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THE RANK OF A COMMUTATIVE SEMIGROUP

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Abstract. The concept of rank of a commutative cancellative semigroup is extended to all commutative semigroups S by defining $\operatorname{rank} S$ as the supremum of cardinalities of finite independent subsets of S. Representing such a semigroup S as a semilattice Y of (archimedean) components S_α, we prove that $\operatorname{rank} S$ is the supremum of ranks of various S_α. Representing a commutative separative semigroup S as a semilattice of its (cancellative) archimedean components, the main result of the paper provides several characterizations of $\operatorname{rank} S$; in particular if $\operatorname{rank} S$ is finite. Subdirect products of a semilattice and a commutative cancellative semigroup are treated briefly. We give a classification of all commutative separative semigroups which admit a generating set of one or two elements, and compute their ranks.

Keywords: semigroup, commutative semigroup, independent subset, rank, separative semigroup, power cancellative semigroup, archimedean component

MSC 2000: 20M10

1. Introduction and summary

We have defined in [2] the rank of a commutative cancellative semigroup S as the number of elements in any maximal independent subset of S if it is finite; otherwise, S has infinite rank. This concept is equivalent to the earlier definition of rank of S as the rank of a group of quotients of S.

The natural generalization of a commutative cancellative semigroup is that of a commutative separative semigroup S since then, and only then, S is a semilattice of commutative cancellative semigroups. Whereas commutative cancellative semigroups are precisely those semigroups which are embeddable into abelian groups,

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This opens two avenues for a definition of rank of a commutative separative semigroup S: either by means of maximal independent subsets or by means of a (Clifford) semigroup of quotients of S. The purpose of the present work is to define rank of an arbitrary commutative semigroup which, for a commutative cancellative semigroup, is equivalent to the one in [2], and to explore its elementary properties. These properties include the behavior of rank relative to certain congruences, as well as direct and certain semidirect products and semigroups with small generating sets.

In Section 2 we list a number of concepts, some notation, and a few results from the literature needed in the main body of the paper. In addition, we introduce the notion of rank of a commutative semigroup, the central concept throughout the paper. A characterization of rank in terms of semilattice decompositions can be found in Section 3. Our main result, in Section 4, provides several characterizations of rank of commutative separative semigroups in terms of the rank of their (Clifford) semigroups of quotients, with special attention to the case of finite rank. The rank of a subdirect product of a semilattice and a commutative cancellative semigroup is briefly discussed in Section 5. The last Section 6 consists of a classification of commutative separative semigroups which admit a generating set of one or two elements in terms of their greatest semilattice decomposition.

2. Preparation

For undefined notation and terminology, we refer the reader to books [4], [5] and [10]. For an excellent treatment of commutative semigroups, we recommend [5].

Throughout the paper S denotes a commutative semigroup. We will often use the following semigroups (under addition):

$\mathbb{P}$—positive integers,

$\mathbb{N}$—nonnegative integers,

$\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$—integers modulo $m \geqslant 0$, where $\mathbb{Z}_1 = \{0\}$ is the trivial group and $\mathbb{Z}_0 = \mathbb{Z}$ is the group of integers.

For A any finite set, $|A|$ denotes the number of its elements.

The relation η on S defined by

$$a \;\eta\; b \Longleftrightarrow a^m = bx,\ b^n = ay \ \text{ for some } \ m,n \in \mathbb{P} \ \text{ and } \ x,y \in S$$

is the least semilattice congruence on S [4, Theorem 4.12]. Its classes are the *(archimedean) components* of S. The semigroup S is *archimedean* if it has only one component. We fix the notation $S = (Y; S_\alpha)$ to mean that $Y \cong S/\eta$ and the various S_α are the archimedean components of S.

The semigroup S is *separative* if it satisfies the implication $a^2 = b^2 = ab \Rightarrow a = b$. The relation σ defined on S by

$$a \ \sigma \ b \Longleftrightarrow a^{m+1} = a^m b, \ b^{m+1} = b^m a \ \text{ for some } \ m \in \mathbb{P},$$

is the least separative congruence on S [4, Theorem 4.14].

The semigroup S is *power cancellative* if it satisfies the implication $a^n = b^n \Rightarrow a = b$. The relation τ defined by

$$a\tau b \ \text{ if } \ a^n = b^n \ \text{ for some } \ n \in \mathbb{P}$$

is the least power cancellative congruence on S (straightforward).

Let S be separative. Then $S = (Y; S_\alpha)$ where each archimedean component S_α is cancellative, see [4, Theorem 4.16], [10, Corollary II.6.5]. Let

$$SQ = \bigcup_{\alpha \in Y} (S_\alpha \times S_\alpha)/{\sim},$$

where $\sim$ is defined by

$$(a, b) \sim (c, d) \text{ if } a, c \in S_\alpha \text{ for some } \alpha \in Y \text{ and } ad = bc,$$

with multiplication of $\sim$-classes $[a, b][c, d] = [ac, bd]$. Then SQ is a *semigroup of quotients of* S and the function

$$\varphi_S : \ a \mapsto [a^2, a] \qquad (a \in S)$$

is the *canonical embedding* of S into SQ. We will often identify S and its image $S\varphi_S$. For each $\alpha \in Y$,

$$G_\alpha = \{[a, b] \in SQ; \ a \in S_\alpha\}$$

is a group of quotients of S_α, and SQ is a *semilattice of groups*, or a *Clifford semigroup*, in notation

$$SQ = [Y; G_\alpha, \varphi_{\alpha,\beta}].$$

We will tacitly use the notation $S_\alpha Q = G_\alpha$. This is an example of a *strong semilattice of semigroups*, which may be defined in the usual way, see [10, Definition III.7.8].

