MULTIPICLITY OF POSITIVE SOLUTIONS FOR SECOND ORDER QUASILINEAR EQUATIONS

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Received April 20, 2018. Published online June 13, 2019. Communicated by Jiří Sremr

Abstract. We discuss the existence and multiplicity of positive solutions for a class of second order quasilinear equations. To obtain our results we will use the Ekeland variational principle and the Mountain Pass Theorem.

Keywords: critical point; Ekeland variational principle; Mountain Pass Theorem; Palais-Smale condition; positive solution

MSC 2010: 35A15, 35B38, 30E25, 58E30, 49K35

1. Introduction

Our aim in this paper is to obtain at least two positive solutions for the problem

\[\begin{align*}
-u'' + u &= \lambda h(x)|u|^{\beta-2}u + q(x)f(u), \quad x \in (0, \infty), \\
u(0) &= u(\infty) = 0,
\end{align*}\]

where \(f \in C(\mathbb{R}, \mathbb{R})\), \(\beta\) and \(\lambda\) are real parameters with \(1 < \beta < 2\) and \(\lambda > 0\).

Throughout this paper we assume the following hypotheses are satisfied:

(H\(_0\)) \(h\) and \(q\): \([0, \infty) \to (0, \infty)\) belong to \(L^1(0, \infty) \cap L^\infty(0, \infty)\);

(H\(_1\)) there is a continuously differentiable and bounded function \(p\): \([0, \infty) \to (0, \infty)\) belonging to \(L^1(0, \infty) \cap L^\infty(0, \infty)\) such that the functions \(q/p\), \(q/p^2\), \(q/p^\beta\), \(q/p^{\beta+1}\), \(h/p^{\beta-1}\) and \(h/p^\beta\) all belong to \(L^1(0, \infty)\);

(H\(_2\)) \(M = \max(\|p\|_{L^2}, \|p'\|_{L^2}) < \infty\),

\[M_{r,g} = \|p\|^{1/2}_{L^\infty} \left( \int_0^\infty g(x) \left( \int_0^x \frac{ds}{p(s)} \right)^{r/2} \ dx \right)^{1/r} < \infty\]

DOI: 10.21136/MB.2019.0051-18
for all $r \in \{\beta, 2, \beta + 1\}$ and all $g \in \{q, h\}$ and

$$M_{2,q} = \| p \|_{\infty}^{1/2} \left( \int_0^\infty q(x) \left( \int_0^x \frac{ds}{p(s)} \right) dx \right)^{1/2} < \frac{1}{\sqrt{A}},$$

where the constant $A$ satisfies

$$(H_3) \quad \lim_{u \to 0^+} f(u)/|u| = A \in (0, \lambda_{2,q}^2) \quad \text{and} \quad \lim_{u \to \infty} f(u)/|u|^\beta = B \in (\lambda_{2,q}^2, \infty),$$

where $\lambda_{2,q}$ is the first eigenvalue of problem (2) which is defined in Lemma 1.3;

$$(H_4) \quad \text{there exists } \mu > \beta + 1 \text{ such that }$$

$$F(s) \leq \frac{1}{\mu} s f(s) \quad \forall |s| > 0,$$

where $F(s) = \int_0^s f(t) \, dt$.

Now we introduce the Hilbert space $H_0^1(0, \infty)$ which is suitable for the study of our problem. Let

$$H_0^1(0, \infty) = \{ u \text{ measurable: } u, u' \in L^2(0, \infty), \ u(0) = u(\infty) = 0 \}$$

equipped with the norm

$$\| u \| = \left( \int_0^\infty |u'(x)|^2 \, dx + \int_0^\infty |u(x)|^2 \, dx \right)^{1/2}$$

and endowed with the inner product

$$(u, v) = \int_0^\infty u'(x) \cdot v'(x) \, dx + \int_0^\infty u(x) \cdot v(x) \, dx.$$

We consider the spaces $L_g^r(0, \infty)$ which are defined by

$$L_g^r(0, \infty) = \left\{ u: (0, \infty) \to \mathbb{R} \text{ measurable such that } \int_0^\infty g(x)|u(x)|^r \, dx < \infty \right\}$$

for all $r \in \{\beta, 2, \beta + 1\}$ and all $g \in \{h, q\}$ equipped, respectively, with the norms

$$\| u \|_{r,g} = \left( \int_0^\infty g(x)|u(x)|^r \, dx \right)^{1/r}.$$

Let the space $C_{l,p}[0, \infty)$ be defined by

$$C_{l,p}[0, \infty) = \left\{ u \in C([0, \infty), \mathbb{R}) : \lim_{x \to \infty} p(x)u(x) \text{ exists} \right\}.$$
The corresponding norm is defined by

\[ \|u\|_{\infty,p} = \sup_{x \in [0, \infty)} p(x)|u(x)|. \]

Now we give some necessary lemmas and corollaries, which are used below.

**Lemma 1.1** ([5]). \( H^1_0(0, \infty) \) embeds continuously and compactly in \( C_{l,p}[0, \infty) \), i.e.

\[ \|u\|_{\infty,p} \leq \sqrt{2} M \|u\| \quad \forall \ u \in H^1_0(0, \infty). \]

**Lemma 1.2** ([2]). \( C_{l,p}[0, \infty) \) is continuously embedded in \( L^{r_g}(0, \infty) \) for all \( r \in \{\beta, 2, \beta + 1\} \) and all \( g \in \{h, q\} \).

**Corollary 1.1** ([2]). \( H^1_0(0, \infty) \) embeds continuously and compactly in \( L^{r_g}(0, \infty) \) with the embedding constant \( M_{r,g} \).

