ON THE CLASS OF b-L-WEAKLY AND ORDER M-WEAKLY COMPACT OPERATORS

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Abstract. In this paper, we introduce and study new concepts of b-L-weakly and order M-weakly compact operators. As consequences, we obtain some characterizations of KB-spaces.

Keywords: L-weakly compact operator; M-weakly compact operator; b-order bounded operator; b-weakly compact operator; b-L-weakly compact operator; order M-weakly compact operator; KB-space

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1. INTRODUCTION AND NOTATION

Throughout this paper X and Y denote real Banach spaces, and E and F denote real Banach lattices. \( B_X \) is the closed unit ball of \( X \) and \( \text{sol}(A) \) denotes the solid hull of a subset \( A \) of a Banach lattice. We use the term operator between two Banach spaces to mean a bounded linear mapping. Let us recall some notions and results from [2] and [7]. \( E \) is called a KB-space (Kantorovich-Banach), if every increasing norm bounded sequence of \( E^+ \) is norm convergent. Note that every KB-space has order continuous norm. A nonempty bounded subset \( A \) of \( E \) is said to be L-weakly compact if \( \lim \|x_n\| = 0 \) for every disjoint sequence \( (x_n) \) contained in the solid hull of \( A \). Note that every L-weakly compact subset \( A \subset E \) is relatively weakly compact (see [7], Proposition 3.6.5).

Recall from [3] that a subset \( A \) of \( E \) is called b-order bounded if it is order bounded in the topological bidual \( E'' \). Note that every order bounded subset of \( E \) is b-order bounded, however, the converse is not true in general. But a Banach lattice \( E \) is said
to have property (b) if each subset $A$ of $E$ is order bounded whenever it is b-order bounded. Note that every topological dual of a Banach lattice has property (b).

Based on this concept, the class of b-weakly compact operators is defined in [3]. In fact, an operator $T$ from a Banach lattice $E$ into a Banach space $Y$ is called b-weakly compact if it maps each b-order bounded subset of $E$ into a relatively weakly compact subset of $Y$. The space of b-weakly compact operators is bigger than the class of weakly compact operators, but smaller than the class of order weakly compact operators, which was introduced by Dodds in [5]. Also, an operator $T: E \to F$ is called b-order bounded if it maps b-order bounded subsets of $E$ into b-order bounded subsets of $F$.

The classes of L-weakly and M-weakly compact operators were introduced by Meyer-Nieberg (see [6]). An operator $T$ from $X$ into $F$ is called L-weakly compact if $T(B_X)$ is an L-weakly compact subset of $F$. An operator $T$ from $E$ into $Y$ is called M-weakly compact if $\lim T(x_n) = 0$ holds for every norm bounded disjoint sequence $(x_n)$ in $E$.

We introduce new classes of b-L-weakly and order M-weakly compact operators. An operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is called b-L-weakly compact if it maps b-order bounded subsets of $E$ into L-weakly compact subsets of $F$, and an operator $T$ from a Banach lattice $E$ into a Banach lattice $F$ is called order M-weakly compact if for every disjoint sequence $(x_n)$ in $B_E$ and every order bounded sequence $(f_n)$ of $F'$ we have $f_n(T(x_n)) \to 0$.

Note that the class of b-L-weakly compact operators contains strictly that of L-weakly compact operators, and the class of order M-weakly compact operators contains strictly that of M-weakly compact operators. On the other hand, it is easy to see that every b-L-weakly compact operator is b-weakly compact but the converse is false in general. We begin by establishing a sequential characterization of b-L-weakly compact operators. As consequences, we give some interesting results. We know that the classes of L-weakly and M-weakly compact operators are in duality with each other (an operator $T$, between two Banach lattices, is L-weakly compact (or M-weakly compact) if and only if its adjoint $T'$ is M-weakly compact (or L-weakly compact), see [7], Proposition 3.6.11). As we shall see, a similar result for the classes of b-L-weakly and order M-weakly compact operators are proved. Finally, we close this paper by presenting a necessary and sufficient condition on which every b-order bounded operator is b-L-weakly (or order M-weakly) compact.

