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CONTROLLABILITY AND OBSERVABILITY OF TIME-INVARIANT LINEAR DYNAMIC SYSTEMS

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Abstract. In the paper, we unify and extend some basic properties for linear control systems as they appear in the continuous and discrete cases. In particular, we examine controllability, reachability, and observability for time-invariant systems and establish a duality principle.

Keywords: time scale, dynamic equation, exponential function, controllability, reachability, observability, duality principle, time invariance

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1. Introduction

In the early 1960s, R. E. Kalman introduced two concepts that have since become the backbone of modern control theory (see [10]–[13]). With "controllability" and "observability", one can classify a control system without first finding the solution in closed form. A linear system is said to be controllable if there exists at least one input that drives the state vector to the origin. On the other hand, a linear system is said to be observable if there exists at least one output such that the initial state can be determined. These properties have been studied in depth in both the continuous and discrete cases, where one can see striking similar, if not identical, results. Yet until recently there did not exist a method to relate these results in one case with the results in the other.

Then in 1988, Stefan Hilger under the direction of Bernd Aulbach introduced calculus on time scales in his PhD thesis [1]. The study of dynamic equations unifies both the continuous and discrete mathematical analysis. As a result, one can generalize a process to account for both cases, or any combination of the two. Since its inception, this area of mathematics has gained a great deal of international attention. Researchers have since found applications of time scales to include heat transfer, population dynamics, as well as economics. For a more indepth study of time scales, it is suggested that one should see Bohner and Peterson's books [4], [5].

The purpose of this paper is to lay down the foundation of linear control systems on time scales. Here we examine controllability, reachability, and observability in the time-invariant case. It should be noted that there have been other excellent attempts to do so, e.g., in [2], [3], [7], [8]. They all examine the linear system

$$\begin{cases} x^{\Delta}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases} \tag{1.1}$$

in an effort to generalize controllability and observability for dynamic equations. At first, this seems to be a very natural extension from the continuous and discrete cases. However, when studying controllability of the linear system (1.1) in a way following the corresponding proofs for the continuous and discrete systems, one must assume that the graininess function is differentiable, an assumption that is not satisfied in general for all time scales (see [4, Example 1.56]). To stepside this issue, we have altered the linear system so that it appears as

$$\begin{cases} x^{\Delta}(t) = -Ax^{\sigma}(t) + Bu(t), \\ y(t) = Cx^{\sigma}(t). \end{cases} \tag{1.2}$$

The study of controllability of the linear system (1.2) turns out to be feasible using the classical techniques without assuming differentiability of the graininess function. However, when examining observability of the linear system (1.2) using classical methods, one must again assume differentiability of the graininess function. But the observability study of the linear system (1.1) does not feature this problem.

Hence we present in this paper a study of controllability of the linear system (1.2) and a study of observability of the linear system (1.1). Then we proceed to draw a connection between the two linear systems. As a result, we see that this connection of controllability and observability for linear systems on time scales is more compelling than previously realized.

The remainder of this paper is organized as follows: Section 2 offers a very brief introduction into the time scales concepts that are needed in this paper. In Section 3 we introduce the necessary terminology and present the assumptions. In Sections 4 and 5, we study controllability and reachability of (1.2), while observability of (1.1) is investigated in Section 6. The final Section 7 relates these concepts to each other. The results presented in this paper are part of the second author's PhD thesis [15].

2. Preliminaries

In this section we present the relevant concepts from the theory of dynamic equations on time scales.

Definition 2.1. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We let $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Example 2.2. The most common examples of time scales are $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, and $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$.

Definition 2.3. We define the *forward jump operator* $\sigma \colon \mathbb{T}^\kappa \to \mathbb{T}$ and the *graininess function* $\mu \colon \mathbb{T}^\kappa \to [0, \infty)$ by

$$\sigma(t) := \inf \{s \in \mathbb{T} \colon s > t\} \quad \text{and } \mu(t) = \sigma(t) - t \quad \text{for all } t \in \mathbb{T}^\kappa.$$

Definition 2.4. For any function $f \colon \mathbb{T} \to \mathbb{R}$, we define $f^\sigma \colon \mathbb{T}^\kappa \to \mathbb{R}$ by $f^\sigma = f \circ \sigma$.