We now introduce some new concepts.

Definition 2.1. A finite subset $A = \{a_1, \dots, a_k\}$ of S is *independent* if

$$a_1^{p_1} \dots a_k^{p_k} = a_1^{q_1} \dots a_k^{q_k}$$

for some $p_i, q_i \in \mathbb{P}$ implies that $p_i = q_i$ for $i = 1, \dots, k$. The *rank* of S is defined by

$$\operatorname{rank} S = \sup\{\, |A|;\ A \text{ is an independent finite subset of } S\}.$$

When the semigroup S is cancellative, this notion of rank is equivalent to that defined in [2, Section 3].

Fact 2.2. *If $S = (Y; S_\alpha)$ is separative and $SQ = [Y; G_\alpha, \varphi_{\alpha,\beta}]$ is its semigroup of quotients, then*

$$\operatorname{rank} S_\alpha = \operatorname{rank} G_\alpha \quad (\alpha \in Y).$$

Proof. See [1, Corollary 4.3]. □

3. Congruences η, σ and τ

The first result, of crucial importance in this context, concerns compatibility of rank with an arbitrary semilattice decomposition of S.

Theorem 3.1. *Let S be a semilattice Y of semigroups S_α. Then*

$$\operatorname{rank} S = \sup\{\operatorname{rank} S_\alpha;\ \alpha \in Y\}.$$

Proof. Since each S_α is a subsemigroup of S, we have $\operatorname{rank} S_\alpha \leqslant \operatorname{rank} S$ and thus

$$\sup\{\operatorname{rank} S_\alpha;\ \alpha \in Y\} \leqslant \operatorname{rank} S.$$

We may therefore restrict our attention to the case

$$\sup\{\operatorname{rank} S_\alpha;\ \alpha \in Y\} = m \in \mathbb{P}$$

and show that every subset A of S with $m+1$ elements, say $A = \{a_1, \dots, a_{m+1}\}$, is necessarily dependent.

For every $1 \leqslant i \leqslant m+1$, let $b_i = a_1 \dots a_{i-1} a_i^2 a_{i+1} \dots a_{m+1}$ and set $B = \{b_1, \dots, b_{m+1}\}$. If $a_i \in S_{\alpha_i}$ for every i, then $B \subseteq S_\alpha$ where $\alpha = \alpha_1 \dots \alpha_{m+1}$. Since $\operatorname{rank} S_\alpha \leqslant m$, we have that B is necessarily dependent. Hence

$$b_1^{p_1} \dots b_{m+1}^{p_{m+1}} = b_1^{q_1} \dots b_{m+1}^{q_{m+1}}$$

for some $p_i, q_i \in \mathbb{P}$ where $(p_1, \dots, p_{m+1}) \neq (q_1, \dots, q_{m+1})$. But then we also have

$$a_1^{p'_1} \dots a_{m+1}^{p'_{m+1}} = a_1^{q'_1} \dots a_{m+1}^{q'_{m+1}}$$

where

$$p'_i = p_1 + \dots + p_{i-1} + 2p_i + p_{i+1} + \dots + p_{m+1},$$
$$q'_i = q_1 + \dots + q_{i-1} + 2q_i + q_{i+1} + \dots + q_{m+1}.$$

Since $(p'_1, \dots, p'_{m+1}) \neq (q'_1, \dots, q'_{m+1})$, we conclude that A is dependent. □

In order to treat the congruences σ and τ, we first prove a lemma which further elucidates the situation.

Lemma 3.2. *Let $A = \{a_1, \dots, a_k\} \subseteq S$ and, for $\theta \in \{\sigma, \tau\}$, let $A\theta = \{a_1\theta, \dots, a_k\theta\}$ so that $A\theta \subseteq S/\theta$. Then A is dependent if and only if $A\theta$ is dependent.*

Proof. Necessity is obvious.

Sufficiency. We assume that there exist $p_1, \dots, p_k, q_1, \dots, q_k \in \mathbb{P}$ such that

$$(a_1\theta)^{p_1} \dots (a_k\theta)^{p_k} = (a_1\theta)^{q_1} \dots (a_k\theta)^{q_k},$$

where $(p_1, \dots, p_k) \neq (q_1, \dots, q_k)$.

Now let $\theta = \sigma$. By definition, there exists $m \in \mathbb{P}$ such that

$$(a_1^{p_1} \dots a_k^{p_k})^{m+1} = (a_1^{p_1} \dots a_k^{p_k})^m a_1^{q_1} \dots a_k^{q_k}$$

so that

$$a_1^{p_1(m+1)} \dots a_k^{p_k(m+1)} = a_1^{p_1 m + q_1} \dots a_k^{p_k m + q_k}.$$

Since for some i we have $p_i \neq q_i$, which evidently implies that $p_i(m+1) \neq p_i m + q_i$, we conclude that A is dependent.

Next let $\theta = \tau$. By definition there exists $m \in \mathbb{P}$ such that

$$a_1^{p_1 m} \dots a_k^{p_k m} = a_1^{q_1 m} \dots a_k^{q_k m}.$$

Since $(p_1 m, \dots, p_k m) \neq (q_1 m, \dots, q_k m)$, A is dependent. □

We now deduce the desired result.