In what follows:

- $L(X, Y)$ denotes the space of all operators from $X$ into $Y$,
- $LW(X, F)$ denotes the space of all L-weakly compact operators from $X$ into $F$,
- $MW(E, Y)$ denotes the space of all M-weakly compact operators from $E$ into $Y$,
- $bLW(E, F)$ denotes the space of all b-L-weakly compact operators from $E$ into $F$,
\( oMW(E, F) \) denotes the space of all order M-weakly compact operators from \( E \) into \( F \).

For the theory of Banach lattices and operators, we refer the reader to the monographs [2], [7], [8].

2. Main results

We start by the following definitions.

**Definition 2.1.** An operator \( T \) from \( E \) into \( F \) is called b-L-weakly compact if it maps b-order bounded subsets of \( E \) into L-weakly compact subsets of \( F \).

**Definition 2.2.** An operator \( T \) from \( E \) into \( F \) is called order M-weakly compact if for every disjoint sequence \((x_n)\) in \( B_E \) and every order bounded sequence \((f_n)\) of \( F' \), we have \( f_n(T(x_n)) \to 0 \).

**Remark 2.1.** Note that as the topological dual \( E' \) has always the property (b), in the previous definition one can replace “every order bounded sequence \((f_n)\)” with “every b-order bounded sequence \((f_n)\)”.

**Proposition 2.1.** The following assertions are equivalent:

1. The identity operator \( \text{Id}_E : E \to E \) is b-L-weakly compact.
2. Every b-order bounded subset of \( E \) is L-weakly compact.
3. \( E \) is a KB-space.

**Proof.** (1) \( \iff \) (2): The proof is obvious.

(2) \( \implies \) (3): According to Proposition 2.8 and Proposition 2.10 of [3], it suffices to show that each b-order bounded disjoint sequence of \( E \) is norm convergent to zero. Given such a sequence \((x_n)\) of \( E \), the set \( A = \{x_n : n \in \mathbb{N}\} \) is b-order bounded, and so by (2) \( A \) is L-weakly compact. Thus \( \|x_n\| \to 0 \).

(3) \( \implies \) (2): Let \( A \) be a b-order bounded subset of \( E \) and \((x_n)\) a disjoint sequence in the solid hull of \( A \). Note that the sequence \((x_n)\) is b-order bounded. In fact, pick some \( 0 \leq x'' \in E'' \) such that \( |x| \leq x'' \) for all \( x \in A \). \( |x_n| \leq |y_n| \) for some \( y_n \in A \) and hence \( |x_n| \leq |y_n| \leq x'' \). So, \( |x_n| \leq x'' \) for all \( n \in \mathbb{N} \), i.e. \((x_n)\) is b-order bounded. Then, by Proposition 2.8 and Proposition 2.10 of [3], we have \( \|x_n\| \to 0 \) and hence \( A \) is L-weakly compact.

**Remark 2.2.** Clearly, every L-weakly compact operator is b-L-weakly compact (it suffices to note that every b-order bounded subset of \( E \) is norm bounded), but the converse is not true in general. For instance, consider the operator \( \text{Id}_{\ell_1} : \ell_1 \to \ell_1 \). Since \( \ell_1 \) is a KB-space, \( \text{Id}_{\ell_1} \) is b-L-weakly compact. On the other hand, \( B_{\ell_1} \) is not
relatively weakly compact and therefore is not L-weakly compact. Hence \( \text{Id}_{\ell_1} \) is not L-weakly compact.

On the other hand, it is easy to see that every b-L-weakly compact operator is b-weakly compact (it suffices to note that every L-weakly compact subset is relatively weakly compact). The converse, however, need not be true. For instance, consider the operator \( T: \ell_1 \to \ell_\infty \) defined by

\[
\forall (\alpha_n) \in \ell_1, \quad T((\alpha_n)) = \left( \sum_{n=1}^{\infty} \alpha_n \right) (1, 1, 1, \ldots).
\]

Clearly, \( T \) is a compact operator (it has rank one) and hence \( T \) is b-weakly compact.