Definition 2.5. Let $f \colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The *delta derivative* $f^\Delta(t)$ is the number (when it exists) such that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

Example 2.6. When $\mathbb{T} = \mathbb{R}$, then (if the limit exists)

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

When $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t) = \Delta f(t).$$

Definition 2.7. A function $f \colon \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* on $\mathbb{T}$ if A is continuous at points $t \in \mathbb{T}$ with $\sigma(t) = t$ and has finite left-sided limits at points $t \in \mathbb{T}$ with $\sup\{s \in \mathbb{T} \colon s < t\} = t$.

Theorem 2.8 (see [4, Theorem 1.74])**.** *Any rd-continuous function $f \colon \mathbb{T} \to \mathbb{R}$ has an antiderivative F, i.e., $F^\Delta = f$ on $\mathbb{T}^\kappa$.*

Definition 2.9. Let $f \colon \mathbb{T} \to \mathbb{R}$ be an rd-continuous function and let F be an antiderivative of f. Then the *Cauchy integral* of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

Example 2.10. Let $a, b \in \mathbb{T}$ with $a < b$ and suppose $f\colon\ \mathbb{T} \to \mathbb{R}$ is rd-continuous. When $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t.$$

When $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

Finally, we introduce the matrix exponential on $\mathbb{T}$ and some of its properties.

Definition 2.11. An $m \times n$-matrix-valued function A on $\mathbb{T}$ is called *rd-continuous* if each of its scalar entry functions are rd-continuous. Moreover, if $m = n$, then A is called *regressive* provided

$$I + \mu(t)A(t) \quad \text{is invertible for all } t \in \mathbb{T}^\kappa,$$

where I is the $n \times n$-identity matrix.

Theorem 2.12 (see [4, Theorem 5.8])**.** *Let $t_0 \in \mathbb{T}$. If A is an rd-continuous and regressive $n \times n$-matrix-valued function on $\mathbb{T}$, then the initial value problem*

$$X^\Delta(t) = A(t)X(t), \quad X(t_0) = I$$

has a unique $n \times n$-matrix-valued solution X.

Definition 2.13. The solution X from Theorem 2.12 is called the *matrix exponential function* on $\mathbb{T}$ and is denoted by $e_A(\cdot, t_0)$.

Theorem 2.14 (see [4, Theorem 5.21])**.** *If A is rd-continuous and regressive, then*

(a) $e_A(t,t) = I$ *for all* $t \in \mathbb{T}$,
(b) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t,s)$ *for all* $s \in \mathbb{T}$, $t \in \mathbb{T}^\kappa$,
(c) $e_A(t,s)e_A(s,r) = e_A(t,r)$ *for all* $r, s, t \in \mathbb{T}$,
(d) $e_A^{-1}(t,s) = e_A(s,t)$ *for all* $s, t \in \mathbb{T}$.

3. Terminology and assumptions

Let $m, n, r \in \mathbb{N}$ and suppose that A, B, and C are real-valued matrices of dimensions $n \times n$, $n \times m$, and $r \times n$, respectively. A is assumed to be regressive. In the linear systems (1.1) and (1.2), $x\colon \mathbb{T} \to \mathbb{R}^n$, $u\colon \mathbb{T} \to \mathbb{R}^m$, and $y\colon \mathbb{T} \to \mathbb{R}^r$ are called the *state*, *input* (or *control*), and *output*, respectively. The control is assumed to be rd-continuous. The first equation in each of the linear systems (1.1) and (1.2) is called the *state equation*. Throughout this paper, we will make the following assumption:

$$t_0, t_\mathrm{f} \in \mathbb{T} \quad \text{such that } t_\mathrm{f} > \sigma^n(t_0).$$

4. Controllability

In this section we consider the linear system (1.2). We need the following variation of parameters result for the state equation of the linear system (1.2).