Corollary 3.3. $\operatorname{rank} S = \operatorname{rank}(S/\sigma) = \operatorname{rank}(S/\tau)$.

Next we consider the direct product of two semigroups in the context of rank. But we first prove a lemma of some independent interest.

Lemma 3.4. *Let S and T be semigroups and $\theta \in \{\eta, \sigma, \tau\}$. Then for any $(a,b),(c,d) \in S \times T$ we have*

$$(a,b)\,\theta\,(c,d) \Longleftrightarrow a\ \theta\, b, c\,\theta\, d.$$

Proof. The assertion follows from

$$\begin{aligned} x^m \in yS &\Longrightarrow x^n \in yS \text{ for all } n \geqslant m, \\ x^{m+1} = x^n y &\Longrightarrow x^{n+1} = x^n y \text{ for all } n \geqslant m, \\ x^m = y^m &\Longrightarrow x^{nm} = y^{nm} \text{ for all } n \in \mathbb{P}. \end{aligned}$$

We omit the details. □

We are now ready for the rank of the direct product.

Proposition 3.5. *For any semigroups S and T, we have*

$$\operatorname{rank}(S \times T) = \operatorname{rank} S + \operatorname{rank} T.$$

Proof. By Corollary 3.3 and Lemma 3.4 for σ, it suffices to consider the case when both S and T are separative. We first consider the case when they are cancellative. In such a case $S \times T$ is also cancellative and

$$(S \times T)Q \cong SQ \times TQ.$$

But then

$$\operatorname{rank}(S \times T) = \operatorname{rank}(SQ \times TQ) = \operatorname{rank}(SQ) + \operatorname{rank}(TQ) = \operatorname{rank} S + \operatorname{rank} T.$$

If now S and T are separative, then Theorem 3.1 and Lemma 3.4 for η yield

$$\begin{aligned} \operatorname{rank}(S \times T) &= \sup\{\operatorname{rank}(A \times B);\ (A,B) \in S/\eta \times T/\eta\} \\ &= \sup\{\operatorname{rank} A + \operatorname{rank} B;\ A \in S/\eta,\ B \in T/\eta\} \\ &= \sup\{\operatorname{rank} A;\ A \in S/\eta\} + \sup\{\operatorname{rank} B;\ B \in T/\eta\} \\ &= \operatorname{rank} S + \operatorname{rank} T. \end{aligned}$$

□

Our basic notation in this section follows.

Definition 4.1. On any semigroup S define a relation π_S by

$$a\,\pi_S\,b \Longleftrightarrow a^2bc = ab^2c \text{ for some } c \in S.$$

If there is no danger of confusion, we will set $\pi = \pi_S$.

We start with several auxiliary statements.

Lemma 4.2. *Let $S = [Y; G_\alpha, \varphi_{\alpha,\beta}]$ be a Clifford semigroup. Then for $g \in G_\alpha$ and $h \in G_\beta$ we have*

$$g\,\pi\,h \Longleftrightarrow g\varphi_{\alpha,\gamma} = h\varphi_{\beta,\gamma} \text{ for some } \gamma \leqslant \alpha\beta.$$

Hence π is the least group congruence on S.

Proof. Straightforward. □

Lemma 4.3. *Assume that a separative semigroup $S = (Y; S_\alpha)$ is embedded in its semigroup of quotients $SQ = [Y; G_\alpha, \varphi_{\alpha,\beta}]$.*

(i) *π_S is a congruence on S and S/π_S is archimedean and cancellative.*

(ii) $\pi_{SQ}|_S = \pi_S$.

(iii) $(S/\pi_S)Q \cong (SQ)/\pi_{SQ}$.

Proof. For any $a \in S_\alpha$ and $b \in S_\beta$, by Lemma 4.2 we obtain

$$\begin{aligned}
[a^2, a]\,\pi_{SQ}\,[b^2, b] &\Longleftrightarrow [a^2, a]\varphi_{\alpha,\gamma} = [b^2, b]\varphi_{\beta,\gamma} \text{ for some } \gamma \leqslant \alpha\beta \\
&\Longleftrightarrow [a^2, a][c, c] = [b^2, b][c, c] \text{ for some } c \in S_\gamma \\
&\Longleftrightarrow [a^2c, ac] = [b^2c, bc] \text{ for some } c \in S_\gamma \\
&\Longleftrightarrow a^2cbc = acb^2c \text{ for some } c \in S_\gamma \\
&\Longleftrightarrow a\,\pi_S\,b.
\end{aligned}$$

In particular, this implies that π_S is a congruence whose classes are cancellative. Further, $a \in S_\alpha$ and $b \in S_\beta$ imply that $a\varphi_{\alpha,\alpha\beta}, b\varphi_{\beta,\alpha\beta} \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is archimedean, we have

$$(a\varphi_{\alpha,\alpha\beta})^m = (b\varphi_{\beta,\alpha\beta})c, \quad (b\varphi_{\beta,\alpha\beta})^n = (a\varphi_{\alpha,\alpha\beta})d$$

for some $m, n \in \mathbb{P}$ and $c, d \in S_{\alpha\beta}$. But then

$$(a\pi_S)^m = (b\pi_S)(c\pi_S), \quad (b\pi_S)^n = (a\pi_S)(d\pi_S)$$

and S/π_S is also archimedean.