Let \( e = (1/n^2)_{n \in \mathbb{N}} \). The sequence \( (e_n) \) of standard unit vectors is a disjoint sequence in the solid hull of \( T[0, e], |e_n| \leq T(e) \). From \( \|e_n\| = 1 \to 0 \) we see that \( T \) fails to be b-L-weakly compact.

Clearly, every M-weakly compact operator is order M-weakly compact (for every sequence \( (y_n) \) of \( F \), if \( \|y_n\| \to 0 \), then \( f_n(y_n) \to 0 \) for every order bounded sequence \( (f_n) \) of \( F' \)), but the converse is not true in general. For instance, consider the operator \( \text{Id}_{c_0} \). Since \( \ell_1 = c_0' \) is a KB-space, \( \text{Id}_{c_0} \) is order M-weakly compact (see Corollary 2.2). And since \( \text{Id}_{c_0} \) is not L-weakly compact, \( \text{Id}_{\ell_1} \) is not M-weakly compact.

The following lemmas are used throughout this paper.

**Lemma 2.1** ([2], Theorem 5.63). For any two nonempty bounded sets \( A \subset E \) and \( B \subset E' \), the following statements are equivalent:

(1) Every disjoint sequence in the solid hull of \( A \) converges uniformly to zero on \( B \).
(2) Every disjoint sequence in the solid hull of \( B \) converges uniformly to zero on \( A \).

**Lemma 2.2.** For every nonempty bounded subset \( A \subset E \), the following assertions are equivalent:

(1) \( A \) is L-weakly compact.
(2) \( f_n(x_n) \to 0 \) for every sequence \( (x_n) \) of \( A \) and every disjoint sequence \( (f_n) \) of \( B_E' \).

**Proof.** Let \( A \) be a nonempty bounded subset of \( E \). \( A \) is L-weakly compact if and only if \( \|x_n\| \to 0 \) holds for every disjoint sequence \( (x_n) \) of \( \text{sol}(A) \). Thus \( A \) is L-weakly compact if and only if every disjoint sequence \( (x_n) \) of \( \text{sol}(A) \) converges uniformly to zero on \( B_E \) (i.e. \( \sup\{|f(x_n)|: f \in B_E'\} \to 0 \)). By Lemma 2.1, this is equivalent to saying that every disjoint sequence \( (f_n) \) of \( B_E' \) converges uniformly to zero on \( A \) (i.e. \( \sup\{|f_n(x)|: x \in A\} \to 0 \)).
Let us now prove the equivalence
\[ \sup\{|f_n(x)|: x \in A\} \to 0 \text{ if and only if for each sequence } (x_n) \text{ of } A, f_n(x_n) \to 0. \]
Indeed, if \( \sup\{|f_n(x)|: x \in A\} \to 0 \), then for each sequence \( (x_n) \) of \( A \)
\[ |f_n(x_n)| \leq \sup\{|f_n(x)|: x \in A\} \to 0. \]
Therefore \( f_n(x_n) \to 0 \).

Conversely, assume that \( f_n(x_n) \to 0 \) for each sequence \( (x_n) \) of \( A \). Assume by way of contradiction that \( \sup\{|f_n(x)|: x \in A\} \nRightarrow 0 \). Then there exist some \( \varepsilon > 0 \) and a subsequence \( (f_{\varphi(n)}) \) of \( (f_n) \) satisfying \( \sup\{|f_{\varphi(n)}(x)|: x \in A\} > \varepsilon \) for all \( n \in \mathbb{N} \). Thus for each \( n \in \mathbb{N} \), there exists some \( x_{\varphi(n)} \in A \) with \( |f_{\varphi(n)}(x_{\varphi(n)})| > \varepsilon \), from our hypothesis it follows that \( f_{\varphi(n)}(x_{\varphi(n)}) \to 0 \), which is impossible, and the proof of the lemma is finished. \( \square \)

In a similar way we may prove the following result.

**Lemma 2.3.** For every nonempty bounded subset \( A \subset E' \) the following assertions are equivalent:
(1) \( A \) is L-weakly compact.
(2) \( f_n(x_n) \to 0 \) for every sequence \( (f_n) \) of \( A \) and every disjoint sequence \( (x_n) \) of \( B_E \).