Let \( \overline{\lambda}_{r,g} \) be the first eigenvalue of the problem

\[
\begin{cases}
-u''(x) + u(x) = \lambda g(x)|u(x)|^{r-2}u(x), & x > 0, \\
u(0) = u(\infty) = 0,
\end{cases}
\]

and note

\[ \overline{\lambda}_{r,g} = \inf_{u \in H^1_0 \setminus \{0\}} \frac{\|u\|}{\|u\|_{r,g}}. \]

**Lemma 1.3** ([2]). The first eigenvalue \( \overline{\lambda}_{r,g} \) is positive and is achieved for some positive function \( \psi_{r,g} \in H^1_0(0, \infty) \setminus \{0\} \), i.e.

\[ \overline{\lambda}_{r,g} := \inf_{u \in H^1_0 \setminus \{0\}} \frac{\|u\|}{\|u\|_{r,g}} = \frac{\|\psi_{r,g}\|}{\|\psi_{r,g}\|_{r,g}}. \]

**Theorem 1.1** ([4], Weak Ekeland variational principle). Let \( (E, d) \) be a complete metric space and let \( J : E \to \mathbb{R} \) be a functional that is lower semi-continuous and bounded from below. Then for each \( \varepsilon > 0 \) there exists \( u_\varepsilon \in E \) with

\[ J(u_\varepsilon) \leq \inf_E J + \varepsilon, \]

and whenever \( w \in E \) with \( w \neq u_\varepsilon \), then

\[ J(u_\varepsilon) < J(w) + \varepsilon d(u_\varepsilon, w). \]
Definition 1.1 ([6]). Let $E$ be a Banach space and $J: E \to \mathbb{R}$ a $C^1$-functional and $c \in \mathbb{R}$. The functional $J$ is said to satisfy the (local) Palais-Smale condition at the level $c$, denoted by $(P.S)_c$, if any sequence $(u_n)$ in $E$ such that

(3) \[J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0,\]

admits a convergent subsequence.

Lemma 1.4 (Mountain Pass Theorem). Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$ with $J(0) = 0$. Suppose $J(u)$ satisfies $(P.S)_c$ condition and

(a) there are $\varrho, \alpha > 0$ such that $J(u) \geq \alpha$ when $\|u\|_E = \varrho$,
(b) there is a $e \in E$, $\|e\|_E > \varrho$ such that $J(e) < 0$.

Define

(4) \[\Gamma = \{ \gamma \in C^1([0, 1], E) : \gamma(0) = 0, \gamma(1) = e \}.\]

Then

(5) \[c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) \geq \alpha\]

is a critical value of $J(u)$.

2. MAIN EXISTENCE RESULTS

Now we define the Euler-Lagrange functional associated to problem (1). Let $J_\lambda: H_0^1(0, \infty) \to \mathbb{R}$ be defined by

(6) \[J_\lambda(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta(x) \, dx - \int_0^\infty q(x)F(u) \, dx.\]

Proposition 2.1. Suppose that the conditions $(H_0)$–$(H_3)$ hold. Then the functional $J_\lambda$ is continuously differentiable. The Fréchet derivative of $J_\lambda$ has the form

(7) \[\langle J_\lambda'(u), v \rangle = \int_0^\infty u'(x)v'(x) \, dx + \int_0^\infty u(x)v(x) \, dx - \lambda \int_0^\infty h(x)|u|^\beta-2(x)u(x)v(x) \, dx - \int_0^\infty q(x)f(u)v(x) \, dx\]

for all $v \in H_0^1(0, \infty)$. 

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Proof. The proof of the proposition will be done consecutively.

Claim 2.1. $J_{\lambda}$ is Gâteaux-differentiable.

For all $v \in H^1_0(0, \infty)$ and for any $t > 0$ we have

\[
J_{\lambda}(u + tv) - J_{\lambda}(u) = \frac{1}{2} \int_0^\infty (u + tv)'^2 \, dx + \frac{1}{2} \int_0^\infty |u + tv|^2 \, dx - \frac{\lambda}{\beta} \int_0^\infty h(x)|u + tv|^\beta \, dx
\]

\[
- \int_0^\infty q(x)F(u + tv) \, dx - \frac{1}{2} \int_0^\infty |u'|^2 \, dx - \frac{1}{2} \int_0^\infty |u|^2 \, dx
\]

\[
+ \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta \, dx + \int_0^\infty q(x)F(u) \, dx
\]

\[
= \frac{t^2}{2} \int_0^\infty |v'|^2 \, dx + t \int_0^\infty u'v' \, dx + \frac{t^2}{2} \int_0^\infty |v|^2 \, dx + t \int_0^\infty uv \, dx
\]

\[
- \lambda \int_0^\infty h(x)(|u + tv|^\beta - |u|^\beta) \, dx - \int_0^\infty q(x)(F(u + tv) - F(u)) \, dx
\]

\[
= \frac{t^2}{2} \int_0^\infty |v'|^2 \, dx + t \int_0^\infty u'v' \, dx + \frac{t^2}{2} \int_0^\infty |v|^2 \, dx + t \int_0^\infty uv \, dx
\]

\[
- \lambda \int_0^\infty h(x)|u + tv|^\beta - 2(u + t\theta v)v \, dx - t \int_0^\infty q(x)f(u + t\theta v)v \, dx,
\]

where $0 < \theta < 1$, and then

\[
\frac{J_{\lambda}(u + tv) - J_{\lambda}(u)}{t} = \frac{t}{2} \int_0^\infty |v'|^2 \, dx + \int_0^\infty u'v' \, dx + \frac{t}{2} \int_0^\infty |v|^2 \, dx
\]

\[
+ \int_0^\infty uv \, dx - \lambda \int_0^\infty h(x)|u + t\theta v|^\beta - 2(u + t\theta v)v \, dx
\]

\[
- \int_0^\infty q(x)f(u + t\theta v)v \, dx.
\]

Let $t \to 0$ and we have

\[
\langle J'_{\lambda}(u), v \rangle = \int_0^\infty u'v' \, dx + \int_0^\infty uv \, dx - \lambda \int_0^\infty h(x)|u|^{\beta - 2}uv \, dx - \int_0^\infty q(x)f(u)v \, dx
\]

for all $v \in H^1_0(0, \infty)$.

Claim 2.2. $J'_{\lambda}$ is continuous.

Let $(u_n) \subset H^1_0(0, \infty)$ with $u_n \to u$ when $n \to \infty$, so there exists $R > 0$ such that $\|u_n\| \leq R$ for all $n \in \mathbb{N}$. 