We may view S/π_S as a subsemigroup of SQ/π_{SQ}. For any $[a,b]\pi_{SQ}$, we have $a, b \in S_\alpha$ for some $\alpha \in Y$ and

$$[a,b]\pi_{SQ} = [a\pi_S, b\pi_S] \in (S/\pi_S)Q$$

where $a\pi_S, b\pi_S \in G_\alpha$. This proves all assertions of the lemma. $\square$

The construction of a semilattice of groups $[Y; G_\alpha, \varphi_{\alpha,\beta}]$ admits an obvious generalization to a *strong semilattice of semigroups* by assuming that G_α need not be groups. In such a case, Lemma 4.2 preserves its validity. We now turn back to the central concept of the paper, namely the rank. First we treat the special case of a strong semilattice of semigroups.

Lemma 4.4. *Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroups where all S_α are cancellative and all $\varphi_{\alpha,\beta}$ are injective. Then* $\operatorname{rank} S = \operatorname{rank}(S/\pi)$.

Proof. Let $A = \{a_1, \dots, a_k\}$, where $a_i \in S_{\alpha_i}$, and assume that $A\pi = \{a_1\pi, \dots, a_k\pi\}$ is dependent. It suffices to prove that A is dependent. Hence suppose that

$$(a_1\pi)^{p_1} \dots (a_k\pi)^{p_k} = (a_1\pi)^{q_1} \dots (a_k\pi)^{q_k}$$

where $(p_1, \dots, p_k) \neq (q_1, \dots, q_k)$. Then

$$a_1^{p_1} \dots a_k^{p_k} \,\pi\, a_1^{q_1} \dots a_k^{q_k}$$

which, for some $\gamma \leqslant \alpha = \alpha_1 \dots \alpha_k$, gives

$$(a_1^{p_1} \dots a_k^{p_k})\varphi_{\alpha,\gamma} = (a_1^{q_1} \dots a_k^{q_k})\varphi_{\alpha,\gamma}.$$

By injectivity of $\varphi_{\alpha,\gamma}$, we deduce that

$$a_1^{p_1} \dots a_k^{p_k} = a_1^{q_1} \dots a_k^{q_k}$$

and A is dependent. $\square$

Lemma 4.5. *Let a separative semigroup $S = (Y; S_\alpha)$ be embedded in its semigroup of quotients $SQ = [Y; G_\alpha, \varphi_{\alpha,\beta}]$. Then S satisfies the condition*

$$a\,\eta\,b, ac = bc \;\Rightarrow\; a = b$$

if and only if all $\varphi_{\alpha,\beta}$ are injective.

P r o o f. Necessity. Let $[a,b],[c,d]\in G_\alpha$, $\alpha\geqslant\beta$ and assume that $[a,b]\varphi_{\alpha,\beta}=[c,d]\varphi_{\alpha,\beta}$. Then $[a,b][e,e]=[c,d][e,e]$ for any $e\in S_\beta$ and thus $[ae,be]=[ce,de]$ whence $aede=bece$. Since $ad\,\eta\,bc$, the hypothesis implies that $ad=bc$ so that $[a,b]=[c,d]$.

Sufficiency. Let $a,b\in S_\alpha$ and $c\in S_\gamma$ be such that $ac=bc$. In SQ, we obtain $[a^2,a][c,c]=[b^2,b][c,c]$ whence $[a^2,a][ac,ac]=[b^2,b][ac,ac]$ and thus $[a^2,a]\varphi_{\alpha,\gamma}=[b^2,b]\varphi_{\alpha,\gamma}$. The hypothesis implies that $[a^2,a]=[b^2,b]$ and thus $a=b$. □

We are now ready for one of the principal results of the paper.

Theorem 4.6. *Let S be a separative semigroup satisfying the condition*

$$a\,\eta\,b,\, ac=bc\Rightarrow a=b. \tag{4.1}$$

Then

$$\operatorname{rank} S=\operatorname{rank}(S/\pi_S)=\operatorname{rank}(SQ/\pi_{SQ})$$

where SQ/π_{SQ} is a group. If $\operatorname{rank} S$ *is finite and A is a maximal independent subset of S, then*

$$\operatorname{rank} S=|A|.$$

P r o o f. By Fact 2.2, Theorem 3.1 and Lemmas 4.2–4.4, we obtain

$$\begin{aligned}\operatorname{rank} S&=\sup\{\operatorname{rank} S_\alpha;\ \alpha\in Y\}=\sup\{\operatorname{rank} G_\alpha;\ \alpha\in Y\}\\&=\operatorname{rank} SQ=\operatorname{rank}(SQ/\pi_{SQ})=\operatorname{rank}(S/\pi_S).\end{aligned}$$

Now suppose that $\operatorname{rank} S=k$ is finite and let A be an independent subset of S. Then $|A|\leqslant\operatorname{rank} S$. Let $A=\{a_1,\dots,a_m\}$ and assume that $m<k$. We shall show that this implies that A is not maximal, contradicting the hypothesis.

We let $S=(Y;S_\alpha)$, $SQ=[Y;G_\alpha,\varphi_{\alpha,\beta}]$, where by Lemma 4.5 all $\varphi_{\alpha,\beta}$ are injective, and write φ for the canonical embedding $\varphi_S\colon S\to SQ$. By Theorem 3.1, we know that $k=\sup\{\text{rank } S_\alpha;\ \alpha\in Y\}$. Hence there exists $\beta\in Y$ such that $k=\operatorname{rank} S_\beta$. For any $\gamma\leqslant\beta$, since $\varphi_{\beta,\gamma}$ is injective, we have

$$k=\operatorname{rank} S_\beta=\operatorname{rank} G_\beta\leqslant\operatorname{rank} G_\gamma\leqslant k$$

and hence equality holds.