The following results give a sequential characterization of b-L-weakly compact operators.

**Theorem 2.1.** For an operator \( T: E \to F \), the following statements are equivalent:
(1) \( T \) is b-L-weakly compact.
(2) For every b-order bounded sequence \( (x_n) \) of \( E \) and every disjoint sequence \( (f_n) \) of \( B_F \), we have \( f_n(T(x_n)) \to 0 \).
(3) For every b-order bounded sequence \( (x_n) \) of \( E \) and every disjoint sequence \( (f_n) \) of \( B_{F^+} \), we have \( f_n(T(x_n)) \to 0 \).
(4) For every b-order bounded sequence \( (x_n) \) of \( E^+ \) and every disjoint sequence \( (f_n) \) of \( B_{F^*} \), we have \( f_n(T(x_n)) \to 0 \).
(5) For every b-order bounded sequence \( (x_n) \) of \( E^+ \) and every disjoint sequence \( (f_n) \) of \( B_{F^*+} \), we have \( f_n(T(x_n)) \to 0 \).

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**Proof.** \((1) \Leftrightarrow (2):\) Consider an operator \(T: E \to F.\) \(T\) is \(b\)-\(L\)-weakly compact if and only if for every \(b\)-order bounded subset \(A \subset E, T(A)\) is \(L\)-weakly compact. By Lemma 2.2, this is equivalent to saying that for every \(b\)-order bounded subset \(A \subset X, f_n(T(x_n)) \to 0\) for every sequence \((x_n)\) of \(A\) and every disjoint sequence \((f_n)\) of \(B_{F'}\).

To conclude, it is sufficient to note the equivalence of the following assertions:

(i) For every \(b\)-order bounded subset \(A \subset E, f_n(T(x_n)) \to 0\) for every sequence \((x_n)\) of \(A\) and every disjoint sequence \((f_n)\) of \(B_{F'}\).

(ii) \(f_n(T(x_n)) \to 0\) for every \(b\)-order bounded sequence \((x_n)\) of \(E\) and every disjoint sequence \((f_n)\) of \(B_{F'}\).

(i) \(\Rightarrow\) (ii): Let \((x_n)\) be a \(b\)-order bounded sequence of \(E.\) It is sufficient to apply (i) to the set \(A = \{x_n: n \in \mathbb{N}\}\).

(ii) \(\Rightarrow\) (i): Let \(A\) be a \(b\)-order bounded subset of \(E.\) It is sufficient to note that every sequence \((x_n)\) of \(A\) is \(b\)-order bounded.

The proof of the equivalence of (1) and (2) is finished.

\((2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5):\) The proof is obvious. \(\Box\)

In the same way we prove the following result:

**Theorem 2.2.** For an operator \(\varphi: E' \to F'\) the following statements are equivalent:

(1) \(\varphi\) is \(b\)-\(L\)-weakly compact.

(2) \(\varphi(f_n)(y_n) \to 0\) for every order bounded sequence \((f_n)\) of \(E'\) and every disjoint sequence \((y_n)\) of \(B_{F}\).

(3) \(\varphi(f_n)(y_n) \to 0\) for every order bounded sequence \((f_n)\) of \(E'^+\) and every disjoint sequence \((y_n)\) of \(B_{F^+}\).

As a consequence, we obtain the following characterizations of KB-spaces.

**Corollary 2.1.** For a Banach lattice \(E\) the following statements are equivalent:

(1) \(\text{Id}_E \in bLW(E).\)

(2) \(E\) is a KB-space.

(3) \(f_n(x_n) \to 0\) for every \(b\)-order bounded sequence \((x_n)\) of \(E\) and every disjoint sequence \((f_n)\) of \(B_{E'}\).

**Corollary 2.2.** For a Banach lattice \(E\) the following statements are equivalent:

(1) \(\text{Id}_{E'} \in bLW(E').\)

(2) \(E'\) is a KB-space.
(3) $f_n(x_n) \to 0$ for every order bounded sequence $(f_n)$ of $E'$ and every disjoint sequence $(x_n)$ of $B_E$.

(4) $\text{Id}_E \in oMW(E)$.