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From (H₃), given \( \varepsilon \) small enough, there exists \( \delta_2 > \delta_1 > 0 \) such that

\[
(A - \varepsilon)|s| < f(s) < (A + \varepsilon)|s| \quad \forall 0 < s < \delta_1
\]

and

\[
(B - \varepsilon)|s|^{\beta} < f(s) < (B + \varepsilon)|s|^{\beta} \quad \forall s > \delta_2,
\]

so from (8) and (9) and since \( f(u) \) is continuous on \([\delta_1, \delta_2]\), there exists \( D_1 > 0 \) such that

\[
-D_1 + (A - \varepsilon)|s| + (B - \varepsilon)|s|^{\beta} < f(s) < D_1 + (A + \varepsilon)|s| + (B + \varepsilon)|s|^{\beta}
\]

for all \( s \in (0, \infty) \). This yields

\[
F(s) \leq D_2 + \frac{A + \varepsilon}{2} s^2 + \frac{B + \varepsilon}{\beta} |s|^{\beta+1} \quad \forall s \in (0, \infty)
\]

and

\[
F(s) \geq -D_2 + \frac{A - \varepsilon}{2} s^2 + \frac{B - \varepsilon}{\beta} |s|^{\beta+1} \quad \forall s \in (0, \infty),
\]

where \( D_2 = D_1(\delta_2 - \delta_1) \). Furthermore, from Lemma 1.1, \((H_0)\)–\((H_1)\) and (10) we obtain

\[
q(x)|f(u_n(x))| \leq (A + \varepsilon)q(x)|u_n(x)| + (B + \varepsilon)q(x)|u_n(x)|^{\beta} + D_1q(x)
\]

\[
\leq (A + \varepsilon) \sup_{x \in [0, \infty)} |(pu_n)(x)| \frac{q(x)}{p(x)}
\]

\[
+ (B + \varepsilon) \sup_{x \in [0, \infty)} |(pu_n)(x)|^{\beta} \frac{q(x)}{p^{\beta}(x)} + D_1q(x)
\]

\[
= (A + \varepsilon)\|u_n\|_{\infty,p} \frac{q(x)}{p(x)} + (B + \varepsilon)\|u_n\|_{\infty,p}^{\beta} \frac{q(x)}{p^{\beta}(x)} + D_1q(x)
\]

\[
\leq (A + \varepsilon)\sqrt{2MR} \frac{q(x)}{p(x)} + (B + \varepsilon)(\sqrt{2MR})^{\beta} \frac{q(x)}{p^{\beta}(x)} + D_1q(x) \in L^1(0, \infty)
\]

and

\[
h(x)|u_n(x)|^{\beta-2}|u_n(x)| \leq h(x)|u_n(x)|^{\beta-1} = p^{\beta-1}(x)|u_n(x)|^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)}
\]

\[
\leq \sup_{x \in [0, \infty)} |(pu_n)(x)|^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)} = \|u_n\|_{\infty,p}^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)}
\]

\[
\leq (\sqrt{2MR})^{\beta-1} \frac{h(x)}{p^{\beta-1}(x)} \in L^1(0, \infty).
\]
Then from the Lebesgue dominated convergence theorem we obtain

\begin{equation}
\lim_{n \to \infty} \int_0^\infty q(x) f(u_n(x)) \, dx = \int_0^\infty q(x) f(u(x)) \, dx \tag{13}
\end{equation}

and also

\begin{equation}
\lim_{n \to \infty} \int_0^\infty h(x)|u_n|^{\beta-2}(x)u_n(x) \, dx = \int_0^\infty h(x)|u|^{\beta-2}(x)u(x) \, dx. \tag{14}
\end{equation}

Thus we have

\begin{equation}
\langle J'_\lambda(u_n) - J'_\lambda(u), v \rangle = \int_0^\infty u'_n v' \, dx + \int_0^\infty u_n v \, dx - \lambda \int_0^\infty h(x)|u_n|^{\beta-2}u_n v \, dx \\
- \int_0^\infty q(x)f(u_n)v \, dx - \int_0^\infty u'v' \, dx - \int_0^\infty uv \, dx \\
+ \lambda \int_0^\infty h(x)|u|^{\beta-2}uv \, dx + \int_0^\infty q(x)f(u)v \, dx \\
= \int_0^\infty (u'_n - u')v' \, dx + \int_0^\infty (u_n - u)v \, dx \\
- \lambda \int_0^\infty h(x)(|u_n|^{\beta-2}u_n - |u|^{\beta-2}u)v \, dx \\
- \int_0^\infty q(x)(f(u_n) - f(u))v \, dx, \tag{15}
\end{equation}

and from (13), (14) and the continuity of \( f \), passing to the limit in \( \langle J'_\lambda(u_n) - J'_\lambda(u), v \rangle \) when \( n \to \infty \), we obtain that \( J'_\lambda(u_n) \to J'_\lambda(u) \) as \( n \to \infty \). \( \square \)

**Definition 2.1.** We say that \( u \in H^1_0(0, \infty) \) is a weak solution of problem (1) if for any \( v \in H^1_0(0, \infty) \) we have

\begin{equation}
\langle J'_\lambda(u), v \rangle = \int_0^\infty u' v' \, dx + \int_0^\infty uv \, dx - \lambda \int_0^\infty h(x)|u|^{\beta-2}uv \, dx \\
- \int_0^\infty q(x)f(u)v \, dx = 0.
\end{equation}

**Remark 2.1.** Since the nonlinear term \( f \) is continuous, then a weak solution of problem (1) is a classical solution.

In our next two sections we will prove the main result of this paper.

**Theorem 2.1.** Suppose that (H\(_0\))–(H\(_4\)) hold. Then there exists \( \xi > 0 \) such that for \( 0 < \lambda < \xi \), problem (1) has at least two positive solutions.

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2.1. Existence of a first solution.

**Lemma 2.1.** Suppose that the hypotheses (H$_0$)--(H$_4$) hold. Then there exists $\xi_1 > 0$ such that for $0 < \lambda \leq \xi_1$, the functional $J_{\lambda}$ satisfies the geometric conditions (a) and (b) in Lemma 1.4, i.e.

(a) there are $\varrho, \alpha > 0$ such that $J_{\lambda}(u) \geq \alpha$ when $\|u\| = \varrho$,
(b) there is $e \in H^1_0(0, \infty)$, $\|e\| > \varrho$ such that $J_{\lambda}(e) < 0$.

**Proof.** (a) From (H$_0$)--(H$_3$), (11) and using Corollary 1.1, we have

\[ J_{\lambda}(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta(x) \, dx - \int_0^\infty q(x)F(u) \, dx \]
\[ \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta(x) \, dx - D_2 \int_0^\infty q(x) \, dx \]
\[ - \frac{A + \varepsilon}{2} \int_0^\infty q(x)|u|^2 \, dx - \frac{B + \varepsilon}{\beta + 1} \int_0^\infty q(x)|u|^\beta \, dx \]
\[ \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta(x) \, dx \]
\[ \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{\beta} \int_0^\infty h(x)|u|^\beta(x) \, dx \]
\[ - \frac{1}{\beta + 1} \|A + \varepsilon M_{\beta,h}^{2,q}\|u\|^\beta + 1 \]
\[ - \frac{B + \varepsilon}{\beta + 1} \|M_{\beta+1,h}^{2,q}\|u\|^\beta + 1 \]
\[ \geq \|u\|^2 \left( \frac{1}{2} (1 - (A + \varepsilon)M_{2,q}^2) - \frac{\lambda}{\beta} M_{\beta,h}^{2,q} \|u\|^\beta - \frac{B + \varepsilon}{\beta + 1} M_{\beta+1,h}^{2,q} \|u\|^\beta \right) \]
\[ - D_2 \|q\|_{L^1} \]
\[ \geq \|u\|^2 \left( \frac{1}{2} (1 - (A + \varepsilon)M_{2,q}^2) - \lambda K_1 \|u\|^\beta \right) - K_3, \]

