A SIMPLE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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Abstract. In the paper an elementary and simple proof of the Fundamental Theorem of Algebra is given.

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1. Introduction

The first accepted proof of the Fundamental Theorem of Algebra was furnished by C. F. Gauss; during his life Gauss gave four proofs of this Theorem. Although the statement of the Fundamental Theorem is easily understood by a high school student the Gauss proofs are sophisticated and use advanced mathematics. They are still of interest to contemporary mathematicians and a proof using modern mathematical language can be found in [1], pp. 247–249. There were more than fifty papers written on it, perhaps hundreds. A monograph [8] is devoted to the Fundamental Theorem of Algebra and various proofs of it. Moreover, its first chapter gives a brief but very informative and interesting history of the Theorem. One of Gauss' proofs can be simplified using the Green Theorem, see e.g. [6]. We are interested in a proof which is elementary and whose sophistication is not much above the level of the statement of the theorem. A lot of effort was made to give such a proof. There are roughly two categories of these proofs. One approach establishes that the absolute value of the polynomial in question reaches its minimum over the complex plane at a finite point in the plane and derives a contradiction from the assumption that this minimum is not zero. Examples of such proofs are in [2], [3], [4] and in [5]. The proofs in the other category use the fact that a real polynomial of odd degree always has a root; it also uses (proves) the existence of a square root of a complex number. Examples of proofs of this kind are in [2], pp. 464–467 and [7]. There is a very simple, short and elegant proof of the Fundamental Theorem of Algebra based on Liouville's Theorem. The difficulty with it is that it presumes the knowledge of complex analysis, including complex integration and the Cauchy Theorem. Our proof is based on a similar idea as the proof by the Liouville theorem but replaces the aparatus of complex analysis by the so called Mean Value Property (MVP for short). Our proof is short and requires the following prerequisites. The Fundamental Theorem of Calculus for a complex valued function of a real variable with a continuous derivative. We also use the definition of a derivative to justify in an elementary manner the interchange of differentiation and integration. The other essential ingredient of the proof is the Tayor theorem for polynomials, which of course does not require any advanced analysis because it is an identity between polynomials.

2. The mean value property

We say that f satisfies the Mean Value Property (MVP for short) on a ball $B_a^R = \{z;\ |z-a| < R\}$ if

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + te^{i\varphi})\,d\varphi \quad \text{for } 0 < t < R.$$

It is easy to see that a polynomial has the MVP on any ball of the complex plane. We shall prove that if P is a polynomial which has no roots in a ball B then the rational function $1/P$ has MVP on B. In particular if P has no zeros in B_0^R then

(1) $$\frac{1}{P(0)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{P(te^{i\varphi})} \quad \text{for } 0 < t < R.$$

3. The fundamental theorem of algebra

Theorem. *Every polynomial (with real or complex coefficients) of degree $n \geqslant 1$ has a root.*

P r o o f. Assume, contrary to what we wish to prove, that there is a polynomial P of degree $n \geqslant 1$ which has no zeros in $\mathbb{C}$. Since[1] $\lim_{z\to\infty} 1/P(z) = 0$ there is $r > 0$ such that

$$\frac{1}{|P(z)|} < \frac{1}{2|P(0)|} \quad \text{for } |z| > r.$$

[1] Here we use the assumption that degree of P is at least one.

Applying equation (1) with $t > r$ leads to

$$\frac{1}{|P(0)|} \leqslant \frac{1}{2\pi} \frac{1}{2|P(0)|} 2\pi = \frac{1}{2|P(0)|}.$$

This is a clear contradiction.

4. Proof of MVP

Define

$$M(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{P(t\mathrm{e}^{\mathrm{i}\varphi})}. \tag{2}$$

Our aim is to show that $M'(t) = 0$ for $t > 0$. If

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

then by the Taylor formula

$$P(a + H) = c_0 + c_1 H + \ldots + c_n H^n \tag{3}$$

with $c_0 = P(a)$ and $c_1 = P'(a)$. If Q is a polynomial, then

$$Q(s) = q_0 + q_1 s + \ldots + q_m s^m$$

with $q_0 \neq 0$; then, using 'the long division algorithm', we have

$$1/Q(s) = \Big(\frac{1}{q_0} - \frac{q_1 s}{q_0^2}\Big) + s^2 \frac{R(s)}{Q(s)}$$

with some polynomial R. By using this for $a = t\mathrm{e}^{\mathrm{i}\varphi}$, $s = H = h\mathrm{e}^{\mathrm{i}\varphi}$ and $Q(H) = P(a + H)$ together with equation (3) we have

$$\frac{M(t+h) - M(t)}{h} = -\frac{1}{2\pi} \int_0^{2\pi} \frac{P'(t\mathrm{e}^{\mathrm{i}\varphi})\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{d}\varphi}{P^2(t\mathrm{e}^{\mathrm{i}\varphi})} + \frac{h}{2\pi} \int_0^{2\varphi} \frac{R((t+h)\mathrm{e}^{\mathrm{i}\varphi})\mathrm{e}^{2\mathrm{i}\varphi}\,\mathrm{d}\varphi}{P((t+h)\mathrm{e}^{\mathrm{i}\varphi})}.$$

Taking limit for $h \to 0$ proves that

$$M'(t) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{P'(t\mathrm{e}^{\mathrm{i}\varphi})\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{d}\varphi}{P^2(t\mathrm{e}^{\mathrm{i}\varphi})}. \tag{4}$$

Since

(5)
$$\frac{P'(te^{i\varphi})e^{i\varphi}}{P^2(te^{i\varphi})} = -\frac{1}{ti}\frac{d}{d\varphi}\frac{1}{P(te^{i\varphi})},$$

it follows from the Fundamental Theorem of Calculus and periodicity of the exponential that $M' = 0$. Consequently, M is constant and since

$$\lim_{t\to 0} M(t) = \frac{1}{P(0)},$$

equation (1) follows.

5. Another proof of MVP

By using the change of order of integration in an iterated integral we can simplify the proof of the Mean Value Property.

By the fundamental theorem of calculus we have

$$\frac{1}{P(ve^{i\varphi})} - \frac{1}{P(ue^{i\varphi})} = -\frac{1}{2\pi}\int_u^v \frac{P'(te^{i\varphi})e^{i\varphi}\,dt}{P^2(te^{i\varphi})}.$$

Consequently

(6)
$$M(v) - M(u) = -\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_u^v \frac{P'(te^{i\varphi})e^{i\varphi}\,dt}{P^2(te^{i\varphi})} = \int_u^v dt \int_0^{2\pi} \frac{P'(te^{i\varphi})e^{i\varphi}\,d\varphi}{P^2(te^{i\varphi})} = 0.$$

The rest follows.

A c k n o w l e d g e m e n t. When author completed an earlier version of this paper he was given the opportunity to see the manuscript of a paper by Schep: A simple complex analysis and advanced calculus proof of the fundamental theorem of algebra. As the title indicates this paper uses more advanced mathematics than the present paper but the main idea of the proof is the use of MVP. Schep's paper will appear in the American Mathematical Monthly.

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