Contrary to weakly compact operators (see [1]), we also deduce that the class of b-L-weakly (or order M-weakly) compact operators satisfies the domination problem.

**Corollary 2.3.** Let $S, T : E \to F$ be two positive operators such that $0 \leq S \leq T$. Then $S$ is b-L-weakly compact (or order M-weakly compact) whenever $T$ is one.

**Proposition 2.2.** Let $E$ and $F$ be two Banach lattices. Then:

1. The set of all b-L-weakly compact operators from $E$ to $F$ is a closed vector subspace of $L(E, F)$.
2. The set of all order M-weakly compact operators from $E$ to $F$ is a closed vector subspace of $L(E, F)$.

**Proof.** (1) Let $T_1, T_2 \in bLW(E, F)$, and $\alpha \in \mathbb{R}$. Let $(x_n)$ be a b-order bounded sequence of $E$ and $(f_n)$ a disjoint sequence of $B_{F'}$. Since $T_1, T_2 \in bLW(E, F)$, it follows from Theorem 2.1, that

$$f_n((\alpha T_1 + T_2)(x_n)) = \alpha f_n(T_1(x_n)) + f_n(T_2(x_n)) \to 0.$$ 

Then $\alpha T_1 + T_2 \in bLW(E, F)$. Thus $bLW(E, F)$ is a vector subspace of $L(E, F)$. To see that it is also a closed vector subspace of $L(E, F)$, let $T$ be in the closure of $bLW(E, F)$. Let $(x_n)$ be a b-order bounded sequence of $E$ and $(f_n)$ a disjoint sequence of $B_{F'}$. We have to show that $f_n(T(x_n)) \to 0$. To this end, let $\varepsilon > 0$. Pick a b-L-weakly compact operator $S : E \to F$ with $\|T - S\| < \varepsilon$ and note that from the inequalities

$$|f_n(T(x_n))| \leq |f_n((T - S)(x_n))| + |f_n(S(x_n))| 
\leq \|f_n\|\|T - S\|(x_n)\| + |f_n(S(x_n))|$$

it follows that $\limsup |f_n(T(x_n))| \leq \varepsilon \|(x_n)\|$.

Since $\varepsilon$ is arbitrary, we see that $f_n(T(x_n)) \to 0$ holds as desired.

(2) Clearly, $oMW(E, F)$ is a vector subspace of $L(E, F)$. To see that it is also a closed vector subspace of $L(E, F)$, let $T$ be in the closure of $oMW(E, F)$. Assume that $(x_n)$ is a disjoint sequence of $B_E$ and $(f_n)$ an order bounded sequence of $F'$. We have to show that $f_n(T(x_n)) \to 0$. To this end, let $\varepsilon > 0$. Pick an order M-weakly...
compact operator $S: E \to F$ with $\|T - S\| < \varepsilon$ and note that from the inequalities

$$|f_n(T(x_n))| \leq |f_n((T - S)(x_n))| + |f_n(S(x_n))|$$

$$\leq \|f_n\| \|T - S\| \|x_n\| + |f_n(S(x_n))|$$

it follows that

$$\limsup |f_n(T(x_n))| \leq \varepsilon \|f_n\| \leq \varepsilon \|(f_n)\|.$$ 

Since $\varepsilon$ is arbitrary, we see that $f_n(T(x_n)) \to 0$ holds as desired. □

The classes of L-weakly and M-weakly compact operators are in duality with each other. For the classes of b-L-weakly and order M-weakly compact operators, we have the following result.

**Theorem 2.3.** Let $E$ and $F$ be two Banach lattices. Then the following statements hold:

1. An operator $T: E \to F$ is order M-weakly compact if and only if its adjoint $T'$ is b-L-weakly compact.
2. For an operator $T: E \to F$, if its adjoint $T'$ is order M-weakly compact then $T$ is b-L-weakly compact.

**Proof.** (1) Consider an operator $T: E \to F$. By Theorem 2.2, $T'$ is b-L-weakly compact if and only if $T'(f_n)(x_n) \to 0$ for every order bounded sequence $(f_n)$ of $F'$ and every disjoint sequence $(x_n)$ of $B_E$. This is equivalent to saying that $f_n(T(x_n)) \to 0$ for every order bounded sequence $(f_n)$ of $F'$ and every disjoint sequence $(x_n)$ of $B_E$. In other words, $T': F' \to E'$ is b-L-weakly compact if and only if $T: E \to F$ is order M-weakly compact.