where $K_1 = \beta^{-1}M_{\beta,h}^{\beta}, K_2 = ((B + \varepsilon)/(\beta + 1))M_{\beta+1,h}^{\beta+1}$ and $K_3 = D_2 \|q\|_{L^1};$ here $\varepsilon$ and $D_2$ are given in the proof of Proposition 2.1. Let

\[ g(t) = \lambda K_1 t^\beta - 2 + K_2 t^\beta - 1 \quad \text{for } t \geq 0. \]

Clearly,

\[ g'(t) = \lambda K_1 (\beta - 2) t^{\beta - 3} + K_2 (\beta - 1) t^{\beta - 2} \quad \text{for } t \geq 0. \]

From $g'(t_0) = 0$ we have

\[ t_0 = \frac{\lambda K_1 (2 - \beta)}{K_2 (\beta - 1)}. \]
Then
\[ g(t_0) = \frac{2\lambda^{\beta-1}K_1^{\beta-1}}{(\beta-1)K_2^{\beta-2}}. \]
Thus, there exists \( 0 < \xi_1 < (\frac{(\beta-1)K_2}{4K_1}(1 - (A + \varepsilon)M_{2,q}^2))^{1/\beta-1} \)
such that
\[ g(t_0) < \frac{1}{2}(1 - (A + \varepsilon)M_{2,q}^2) \quad \forall 0 < \lambda \leq \xi_1. \]

Consequently, taking \( \varrho = t_0 \) and choosing \( \lambda \in (0, \xi_1) \) such that
\[ m_0 = \varrho^2\left(\frac{1}{2}(1 - (A + \varepsilon)M_{2,q}^2) - \lambda K_1 \varrho^{\beta-2} - K_2 \varrho^{\beta-1}\right) > K_3, \]
from (16) we have
\[ J_{\lambda}(u) \geq \alpha > 0 \quad \text{when} \quad \|u\| = \varrho, \]
where \( \alpha = m_0 - K_3. \) Thus (a) is proved.

(b) For \( t > 0 \) large enough, from (12) and Lemma 1.3 we have
\[
J_{\lambda}(t \overline{\psi}_{\beta+1,q})
= \frac{1}{2} t^2 \|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta} t^\beta \int_0^\infty h(x) |\overline{\psi}_{\beta+1,q}|^\beta \, dx - \int_0^\infty q(x) F(t \overline{\psi}_{\beta+1,q}) \, dx
\leq \frac{1}{2} t^2 \|\overline{\psi}_{\beta+1,q}\|^2 - \frac{\lambda}{\beta} t^\beta \int_0^\infty h(x) |\overline{\psi}_{\beta+1,q}|^\beta \, dx - \frac{A - \varepsilon}{2} t^2 \int_0^\infty q(x) |\overline{\psi}_{\beta+1,q}|^2 \, dx
- \frac{B - \varepsilon}{\beta+1} t^{\beta+1} \int_0^\infty q(x) |\overline{\psi}_{\beta+1,q}|^{\beta+1} \, dx + D_2 \|q\|_{L^1}
\leq \frac{1}{2} \|\overline{\psi}_{\beta+1,q}\|^2 - (A - \varepsilon) \|\overline{\psi}_{\beta+1,q}\|_{\beta+1} + B - \varepsilon t^{\beta+1} \|\psi_{\beta+1,q}\|_{\beta+1} + D_2 \|q\|_{L^1}

\]
Therefore \( J_{\lambda}(t \overline{\psi}_{\beta+1,q}) \rightarrow -\infty \) as \( t \rightarrow \infty. \) Choose \( t_1 > 0 \) large enough and \( e = t_1 \overline{\psi}_{\beta+1,q}. \) Hence, we conclude that
\[ J_{\lambda}(e) < 0 \quad \text{when} \quad \|e\| > \varrho. \]
Thus (b) is proved. \( \square \)

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From a version of the Mountain Pass Theorem without the Palais-Smale condition (see [7]), there exists a (P.S)\(_c\) sequence \((u_n)\subset H^1_0(0, \infty)\) for \(J_\lambda\) which satisfies (3), i.e.
\[
J_\lambda(u_n) \to c \quad \text{and} \quad J'_\lambda(u_n) \to 0,
\]
where
\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t))
\]
with
\[
\Gamma = \{ \gamma \in C([0,1], H^1_0(0, \infty)) : \gamma(0) = 0, \gamma(1) = e \},
\]
where \(e\) is given in Lemma 2.1.

**Remark 2.2.** Since the sequence \((u^+_n)\) also satisfies (3) (see [1], Lemma 1), we assume, without of loss generality, that \(u_n \geq 0\) for all \(n \in \mathbb{N}\).

**Lemma 2.2.** Suppose that the hypotheses (H\(_0\))–(H\(_4\)) hold. Then the mountain level \(c\) satisfies the following inequality:
\[
c < \left( \frac{\lambda^{\beta+1}}{B - \varepsilon} \right)^{2/(\beta-1)} \left( \frac{1}{2} - \frac{1}{\mu} \right) + K_3;
\]
here \(K_3\) is given in the proof of Lemma 2.1.