Theorem 4.1 (see [4, Theorem 5.27]). *Suppose A is a regressive $n \times n$-matrix and B is an $n \times m$-matrix. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Suppose $u\colon \mathbb{T} \to \mathbb{R}^m$ is rd-continuous. Then the unique solution of the initial value problem*

$$x^\Delta(t) = -Ax^\sigma(t) + Bu(t), \quad x(t_0) = x_0$$

is given by

$$\begin{aligned} x(t) &= e^T_{A^T}(t_0, t)x_0 + \int_{t_0}^{t} e^T_{A^T}(\tau, t)Bu(\tau)\Delta\tau \\ &= e^T_{A^T}(t_0, t)\left[x_0 + \int_{t_0}^{t} e^T_{A^T}(\tau, t_0)Bu(\tau)\Delta\tau\right]. \end{aligned}$$

We refer to a linear system as being “controllable” if there exist inputs such that the state vector “can be steered” to the origin for any given initial condition. The precise definition is as follows.

Definition 4.2. The linear system (1.2) is said to be (*completely*) *controllable* on $[t_0, t_\mathrm{f}]$ if for all $x_0 \in \mathbb{R}^n$, there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = x_0$ satisfies $x(t_\mathrm{f}) = 0$.

Next, we give the generalized controllability criterion for linear systems (1.2) as follows.

Theorem 4.3. *The linear system* (1.2) *is controllable if and only if the controllability Gramian* W_{C} *is invertible, where*

$$W_{\mathrm{C}} := \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_0) B B^T e_{A^T}(\tau, t_0) \Delta\tau.$$

Proof. First assume that (1.2) is controllable and let $\alpha \in \operatorname{Ker} W_{\mathrm{C}}$. Then

$$\begin{aligned} 0 &= \alpha^T W_{\mathrm{C}} \alpha \\ &= \int_{t_0}^{t_{\mathrm{f}}} \alpha^T e^T_{A^T}(\tau, t_0) B B^T e_{A^T}(\tau, t_0) \alpha \Delta\tau \\ &= \int_{t_0}^{t_{\mathrm{f}}} \left\| B^T e_{A^T}(\tau, t_0) \alpha \right\|^2 \Delta\tau, \end{aligned}$$

which implies $B^T e_{A^T}(\tau, t_0)\alpha = 0$ for all $\tau \in [t_0, t_{\mathrm{f}}) \cap \mathbb{T}$. Since (1.2) is controllable, there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = \alpha$ satisfies $x(t_{\mathrm{f}}) = 0$. It follows from Theorem 4.1 that

$$\alpha = x(t_0) = -\int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_0) B u(\tau) \Delta\tau.$$

Then

$$\|\alpha\|^2 = \alpha^T \alpha = -\int_{t_0}^{t_{\mathrm{f}}} u^T(\tau) B^T e_{A^T}(\tau, t_0) \alpha \Delta\tau = 0,$$

which implies $\alpha = 0$. Hence $\operatorname{Ker} W_{\mathrm{C}} = \{0\}$. Therefore W_{C} is invertible.

Now assume that W_{C} is invertible and let $x_0 \in \mathbb{R}^n$. Define $u\colon \mathbb{T} \to \mathbb{R}^m$ by

$$u(t) = -B^T e_{A^T}(t, t_0) W_{\mathrm{C}}^{-1} x_0.$$

Then by Theorem 4.1, the solution x of the state equation of (1.2) with $x(t_0) = x_0$ satisfies

$$\begin{aligned} x(t_{\mathrm{f}}) &= e^T_{A^T}(t_0, t_{\mathrm{f}}) \left[x_0 + \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_0) B u(\tau) \Delta\tau \right] \\ &= e^T_{A^T}(t_0, t_{\mathrm{f}}) \left[x_0 - \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_0) B B^T e_{A^T}(\tau, t_0) W_{\mathrm{C}}^{-1} x_0 \Delta\tau \right] \\ &= e^T_{A^T}(t_0, t_{\mathrm{f}}) [x_0 - W_{\mathrm{C}} W_{\mathrm{C}}^{-1} x_0] \\ &= 0, \end{aligned}$$

which tells us that the linear system is controllable. □

Now we present the generalized Kalman rank condition for controllability of linear systems (1.2).