Let $a_i\in S_{\alpha_i}$ for $i=1,\dots,m$, $\alpha=\alpha_1\dots\alpha_m$, $\gamma=\alpha\beta$ and

$$B=\{a_1\varphi\varphi_{\alpha_1,\gamma},\dots,a_m\varphi\varphi_{\alpha_m,\gamma}\}$$

so that $B\subseteq S_\gamma Q$. We now claim that B is independent.

Indeed, suppose that

$$\prod_{i=1}^{m}(a_i\varphi\varphi_{\alpha_i,\gamma})^{p_i} = \prod_{i=1}^{m}(a_i\varphi\varphi_{\alpha_i,\gamma})^{q_i}$$

where $p_i, q_i \in \mathbb{P}$, $i = 1, \ldots, m$. It follows that

$$\Big(\prod_{i=1}^{m} a_i^{p_i}\varphi\varphi_{\alpha_i,\alpha}\Big)\varphi_{\alpha,\gamma} = \Big(\prod_{i=1}^{m} a_i^{q_i}\varphi\varphi_{\alpha_i,\alpha}\Big)\varphi_{\alpha,\gamma}$$

and since $\varphi_{\alpha,\gamma}$ is injective, we obtain

$$\prod_{i=1}^{m} a_i^{p_i}\varphi\varphi_{\alpha_i,\alpha} = \prod_{i=1}^{m} a_i^{q_i}\varphi\varphi_{\alpha_i,\alpha}.$$

This equality in SQ yields

$$\Big(\prod_{i=1}^{m} a_i^{p_i}\Big)\varphi = \Big(\prod_{i=1}^{m} a_i^{q_i}\Big)\varphi$$

which evidently implies $\prod\limits_{i=1}^{m} a_i^{p_i} = \prod\limits_{i=1}^{m} a_i^{q_i}$. Since A is independent by hypothesis, we get $p_i = q_i$ for $i = 1, \ldots, m$. This proves that B is independent.

We have assumed at the outset that $m < k$, which implies that B is not a maximal independent subset of G_γ. Hence there exists $x \in G_\gamma$ such that the set $B \cup \{x\}$ is independent. Let $x = [a, b]$ where $a, b \in S_\gamma$. Next we claim that either $A \cup \{a\}$ or $A \cup \{b\}$ must be independent. By contrapositive, suppose that both these sets are dependent. In such a case, we have equalities

$$a_1^{p_1} \ldots a_m^{p_m} a^p = a_1^{q_1} \ldots a_m^{q_m} a^q,$$
$$a_1^{r_1} \ldots a_m^{r_m} b^r = a_1^{s_1} \ldots a_m^{s_m} b^s$$

where all the exponents are in $\mathbb{P}$ and $p \neq q$, $r \neq s$. It follows that in G_γ we have

$$(a_1\varphi)^{p_1-q_1} \ldots (a_m\varphi)^{p_m-q_m}(a\varphi)^{p-q} = e_\gamma,$$
$$(a_1\varphi)^{r_1-s_1} \ldots (a_m\varphi)^{r_m-s_m}(b\varphi)^{r-s} = e_\gamma,$$

where e_γ is the identity element of G_γ, which implies that

$$(a_1\varphi)^{(p_1-q_1)(r-s)-(r_1-s_1)(p-q)} \ldots (a_m\varphi)^{(p_m-q_m)(r-s)-(r_m-s_m)(p-q)} x^{(p-q)(r-s)} = e_\gamma$$

where $(p-q)(r-s) \neq 0$. But then the set $B \cup \{x\}$ is dependent, contrary to the assumption.

Therefore either $A \cup \{a\}$ or $A \cup \{b\}$ is independent contradicting the overall hypothesis of maximality of A. Consequently $\operatorname{rank} S = |A|$. $\square$

Note that condition 4.1 is a genuine implication, that is, it can be formulated in terms of elements of S.

5. Strong semilattices of semigroups

We digress slightly from the main theme of the paper in order to complete some statements in the preceding section. In Lemma 4.4, we treated a strong semilattice of semigroups. This generalization of the semilattice of groups was not used in the proof of Theorem 4.6. It is well known that injectivity of structure mappings in a Clifford semigroup S is equivalent to S being a subdirect product of a semilattice and a group. We extend this in the following simple result.

Lemma 5.1. *Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ where S_α are semigroups and all $\varphi_{\alpha,\beta}$ are injective. Then the mapping*

$$\xi\colon\ a \mapsto (\alpha, a\pi) \quad \textit{if} \quad a \in S_\alpha \qquad (a \in S)$$

is an isomorphism of S onto a subdirect product of Y and S/π.

P r o o f. Straightforward. □

As a converse of this lemma, we have the following.

Lemma 5.2. *Let S be a subdirect product of a semilattice Y and a cancellative semigroup T. Then S is a strong semilattice Y of semigroups S_α with all structure homomorphisms injective.*

P r o o f. For every $\alpha \in Y$, let $S_\alpha = \{(\alpha, t) \in S\}$. The verification of the assertions implicit in the contention of the lemma require just a simple argument. □

Corollary 5.3. *Let S be a subdirect product of a semilattice Y and a cancellative (archimedean) semigroup T. Then* $\operatorname{rank} S = \operatorname{rank} T$.

