappa\to\infty} \frac{|w(\kappa)|}{\kappa^{1/\nu}\mathcal{R}(\kappa)} = \limsup_{\kappa\to\infty} \frac{|w(\kappa)|}{\kappa\sum_{\beta=1}^{\kappa-1} 1} < \infty.$$

5. Conclusion

The arrival of discrete fractional calculus in the framework of mathematical modelling has provided researchers with new ideas to model systems with discrete-time features and memory effects, which are quite common in real world scenario. The main difficulty that we have faced lies in putting the main problem within the discrete fractional settings and adopting the terminologies and corresponding definitions to provide a correct platform. Furthermore, getting inequality (3.15) was challenging. To proceed with this, we applied some fundamental techniques as well as mathematical inequalities such as Holders, Youngs and Discrete Gronwalls inequalities to analyse the results. To ensure the validity of theoretical results, three numerical examples are presented.

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