Theorem 4.4. *The linear system* (1.2) *is controllable if and only if the* $n \times (nm)$ *controllability matrix* $\Gamma_{\mathrm{C}}[A, B]$ *has full rank* n, *where*

$$\Gamma_{\mathrm{C}}[A, B] := [\ B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B\ ].$$

P r o o f. First assume that (1.2) is controllable. Let $x_0 \in \mathbb{R}^n$. Then there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = x_0$ satisfies $x(t_{\mathrm{f}}) = 0$. It follows from Theorem 4.1 that

$$x_0 = -\int_{t_0}^{t_{\mathrm{f}}} e_{A^T}^T(\tau, t_0) B u(\tau) \Delta\tau. \tag{4.1}$$

Now by DaCunha [6] or Zafer [16] (using the Cayley-Hamilton theorem), there exist r_j for $0 \leqslant j \leqslant n-1$ such that

$$e_{A^T}(t, t_0) = \sum_{j=0}^{n-1} r_j(t, t_0)(A^T)^j. \tag{4.2}$$

Next, we define F_j for $0 \leqslant j \leqslant n-1$ by

$$F_j = -\int_{t_0}^{t_{\mathrm{f}}} r_j(\tau, t_0) u(\tau) \Delta\tau.$$

Substituting F_j and (4.2) into (4.1), we obtain

$$\begin{aligned} x_0 &= -\int_{t_0}^{t_{\mathrm{f}}} e_{A^T}^T(\tau, t_0) B u(\tau) \Delta\tau \\ &= -\int_{t_0}^{t_{\mathrm{f}}} \left[\sum_{j=0}^{n-1} r_j(\tau, t_0) A^j \right] B u(\tau) \Delta\tau \\ &= \sum_{j=0}^{n-1} A^j B F_j \\ &= \Gamma_{\mathrm{C}}[A, B] \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{n-1} \end{bmatrix} \in \operatorname{Im} \Gamma_{\mathrm{C}}[A, B]. \end{aligned}$$

Then $\mathbb{R}^n \subset \operatorname{Im} \Gamma_{\mathrm{C}}[A, B] \subset \mathbb{R}^n$ and thus $\operatorname{rank} \Gamma_{\mathrm{C}}[A, B] = n$.

Now assume that $\operatorname{rank}\Gamma_{\mathrm{C}}[A,B]=n$ and let $\alpha\in\operatorname{Ker}W_{\mathrm{C}}$. Then

$$\begin{aligned}0&=\alpha^T W_{\mathrm{C}}\alpha\\&=\int_{t_0}^{t_{\mathrm{f}}}\alpha^T e_{A^T}^T(\tau,t_0)BB^T e_{A^T}(\tau,t_0)\alpha\Delta\tau\\&=\int_{t_0}^{t_{\mathrm{f}}}\left\|B^T e_{A^T}(\tau,t_0)\alpha\right\|^2\Delta\tau,\end{aligned}$$

which implies

$$B^T e_{A^T}(\tau,t_0)\alpha=0\quad\text{for all }\tau\in[t_0,t_{\mathrm{f}})\cap\mathbb{T}. \tag{4.3}$$