P r o o f. This follows easily from Lemmas 4.4 and 5.2. □

In Lemma 4.5 we expressed condition (4.1) directly on the separative semigroup S. It is of some interest to express the condition for S to be a strong semilattice of cancellative (archimedean) semigroups directly on S. This we do in the next result.

Theorem 5.4. *Let $S = (Y; S_\alpha)$ where each S_α is archimedean and cancellative. Assume that S is embedded in its semigroup of quotients $SQ = [Y; G_\alpha, \varphi_{\alpha,\beta}]$. Then the following statements are equivalent.*

(i) *In SQ for any $\alpha \geqslant \beta$, $\varphi_{\alpha,\beta}$ maps S_α into S_β.*

(ii) *S is a strong semilattice Y of semigroups S_α.*

(iii) *For any $a, b \in S_\alpha$ and $\alpha \geqslant \beta$, there exists (a unique) $b \in S_\beta$ such that $a^2b = ab^2$.*

(iv) *For any $a, b \in S$ such that $b^n \in aS$ for some $n \in \mathbb{P}$, there exists (a unique) $c \in S$ such that $b\,\eta\,c$ and $a^2c = ac^2$.*

P r o o f. (i) implies (ii). It is immediate that $\psi_{\alpha,\beta} = \varphi_{\alpha,\beta}|_S$ has all the requisite properties for a strong semilattice Y of semigroups S_α.

(ii) implies (iii). Let $S = [Y; S_\alpha, \psi_{\alpha,\beta}]$. For $a \in S_\alpha$, $\alpha \geqslant \beta$ and $b = a\psi_{\alpha,\beta}$, we get

$$a^2b = (a^2\psi_{\alpha,\beta})b = (a\psi_{\alpha,\beta})(a\psi_{\alpha,\beta})b = (a\psi_{\alpha,\beta})b^2 = ab^2.$$

For uniqueness, assume that for $c \in S_\beta$ we have $a^2c = ac^2$. Then $a^2cb = ac^2b$ whence

$$b(acb) = ac^2b = a^2cb = (a^2b)c = (ab^2)c = b(acb)$$

and by cancellation in S_β, we obtain $b = c$.

(iii) implies (i). Let $[a^2, a] \in S_\alpha$, $\alpha \geqslant \beta$ and $b \in S_\beta$. Then

$$[a^2, a]\varphi_{\alpha,\beta} = [a^2, a][b, b] = [a^2b, ab] = [ab^2, ab] = [a, a][b^2, b] = [b^2, b]$$

as required.

Condition (iv) is essentially a restatement of part (iii). $\square$

We now formulate a summary statement.

Corollary 5.5. *A semigroup S is a subdirect product of a semilattice and a cancellative (archimedean) semigroup if and only if it satisfies condition* (4.1) *and the condition in Theorem* 5.4(iv).

6. Separative semigroups with one or two generators

It seems reasonable to expect that a class $\mathcal{C}$ of semigroups could be determined concretely, or at least classified in a certain way, if we limit our considerations to members of $\mathcal{C}$ generated by a single element, or less modestly, by a 2-element set. Recall that Hall [6] classified precisely commutative cancellative semigroups with at most two generators. If we specify that $\mathcal{C}$ be the class of commutative semigroups, then with monogenic ones there is no problem, but already for those with a 2-element generating set there arise great difficulties. We thus limit ourselves to the class $\mathcal{C}$ of commutative separative semigroups. Even this is a daunting problem, but we will provide for it a rough classification.

Recall that for a (not necessarily commutative) semigroup S, the *lower rank* of S is defined in [8] as

$$\operatorname{lrank} S = \min\{k;\ S \text{ has a set of generators of cardinality } k\}.$$

Hence we are interested here in (commutative) separative semigroups of lower rank 1 or 2. Observe that monogenic is synonymous with lower rank equal to 1. We start with a comparison of rank and lower rank, and remind the reader that S denotes a commutative semigroup.

Proposition 6.1. $\operatorname{rank} S \leqslant \operatorname{lrank} S$.

Proof. We assume that $\operatorname{lrank} S = m$ with $m \in \mathbb{P}$, let $A = \{a_1, \dots, a_{m+1}\}$ be any subset of S having $m+1$ elements, and we will show that A must be dependent.

Let $X = \{x_1, \dots, x_m\}$ be a generating set of S having m elements. Hence

$$a_i = x_1^{r_{i,1}} \dots x_m^{r_{i,m}} \qquad (1 \leqslant i \leqslant m+1)$$

for some $r_{i,k} \in \mathbb{N}$. Since $\operatorname{rank} \mathbb{N}^m = m$, the set

$$\{(r_{1,1}, \dots, r_{1,m}), \dots, (r_{m+1,1}, \dots, r_{m+1,m})\}$$

of m-tuples is dependent, and thus there exist $p_i, q_i \in \mathbb{P}$, for $i = 1, \dots, m+1$ such that

$$\sum_{i=1}^{m+1} p_i(r_{i,1}, \dots, r_{i,m}) = \sum_{i=1}^{m+1} q_i(r_{i,1}, \dots, r_{i,m})$$

with $(p_1, \dots, p_m) \neq (q_1, \dots, q_m)$. It follows that

$$\sum_{i=1}^{m+1} p_i\, r_{i,j} = \sum_{i=1}^{m+1} q_i\, r_{i,j} \quad (1 \leqslant j \leqslant m). \tag{6.1}$$