**Proof.** From the proof of Lemma 2.1 we can consider \(\gamma(t) = t_{\delta} \overline{\psi}_{\beta+1,q}\), where \(t_{\delta} > 0\) is sufficiently large such that \(e = t_{\delta} \overline{\psi}_{\beta+1,q}\). Thus, from the definition of \(c\),
\[
c \leq \max_{t \geq 0} J_\lambda(t \overline{\psi}_{\beta+1,q}),
\]
that is,
\[
c \leq \max_{t \geq 0} \left\{ \frac{1}{2} t^2 \| \overline{\psi}_{\beta+1,q} \|^2 - \frac{\lambda}{\beta} t^\beta \| \overline{\psi}_{\beta+1,q} \|_{\beta,h}^\beta - \int_0^\infty q(x) F(t \overline{\psi}_{\beta+1,q}) \, dx \right\}.
\]
From (12),
\[
c \leq \max_{t \geq 0} \left\{ \frac{1}{2} t^2 \| \overline{\psi}_{\beta+1,q} \|^2 - \frac{\lambda}{\beta} t^\beta \| \overline{\psi}_{\beta+1,q} \|_{\beta,h}^\beta - \frac{A - \varepsilon}{2} t^2 \| \overline{\psi}_{\beta+1,q} \|_{2,q}^2 - \frac{B - \varepsilon}{\beta + 1} t^{\beta+1} \| \overline{\psi}_{\beta+1,q} \|_{\beta+1, q}^{\beta+1} + K_3 \right\}
\]
\[
\leq \max_{t \geq 0} \left\{ \frac{1}{2} t^2 \| \overline{\psi}_{\beta+1,q} \|^2 - \frac{B - \varepsilon}{\beta + 1} t^{\beta+1} \| \overline{\psi}_{\beta+1,q} \|_{\beta+1, q}^{\beta+1} \right\} + K_3,
\]
and then
\[
\frac{c}{\|\psi_{\beta+1,q}\|_{\beta+1,q}^2} \leq \max_{t \geq 0} \left\{ \frac{\lambda_{\beta+1,q}^2}{2} t^2 - \frac{B - \varepsilon}{\beta + 1} \|\psi_{\beta+1,q}\|_{\beta+1,q}^{\beta-1} t^{\beta+1} \right\} + \frac{K_3}{\|\psi_{\beta+1,q}\|_{\beta+1,q}^2}.
\]

Let
\[
Z(t) = \frac{\lambda_{\beta+1,q}^2}{2} t^2 - \frac{B - \varepsilon}{\beta + 1} \|\psi_{\beta+1,q}\|_{\beta+1,q}^{\beta-1} t^{\beta+1}.
\]

Clearly,
\[
Z'(t) = \lambda_{\beta+1,q}^2 t - (B - \varepsilon)\|\psi_{\beta+1,q}\|_{\beta+1,q}^{\beta-1} t^\beta.
\]

Since the function \(Z\) attains its maximum at
\[
t = \left( \frac{\lambda_{\beta+1,q}^2}{(B - \varepsilon)\|\psi_{\beta+1,q}\|_{\beta+1,q}^{\beta-1}} \right)^{1/(\beta-1)},
\]

it follows that
\[
c < \left( \frac{\lambda_{\beta+1,q}^2}{B - \varepsilon} \right)^{2/(\beta-1)} \left( \frac{1}{2} - \frac{1}{\beta + 1} \right) + K_3,
\]

and therefore we have
\[
c < \left( \frac{\lambda_{\beta+1,q}^2}{B - \varepsilon} \right)^{2/(\beta-1)} \left( \frac{1}{2} - \frac{1}{\mu} \right) + K_3.
\]

\[\square\]

**Lemma 2.3.** There exists \(\xi_2 > 0\) such that for \(0 < \lambda < \xi_2\), the Palais-Smale sequence \((u_n)\) associated with the functional \(J_\lambda\) satisfies
\[
\limsup_{n \to \infty} \|u_n\|^2 < 2 \left( \frac{\lambda_{\beta+1,q}^2}{B - \varepsilon} \right)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/2 - \beta} + \frac{4K_3\mu}{\mu - 2}.
\]

**Proof.** First, observe that \((u_n)\) is bounded in \(H_0^1(0, \infty)\). In fact, from (3)
\[
J_\lambda(u_n) \to c \quad \text{and} \quad \langle J'_\lambda(u_n), u_n \rangle \to 0 \quad \text{as} \quad n \to \infty.
\]

Notice that from (7) we have
\[
\int_0^\infty q(x) f(u_n) u_n \, dx = \|u_n\|^2 - \lambda \|u_n\|_{\beta,h}^\beta - \langle J'_\lambda(u_n), u_n \rangle.
\]
Using Corollary 1.1 and (H₄), it follows from (3) that

$$(18) \quad c + \varepsilon > J_\lambda(u_n) = \frac{1}{2} \| u_n \|^2 - \frac{\lambda}{\beta} \| u_n \|_{\beta,h}^\beta - \int_0^\infty q(x) F(u_n) \, dx$$

$$\geq \int_0^\infty q(x) f(u_n) u_n \, dx$$

Choosing $\mu$, we get

$$\int_0^\infty q(x) f(u_n) u_n \, dx \geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \| u_n \|^2 - \lambda \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \| u_n \|_{\beta,h}^\beta + \frac{1}{\mu} \| J'_\lambda(u_n) \| u_n \|.$$

This implies that

$$\| u_n \| \geq \lambda M_{\beta,h} \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \| u_n \|_{\beta,h}^\beta + o_n(1) \| u_n \| + c + \varepsilon,$$

and then we have

$$(\| u_n \|^2 - \lambda M_{\beta,h}^\beta \| u_n \|^\beta) + o_n(1) \| u_n \| + c + \varepsilon.$$
Thus
\[
\limsup_{n \to \infty} \|u_n\|^2 \leq \left( \frac{1}{2} \left( \frac{1}{2} - \frac{1}{\mu} \right) \right)^{-1} \left( \frac{1}{2} - \frac{1}{\mu} \right) \left( \frac{\lambda_{\beta+1,q}}{B - \varepsilon} \right)^{2/(\beta-1)} + K_3 + \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \frac{2 - \beta}{2} M_{\beta,h}^{2\beta/(2-\beta)} \\
< 2 \left( \frac{\lambda_{\beta+1,q}}{B - \varepsilon} \right)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/2-\beta} + \frac{4K_3\mu}{\mu - 2}.
\]

\square

Since \((u_n)\) satisfying (3) is bounded in \(H_0^1(0, \infty)\) (see Lemma 2.3), there exists \(u_1 \in H_0^1(0, \infty)\) such that for a subsequence we have

\[(21) \quad u_n \rightharpoonup u_1 \quad \text{in} \quad H_0^1(0, \infty),\]
\[(22) \quad u_n \to u_1 \quad \text{in} \quad L^r_\beta(0, \infty)\]
for all \(r \in \{\beta, 2, \beta + 1\}\) and all \(g \in \{h, q\}\) and

\[(23) \quad u_n(x) \to u_1(x) \quad \text{a.e. in} \quad (0, \infty).\]

In the next lemma we obtain some convergences results involving the sequence \((u_n)\) and its weak limit \(u_1\).