(2) Let $T: E \to F$ be an operator such that $T'$ is order M-weakly compact. Let $(x_n)$ be a b-order bounded sequence of $E$ and $(f_n)$ a disjoint sequence of $B_{E'}$. Let $J: E \to E''$ be the canonical embedding of $E$ into $E''$. Since $T': F' \to E'$ is order M-weakly compact and the sequence $(J(x_n))$ of $E''$ is order bounded, then $J(x_n)(T'(f_n)) = f_n(T(x_n)) \to 0$. Hence $T$ is b-L-weakly compact. □

**Remark 2.3.** However, in general:

$$T$$ is b-L-weakly compact $\iff T'$ is order M-weakly compact.

Indeed, the Banach lattice $E = \ell_1(\ell_\infty_n)$ is a KB-space whose bidual $E''$ fails to have an order continuous norm (see [7], page 95). Therefore the identity operator of $E$ is b-L-weakly compact, but $\text{Id}_{E'} = \text{Id}_{E''}$ is not order M-weakly compact.

We now present another characterization of KB-spaces.

**Theorem 2.4.** A Banach lattice $F$ is a KB-space if and only if for every Banach lattice $E$ and b-order bounded operator $T: E \to F$, $T$ is b-L-weakly compact.
Proof. If the hypothesis on $F$ is true then taking $E = F$ we see that the identity on $E$ is b-L-weakly compact and thus, by Proposition 2.1, $F$ is a KB-space. On the other hand, let $T: E \to F$ be a b-order bounded operator, and let $A$ be a b-order bounded subset of $E$. Since $T$ is b-order bounded, $T(A) \subset F$ is b-order bounded. If $F$ is a KB-space, then by Proposition 2.1, $T(A)$ is L-weakly compact and so $T$ is b-L-weakly compact. □

Theorem 2.5. Let $E$ and $F$ be nonzero Banach lattices. Then the following assertions are equivalent:

1. Every b-order bounded operator $T: E \to F$ is order M-weakly compact.
2. $E'$ is a KB-space.

Proof. (1) ⇒ (2): Assume by the way of contradiction that $E'$ is not a KB-space. We have to construct a b-order bounded operator $T: E \to F$ which is not order M-weakly compact. Since $E'$ is not a KB-space (i.e. the norm of $E'$ is not order continuous), it follows from [2], Theorem 4.14, that there exists some $f \in E'^+$ and there exists a disjoint sequence $(f_n)$ in $[0, f]$ which does not converge to zero in norm. Pick some $c \in F^+$ and $g \in F'^+$ such that $g(c) = 1$.

Now, we consider the positive operator $T: E \to F$ defined by $T(x) = f(x)c$ for every $x \in E$. $T$ is b-order bounded, on the other hand, we claim that $T$ is not order M-weakly compact. By Theorem 2.3, it suffices to show that its adjoint $T': F' \to E'$ is not b-L-weakly compact. Note that $T'(\varphi) = \varphi(c)f$ for every $\varphi \in F'$. In particular, $T'(g) = g(c)f = f$. So, $f \in T'([0, g])$. From $(f_n) \subset [0, f]$ it follows that $(f_n)$ is a disjoint sequence in the solid hull of $T'([0, g])$. Since $(f_n)$ is not norm convergent to zero, then $T'$ is not b-L-weakly compact. Hence $T$ is not order M-weakly compact. But this is in contradiction with our hypothesis (1). So, $E'$ is a KB-space.

(2) ⇒ (1): Let $T: E \to F$ be a b-order bounded operator. By [4], Proposition 1, the adjoint operator $T': F' \to E'$ is order bounded, and hence it is b-order bounded. Since $E'$ is a KB-space, it follows from Theorem 2.4 that $T'$ is b-L-weakly compact and so by Theorem 2.3 the operator $T$ is order M-weakly compact. □

References


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