Now

$$\begin{aligned} a_1^{p_1}\dots a_{m+1}^{p_{m+1}} &= (x_1^{r_{1,1}p_1}\dots x_m^{r_{1,m}p_1})\dots(x_1^{r_{m+1,1}p_{m+1}}\dots x_m^{r_{m+1,m}p_{m+1}}) \\ &= x_1^{r_{1,1}p_1+\dots+r_{m+1,1}p_{m+1}}\dots x_m^{r_{1,m}p_1+\dots+r_{m+1,m}p_{m+1}} \end{aligned}$$

and similarly we get the same kind of expression for $a_1^{q_1}\dots a_{m+1}^{q_{m+1}}$ with q's instead of p's. By (6.1), we finally obtain that

$$a_1^{p_1}\dots a_{m+1}^{p_{m+1}} = a_1^{q_1}\dots a_{m+1}^{q_{m+1}},$$

which shows that A is dependent. □

We are concerned here with semigroups whose lower rank is 1 or 2. To this end, we first consider groups. Recall the notation $\mathbb{Z}_0$ and $\mathbb{Z}_1$ in Section 2.

Fact 6.2. *Let G be a group.*
(i) $\operatorname{lrank} G = 1$ *if and only if* $G \cong \mathbb{Z}_m$ *for some* $m \geqslant 1$.
(ii) $\operatorname{lrank} G \leqslant 2$ *if and only if* $G \cong \mathbb{Z}_m \times \mathbb{Z}_n$ *for some* $m \geqslant 0$, $n \geqslant 1$.

Proof. See [6, Theorem 1]. □

As for idempotent-free archimedean cancellative semigroups, we first recall that they are called *N-semigroups*. For these semigroups we have the following result due to Tamura.

Fact 6.3. *Let G be an abelian group and $I\colon\ G\times G\to\mathbb{N}$ a function satisfying the conditions: for all $a,b,c\in G$,*

$$\begin{aligned} I(a,b)+I(ab,c) &= I(a,bc)+I(b,c), \\ I(a,b) &= I(b,a), \\ I(e,e) &= 1, \end{aligned}$$

where e is the identity of G. On the set $\mathbb{N}\times G$ define a multiplication by

$$(m,a)(n,b) = (m+n+I(a,b),ab),$$

and denote the resulting groupoid by $\mathbb{N}(G,I)$. Then $\mathbb{N}(G,I)$ is an N-semigroup. Conversely, every N-semigroup is isomorphic to some $\mathbb{N}(G,I)$.

Proof. See [5, Theorem III,4.5] and [10, Section II.7]. □

Fact 6.4. $\operatorname{rank} \mathbb{N}(G, I) = \operatorname{rank} G + 1$.

Proof. See [1, Corollary 4.3]. $\square$

In Theorem 6.6 below we will classify separative semigroups of lower rank 1 or 2 according to the number of their archimedean components. In each component we distinguish groups and N-semigroups. In the discussion of Theorem 6.6 and its proof, we omit the details but supply an outline which, for an assiduous reader, should be sufficient for reconstructing a complete argument.

As in every classification, it is of importance to observe how far it goes, namely at which point do we stop explaining the distinction among classified objects. We provide only a necessary condition: the "type" in our case is the rough make up of archimedean components and we do not specify what the products are of elements in different components. Hence our classification can rightly be considered a first step in determining the structure of the semigroups in question; we leave its completion to the next generation.

If there exists a singleton generating set, we have one of the following cases:

$$\begin{aligned} &\mathbb{Z}_m \ \text{ for the group case, } m > 0, \\ &\mathbb{P} \ \text{ for the } N\text{-semigroup case.} \end{aligned}$$

It remains to consider the 2-generator case. If S has only one component, it is a group or it is an N-semigroup with a 2-element generating set. Fact 6.2 provides all group cases. In the N-semigroup case, we have two possibilities for S: $\mathbb{P}$ and the semigroup $N(m_1, m_2)$ below.

Fact 6.5. *Let $2 \leqslant m_1 \leqslant m_2$ and set*

$$N(m_1, m_2) = \{(k_1, k_2);\ k_1 \in \mathbb{N},\ k_2 = 0, \ldots, m_2 - 1,\ k_1 + k_2 > 0\}$$

with multiplication

$$(k_1, k_2)(l_1, l_2) = \begin{cases} (k_1 + l_1, k_2 + l_2) & \text{if } k_2 + l_2 < m_2 \\ (k_1 + l_1 + m_1, k_2 + l_2 - m_2) & \text{otherwise.} \end{cases}$$

Then $N(m_1, m_2)$ is an N-semigroup generated by a 2-element set. Conversely, every N-semigroup generated by a 2-element set, but not by a 1-element set, is isomorphic to some $N(m_1, m_2)$.

Proof. See [5, Proposition VI.6.7], [9, Theorem 5]. $\square$

In the 2- or 3-component case, the upper ones must be monogenic; hence

$$\begin{aligned}&\text{either } \mathbb{Z}_m,\ \ m>0,\ \text{ in the group case,}\\ &\text{or } \mathbb{P}\ \text{ in the } N\text{-semigroup case.}\end{aligned}$$

In the lower component, we either have a group from Lemma 6.2 or an N-semigroup. In both cases, we are faced with an ideal extension of the lower component by one or two upper components with a zero adjoined. For an extensive treatment of ideal extensions, we recommend the book [10].