**Lemma 2.4.** The following limits are satisfied:

(c) \(\int_0^\infty q(x)|f(u_n) - f(u_1)| |u_n - u_1| \, dx = o_n(1),\)

(d) \(\int_0^\infty h(x)||u_n|^{\beta-2}u_n - |u_1|^{\beta-2}u_1|| |u_n - u_1| \, dx = o_n(1).\)

**Proof.** (c) From (10) and using Corollary 1.1 and Lemmas 2.2 and 2.3, we obtain

\[
\int_0^\infty q(x)|f(u_n) - f(u_1)||u_n - u_1| \, dx \\
\leq \int_0^\infty q(x)|f(u_n)||u_n - u_1| \, dx + \int_0^\infty q(x)|f(u_1)||u_n - u_1| \, dx \\
\leq 2D_1 \int_0^\infty q(x)|u_n - u_1| \, dx \\
+ (A + \varepsilon) \int_0^\infty q(x)|u_n||u_n - u_1| \, dx + (A + \varepsilon) \int_0^\infty q(x)|u_1||u_n - u_1| \, dx \\
+ (B + \varepsilon) \int_0^\infty q(x)|u_n|^\beta|u_n - u_1| \, dx + (B + \varepsilon) \int_0^\infty q(x)|u_1|^\beta|u_n - u_1| \, dx
\]

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and using the Cauchy-Schwarz inequality, we have

\[
\int_0^\infty q(x)|f(u_n) - f(u_1)||u_n - u_1| \, dx \\
\leq 2D_1 \left( \int_0^\infty q(x) \, dx \right)^{1/2} \left( \int_0^\infty q(x)|u_n - u_1|^2 \, dx \right)^{1/2} \\
+ (A + \varepsilon) \left( \int_0^\infty q(x)|u_n|^2 \, dx \right)^{1/2} \left( \int_0^\infty q(x)|u_n - u_1|^2 \, dx \right)^{1/2} \\
+ (A + \varepsilon) \left( \int_0^\infty q(x)|u_1|^2 \, dx \right)^{1/2} \left( \int_0^\infty q(x)|u_n - u_1|^2 \, dx \right)^{1/2} \\
+ (B + \varepsilon) \left( \int_0^\infty q(x)|u_n|^{\beta} \, dx \right)^{(\beta - 1)/\beta} \left( \int_0^\infty q(x)|u_n - u_1|^{\beta} \, dx \right)^{1/\beta} \\
+ (B + \varepsilon) \left( \int_0^\infty q(x)|u_1|^{\beta} \, dx \right)^{(\beta - 1)/\beta} \left( \int_0^\infty q(x)|u_n - u_1|^{\beta} \, dx \right)^{1/\beta}.
\]

Thus,

\[
\int_0^\infty q(x)|f(u_n) - f(u_1)||u_n - u_1| \, dx \\
\leq 2D_1 \|q\|_{L^1} \|u_n - u_1\|_{2,q} + (A + \varepsilon) \|u_n\|_{2,q} \|u_n - u_1\|_{2,q} \\
+ (A + \varepsilon) \|u_1\|_{2,q} \|u_n - u_1\|_{2,q} + (B + \varepsilon) \|u_n\|_{\beta,q} \|u_n - u_1\|_{\beta,q} \\
+ (B + \varepsilon) \|u_1\|_{\beta,q} \|u_n - u_1\|_{\beta,q} \\
\leq C_{1,\varepsilon} \|u_n - u_1\|_{2,q} + C_{2,\varepsilon} \|u_n - u_1\|_{\beta,q},
\]

where

\[
C_{1,\varepsilon} = 2(A + \varepsilon)M_{2,q}C_1^{1/2} + 2D_1 \|q\|_{L^1}, \quad C_{2,\varepsilon} = 2(B + \varepsilon)(M_{2,h}C_1^{1/2})^{\beta - 1},
\]

\[
C_1 = 2 \left( \frac{\lambda_{\beta+1,q}}{B - \varepsilon} \right)^{2/(\beta - 1)} + \frac{M_{2,h}^{\beta/2 - \beta}}{\mu - 2}.
\]

Then according to (22) we have

\[
\int_0^\infty h(x)|u_n|^{\beta - 2}u_n - |u_1|^{\beta - 2}u_1| \, dx \\
\leq \int_0^\infty h(x)|u_n|^{\beta - 1}u_n - u_1| \, dx + \int_0^\infty h(x)|u_1|^{\beta - 1}u_n - u_1| \, dx
\]

(d) From Corollary 1.1 and Lemmas 2.2 and 2.3, we have
\[ \begin{align*}
\leq \left( \int_0^\infty h(x)|u_n|^{\beta \over 2} \, dx \right)^{\beta - 1 / \beta} \left( \int_0^\infty h(x)|u_n - u_1|^{\beta \over 2} \, dx \right)^{1 / \beta} \\
+ \left( \int_0^\infty h(x)|u_1|^{\beta \over 2} \, dx \right)^{\beta - 1 / \beta} \left( \int_0^\infty h(x)|u_n - u_1|^{\beta \over 2} \, dx \right)^{1 / \beta}
\leq \|u_n\|_{\beta,h}^{\beta - 1} \|u_n - u_1\|_{\beta,h} + \|u_1\|_{\beta,h}^{\beta - 1} \|u_n - u_1\|_{\beta,h} \leq 2C_3 \|u_n - u_1\|_{\beta,h},
\end{align*} \]

where \( C_3 = (M_{\beta,h}C_1^{1/2})^{\beta - 1}. \) Then according to (22) we have

\[ \int_0^\infty h(x)|u_n|^{\beta - 2}u_n - |u_1|^{\beta - 2}u_1| \, dx = o_n(1). \]

\[ \square \]

**Proposition 2.2.** Suppose that \( f \) is a function satisfying \( (H_0)-(H_4) \). Then there exists a constant \( \overline{\xi} > 0 \) such that for \( 0 < \lambda < \overline{\xi} \), problem (1) has a positive solution \( u_1 \) satisfying \( J_{\lambda}(u_1) > 0 \).

**Proof.** Let \( u_1 \) be the weak limit of the sequence \( (u_n) \) that satisfies (3). Consider \( \overline{\xi} = \min\{\xi_1, \xi_2\} \), where \( \xi_1 \) and \( \xi_2 \) are given in Lemmas 2.1 and 2.3, respectively. We will prove that \( u_n \to u_1 \) in \( H_0^1(0, \infty) \).

From (15) we have

\[ \langle J'_\lambda(u_n) - J'_\lambda(u_1), u_n - u_1 \rangle \]
\[ = \int_0^\infty (u_n' - u_1')(u_n' - u_1') \, dx + \int_0^\infty (u_n - u_1)(u_n - u_1) \, dx \]
\[ - \lambda \int_0^\infty h(x)(|u_n|^{\beta - 2}u_n - |u_1|^{\beta - 2}u_1)(u_n - u_1) \, dx \]
\[ - \int_0^\infty q(x)(f(u_n) - f(u_1))(u_n - u_1) \, dx. \]

Thus,

\[ \|u_n - u_1\|^2 \leq \langle J'_\lambda(u_n) - J'_\lambda(u_1), u_n - u_1 \rangle \]
\[ + \lambda \int_0^\infty h(x)|u_n|^{\beta - 2}u_n - |u_1|^{\beta - 2}u_1| \|u_n - u_1\| \, dx \]
\[ + \int_0^\infty q(x)|f(u_n) - f(u_1)||u_n - u_1| \, dx. \]

Therefore, from Lemma 2.4 above and taking into account that \( J'_\lambda \) is continuous (see Proposition 2.1), we have

\[ \|u_n - u_1\|^2 \leq \int_0^\infty |u_n' - u_1'|^2 \, dx + \int_0^\infty |u_n - u_1|^2 \, dx = o_n(1). \]
Consequently,
\[
\lim_{n \to \infty} \left( \int_0^\infty |u_n' - u_1'|^2 \, dx + \int_0^\infty |u_n - u_1|^2 \, dx \right) = 0.
\]
That is, \( u_n \to u_1 \) as \( n \to \infty \) in \( H_0^1(0, \infty) \), i.e. \( (u_n) \) satisfies the Palais-Smale condition.

Now by applying the Mountain Pass Theorem, we obtain
\[
J'_\lambda(u_1) = 0 \quad \text{and} \quad J_\lambda(u_1) = c > 0.
\]

\[\square\]

2.2. Existence of a second solution. Now we apply the Ekeland variational principle to prove the existence of a weak solution \( u_2 \) which is different from the solution \( u_1 \).

**Lemma 2.5.** Suppose that \((H_0)-(H_4)\) hold. Then there exists a constant \( \xi_3 > 0 \) such that for \( 0 < \lambda < \xi_3 \), the functional \( J_\lambda \) satisfies \((P.S)_d\) condition with \( d < 0 \).

**Proof.** Fix \( d < 0 \) and suppose that \( (u_n) \subset H_0^1(0, \infty) \) satisfies
\[
(24) \quad J_\lambda(u_n) \to d \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
We need to show that \( (u_n) \) admits a subsequence converging strongly in \( H_0^1(0, \infty) \).

Proceeding as in (20) we get
\[
\left( \frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|^2 \leq \lambda M_{\beta,h}^\beta \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \|u_n\|^\beta + o_n(1) \|u_n\| + d + \varepsilon.
\]
Thus, for a subsequence we have
\[
\limsup_{n \to \infty} \left( \frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|^2 - \lambda M_{\beta,h}^\beta \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \|u_n\|^\beta \leq d < 0.
\]
Hence,
\[
(25) \quad \limsup_{n \to \infty} \|u_n\|^2 \leq \left( \frac{\lambda M_{\beta,h}^\beta \left( \frac{\beta - 1}{\beta} - \frac{1}{\mu} \right)}{\left( \frac{1}{2} - \frac{1}{\mu} \right)} \right)^{2/(2-\beta)} < (\lambda M_{\beta,h}^\beta)^{2/(2-\beta)}.
\]
Choosing
\[
(\xi_3 = M_{\beta,h}^{-\beta} \left( 2 \frac{M_{\beta,h}^{\beta+1} \left( \beta^{-1} - \frac{1}{\mu} \right)}{(B - \varepsilon)} \right)^{2/(\beta-1)} + M_{\beta,h}^{2\beta/(\beta-1)} + 4K_3 \mu + 2 \right)^{(2-\beta)/2},
\]
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we have that for $\lambda < \xi_3$,

$$\limsup_{n \to \infty} \|u_n\|^2 < 2 \left( \frac{\lambda^{3+1}_{\beta+1,q}}{B - \varepsilon} \right)^{2/(\beta - 1)} + M^{2\beta/(2-\beta)}_{\beta,h} + \frac{4K_3\mu}{\mu - 2}. \tag{26}$$

From (26) we have that $(u_n)$ is bounded in $H^1_0(0, \infty)$ and there exists $u \in H^1_0(0, \infty)$ such that $u_n \rightharpoonup u$ in $H^1_0(0, \infty)$. Now, we can repeat the same arguments employed in the proofs of Lemma 2.3 and Proposition 2.2 to conclude that $u_n \rightarrow u$ in $H^1_0(0, \infty)$. \hfill \Box

**Proposition 2.3.** Suppose that $f$ is a function satisfying $(H_0)$–$(H_4)$. Then there exists a constant $\hat{\xi} > 0$ such that for $0 < \lambda < \hat{\xi}$, problem (1) has a positive solution $u_2$ satisfying $J_\lambda(u_2) < 0$.

**Proof.** Consider the complete metric space

$$\overline{B}_\varrho(0) := \{ u \in H^1_0(0, \infty) : \|u\| \leq \varrho \}$$

with a metric given by $d(u, w) = \|u - w\|$. The functional $J_\lambda$ is bounded from below on $\overline{B}_\varrho(0)$ for $\lambda < \xi_1$ (see Lemma 1.4). Note that

$$\forall t < \min \left\{ \frac{\delta_1}{\|\psi_{\beta,h}\|_1} \frac{1}{\|\psi_{\beta,h}\|_{\beta,h}} \left( \frac{2\lambda}{\beta \lambda_{\beta,h}} \right)^{1/2 - \beta} \right\}$$

$(t$ near $0)$ using $(H_3)$ in (8), we get

$$J_\lambda(t \psi_{\beta,h}) = \frac{1}{2} t^2 \|\psi_{\beta,h}\|^2 - \frac{\lambda}{\beta} t^\beta \|\psi_{\beta,h}\|_{\beta,h}^\beta - \int_0^\infty q(x) F(t \psi_{\beta,h}) \, dx$$

$$\leq \frac{1}{2} t^2 \|\psi_{\beta,h}\|^2 - \frac{\lambda}{\beta} t^\beta \|\psi_{\beta,h}\|_{\beta,h}^\beta - A - \frac{\varepsilon}{2} t^2 \|\psi_{\beta,h}\|_{2,q}^2$$

$$= \frac{1}{2} t^2 \|\psi_{\beta,h}\|^2 \left( 1 - \frac{2\lambda \beta - 2}{\beta \lambda_{\beta,h}} \|\psi_{\beta,h}\|_{\beta,h}^{\beta - 2} \right) - A - \frac{\varepsilon}{2} t^2 \|\psi_{\beta,h}\|_{2,q}^2 < 0,$$

by (17). Then, in view of (27), we see that

$$\inf_{u \in \overline{B}_\varrho(0)} J(u) < 0 < \inf_{u \in \partial \overline{B}_\varrho(0)} J(u). \tag{28}$$