If the lower component is a group, the ideal extension is determined by homomorphisms from the upper components to the lower component. The case when the lower component is an N-semigroup, we may set it equal to $\mathbb{N}(G,I)$, which is much more complex. The extension is constructed by means of homomorphisms of the upper components into the semigroup of (left) translations $\Lambda(\mathbb{N}(G,I))$. The semigroup $\Lambda(\mathbb{N}(G,I))$ was constructed by Hall [7, Theorem 1.1 and Proposition 1.1] and homomorphisms from a group into it in [7, Theorem 2.1]. One of the parameters of this homomorphism $H \to \Lambda(\mathbb{N}(G,I))$ is a homomorphism $\psi\colon\ H \to G$ which implies that G is also of the type which occurs in Fact 6.2. The difficulty arises from the conditions that ψ must satisfy in relation to the function I.

The ranks of the groups that occur in this discussion are easily determined. For the same purpose with $\mathbb{N}(G,I)$ we use Fact 6.4. In all cases, we use Theorem 3.1 repeatedly. In the next theorem, for the case 4, we need the reference [3, Lemma 4.1] and for the case 9, [2, Theorem 4.2].

Theorem 6.3. *Every separative semigroup S with* $\operatorname{lrank} S \leqslant 2$ *is exactly one of the following types. We set*

$$G = \mathbb{Z}_p \times \mathbb{Z}_q, \quad p,q \geqslant 0,\ p+q>0.$$

(i) *S has 1 component:*

$$1\ \ {\bullet}\mathbb{Z}_m \qquad 2\ \ {\bullet}\mathbb{Z}_m \times \mathbb{Z}_n \qquad 3\ \ {\bullet}\mathbb{P} \qquad 4\ \ {\bullet}N(m_1,m_2)$$

where $m \geqslant 0$, $n>0$, $2 \leqslant m_1 \leqslant m_2$.

(ii) *S has 2 components:*

$$5\ \begin{array}{l} \bullet\mathbb{Z}_m \\ \ | \\ \bullet G \end{array} \qquad 6\ \begin{array}{l} \bullet\mathbb{P} \\ \ | \\ \bullet G \end{array} \qquad 7\ \begin{array}{l} \bullet\mathbb{Z}_m \\ \ | \\ \bullet\mathbb{N}(G,I) \end{array} \qquad 8\ \begin{array}{l} \bullet\mathbb{P} \\ \ | \\ \bullet\mathbb{N}(G,I) \end{array}$$

where $m>0$.

(iii) *S has 3 components:*

$$
9\quad
\begin{tikzcd}[column sep=small]
\bullet\mathbb{Z}_m \arrow[dr, dash] & & \bullet\mathbb{Z}_n \arrow[dl, dash] \\
& \bullet G &
\end{tikzcd}
\qquad
10\quad
\begin{tikzcd}[column sep=small]
\bullet\mathbb{Z}_m \arrow[dr, dash] & & \bullet\mathbb{P} \arrow[dl, dash] \\
& \bullet G &
\end{tikzcd}
\qquad
11\quad
\begin{tikzcd}[column sep=small]
\bullet\mathbb{P} \arrow[dr, dash] & & \bullet\mathbb{P} \arrow[dl, dash] \\
& \bullet G &
\end{tikzcd}
$$

$$
12\quad
\begin{tikzcd}[column sep=small]
\bullet\mathbb{Z}_m \arrow[dr, dash] & & \bullet\mathbb{P} \arrow[dl, dash] \\
& \bullet\mathbb{N}(G,I) &
\end{tikzcd}
\qquad
13\quad
\begin{tikzcd}[column sep=small]
\bullet\mathbb{P} \arrow[dr, dash] & & \bullet\mathbb{P} \arrow[dl, dash] \\
& \bullet\mathbb{N}(G,I) &
\end{tikzcd}
$$

where $m, n > 0$.

$$
\begin{aligned}
\operatorname{rank} S = 0\colon\ & (1,\ m>0),\ (2,\ m>0), (5,\ m>0),\ (9,\ m,p,q>0, n \geqslant 0),\\
\operatorname{rank} S = 1\colon\ & (1,\ m=0),\ (2,\ m=0),\ 3,\ 4,\ 6,\ (7,\ p,q>0),\ (8,\ p,q>0)\\
& (9,\ p=0),\ 10,\ 11,\ (12,\ m,p,q>0),\ (13,\ p,q>0),\\
\operatorname{rank} S = 2\colon\ & (7,\ p=0),\ (8,\ p=0),\ (12,\ p=0),\ (13,\ p=0).
\end{aligned}
$$

Observe that in Theorem 6.6, the case of two upper components being groups and the lower component an N-semigroup is missing since this case cannot occur. Simple examples show that all other types indeed occur.

Of course there are special cases where we can be more precise in describing their structure. We show this on the following example.

Proposition 6.7. *Let S be a separative semigroup with two archimedean components: the upper one $A = \mathbb{Z}_m$ for $m > 0$, and the lower one B a power cancellative N-semigroup. Then for any $a \in A$ and $b \in B$, we have $a \star b = b$.*

P r o o f. Since B is power cancellative, its group of quotients BQ is torsion free. The semigroup of translations $\Lambda(B)$ is embeddable into BQ and is thus torsion free. Any ideal extension of B by A is determined by a homomorphism $\varphi\colon A \to \Lambda(B)$. Since A is finite cyclic and $\Lambda(B)$ is torsion-free, φ must map A onto the identity of $\Lambda(B)$. As a result, we get the assertion of the proposition. $\square$

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