Consequently, by applying Ekeland’s variational principle in $\overline{B}_\varrho(0)$, there is a minimizing sequence $(u_n)_{n \geq 1} \subset \overline{B}_\varrho(0)$ such that

$$J_\lambda(u_n) \rightarrow d := \inf \{ J_\lambda(u) : u \in \overline{B}_\varrho(0) \}, \tag{29}$$

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i.e.

$$J(u_n) \leq \inf_{u \in \overline{B}_\varrho(0)} J(u) + \frac{1}{n} \quad \forall n \geq 1,$$

and for every $w \in \overline{B}_\varrho(0)$ with $w \neq u_n$,

(30) \hspace{1cm} J_\lambda(w) - J_\lambda(u_n) + \frac{1}{n} \|u_n - w\| > 0.

Let $v \in H^1_0(0, \infty)$. We consider the sequence $w_n := u_n + tv \subset \overline{B}_\varrho(0)$, $t$ near 0 (small enough), and for all $n \geq 1$. From (30) we obtain

$$\frac{1}{t}(J_\lambda(u_n + tv) - J_\lambda(u_n)) > -\frac{1}{n} \|v\|.$$

Thus, $\langle J_\lambda(u_n), v \rangle \geq -n^{-1} \|v\|$ and similarly, $\langle J_\lambda(u_n), (-v) \rangle \geq -n^{-1} \|v\|$. Therefore

$$|\langle J_\lambda(u_n), v \rangle| < \frac{1}{n} \|v\| \quad \forall v \in H^1_0(0, \infty).$$

Consequently,

(31) \hspace{1cm} \|J_\lambda'(u_n)\| \to 0 \quad \text{as } n \to \infty.

Fix $\hat{\xi} := \min\{\xi_1, \xi_3\}$, where $\xi_1$ and $\xi_3$ are given by Lemmas 2.1 and 2.5, respectively. Then from (29) and (31) it follows that $(u_n)_{n \geq 1}$ is a (P.S)$_d$ sequence for the functional $J_\lambda$ for all $0 < \lambda < \hat{\xi}$.

Using Lemma 2.5 and Propositions 2.2, we obtain a subsequence, still denoted by $(u_n)_{n \geq 1}$, which converges strongly to a function $u_2 \in H^1_0(0, \infty)$. In this case

$$J_\lambda'(u_2) = 0.$$

Now we will check $J_\lambda(u_2) < 0$ to complete the proof. Note that using (H$_4$) and (7) we obtain

$$d + o_n(1) = J_\lambda(u_n) - \frac{1}{\mu} J_\lambda'(u_n) u_n = \left( \frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|^2 - \lambda \left( \frac{1}{\beta} - \frac{1}{\mu} \right) \int_0^\infty h(x) \|u_n\|^\beta$$

$$- \int_0^\infty \left( F(u_n) - \frac{1}{\mu} f(u_n) u_n \right) + o_n(1).$$

From Fatou’s lemma (see [3], Lemma 4.1) we conclude that

$$d = \liminf_{n \to \infty} \left( J_\lambda(u_n) - \frac{1}{\mu} J_\lambda'(u_n) u_n \right) \geq J_\lambda(u_2) - \frac{1}{\mu} J_\lambda'(u_2) u_2.$$

Thus

$$J_\lambda(u_2) = d < 0.$$

\[\square\]
Remark 2.3. If $u$ is a nontrivial solution for problem (1), by Remark 2.2, $u \geq 0$. Furthermore, as a consequence of (28) and $J_{\lambda}(0) = 0$, we have $u > 0$ in $(0, \infty)$.

Proof of Theorem 2.1. We take $\xi := \min\{\widehat{\xi}, \bar{\xi}\}$ and then the proof of Theorem 2.1 follows directly from Propositions 2.2, 2.3 and Remark 2.3. \hfill $\Box$

3. Example

In this section we give an example to illustrate our results.

Example 3.1. Consider the problem

\begin{equation}
\begin{aligned}
-u'' + u = \lambda h(x)|u|^\beta - 2u + q(x)f(u), & \quad x \in [0, \infty), \\
\quad u(0) = u(\infty) = 0,
\end{aligned}
\end{equation}

where

\[ f(u) = \begin{cases} 
\frac{1}{2M_{2,q}^2} |u| + (\lambda_{2,q}^2 + 1)|u|^\beta & \text{if } |u| \leq 1, \\
\frac{(\lambda_{2,q}^2 + 1)|u|^\beta + 1}{2M_{2,q}^2} & \text{if } |u| \geq 1,
\end{cases} \]

\[ q(x) = \frac{1}{4} D_2^{-1} e^{-3x/2} \text{ and } h(x) = e^{-4x/3}. \]

Choose $p(x) = e^{-x/4}$ and we see that

\[ \frac{q}{p}(x) = \frac{1}{4D_2} e^{-5x/4}, \quad \frac{q}{p^2}(x) = \frac{1}{4D_2} e^{-x}, \quad \frac{h}{p^{\beta-1}}(x) = e^{(3\beta-19)x/12}, \]

\[ \frac{q}{p^\beta}(x) = \frac{1}{4D_2} e^{(\beta-6)x/4}, \quad \frac{h}{p^\beta}(x) = e^{x(\beta/4-4/3)} \quad \text{and} \quad \frac{q}{p^{\beta+1}}(x) = \frac{1}{4D_2} e^{(\beta-5)x/4} \]

are in $L^1[0, \infty)$ for all $\beta \in (1, 2)$. Note that $\lambda_{2,q} > M_{2,q}^{-1}$, and we also obtain that

\[ M_{2,q} = \frac{\sqrt{2}}{\sqrt{15D_2}}, \quad A := \lim_{u \to 0^+} \frac{f(u)}{|u|} = \frac{1}{2M_{2,q}^2} \quad \text{and} \quad B := \lim_{u \to \infty} \frac{f(u)}{|u|^\beta} = \lambda_{2,q}^2 + 1. \]

It is easy to see that conditions (H$_0$)–(H$_4$) hold. Thus from Theorem 2.1, (32) has at least two positive solutions for each $\lambda \in (0, \xi)$. 

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References


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