Differentiating (4.3) m times, where $0\leqslant m\leqslant n-1$, we have

$$B^T(A^m)^T e_{A^T}(\tau,t_0)\alpha=0\quad\text{for all }\tau\in[t_0,t_{\mathrm{f}})^{\kappa^m}\cap\mathbb{T}$$

and hence

$$(A^mB)^T e_{A^T}(\tau,t_0)\alpha=0\quad\text{for all }\tau\in[t_0,t_{\mathrm{f}})^{\kappa^m}\cap\mathbb{T}.$$

Then picking $\tau=t_0\in[t_0,t_{\mathrm{f}})^{\kappa^m}\cap\mathbb{T}$ for all $0\leqslant m\leqslant n-1$ (since $t_{\mathrm{f}}>\sigma^n(t_0)$) and using Theorem 2.14 part (a), we have

$$(A^mB)^T\alpha=0\quad\text{for all }0\leqslant m\leqslant n-1,$$

which implies $\Gamma_{\mathrm{C}}^T[A,B]\alpha=0$. Then $\alpha\in\operatorname{Ker}\Gamma_{\mathrm{C}}^T[A,B]=\{0\}$. Hence $\operatorname{Ker}W_{\mathrm{C}}=\{0\}$ and thus W_{C} is invertible. Then by Theorem 4.3, (1.2) is controllable. □

Remark 4.5. We have included Theorem 4.3 and its proof here for completeness, although it may be derived from [7, Theorem 2.2] with the $A(t)$ there replaced by $(\ominus A)(t)$ and the $B(t)$ there replaced by $(I+\mu(t)A)^{-1}B$. On the other hand, Theorem 4.4 cannot be derived from [7, Theorem 2.4] since two of the assumptions of [7, Theorem 2.4] are in general not satisfied: μ is in general not rd-continuously differentiable on the time scale, not even differentiable, and thus $(I+\mu(t)A)^{-1}B$ does not satisfy this assumption in general either. It is exactly this requirement needed in the proof of [7, Theorem 2.4] that led to the use of system (1.2) rather than system (1.1). As seen in the proof of Theorem 4.4 above, the classical proof method for system (1.2) goes through without requiring the differentiability condition on the graininess, in contrast to the situation of (1.1) in [7, Theorem 2.4].

Remark 4.6. In [9, Definition 4.2], another related notion of M-controllability is given for the same linear system (except the minus sign) as in the first equation of (1.2). However, this notion is also related to the corresponding time scale optimal control problem, so it cannot be directly compared to Definition 4.2 above. The controllability criterion also plays a rôle in determining an optimal control when the final state is fixed. It does, however, appear that in the absence of constraints, the notion of M-controllability can be expressed in terms of a Gramian matrix. The connection between the existence of an optimal control and the invertibility of the controllability Gramian is investigated in two forthcoming papers of the authors.

5. Reachability

In this section we consider the linear system (1.2) and discuss reachability, a similar concept to controllability.

Definition 5.1. The linear system (1.2) is said to be (*completely*) *reachable* on $[t_0, t_{\mathrm{f}}]$ if for all $x_{\mathrm{f}} \in \mathbb{R}^n$, there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = 0$ satisfies $x(t_{\mathrm{f}}) = x_{\mathrm{f}}$.

We give the relationship between controllability and reachability of (1.2) in the following theorem.

Theorem 5.2. *The linear system* (1.2) *is controllable if and only if it is reachable.*

Proof. First assume that (1.2) is controllable. Let $x_{\mathrm{f}} \in \mathbb{R}^n$. Then there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution $\tilde{x}$ of the state equation of (1.2) with $\tilde{x}(t_0) = -e^T_{A^T}(t_{\mathrm{f}}, t_0)x_{\mathrm{f}}$ satisfies $\tilde{x}(t_{\mathrm{f}}) = 0$. Then by Theorem 4.1, we have

$$\begin{aligned} 0 &= \tilde{x}(t_{\mathrm{f}}) \\ &= e^T_{A^T}(t_0, t_{\mathrm{f}}) \left[\tilde{x}(t_0) + \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_0) B u(\tau) \Delta\tau \right] \\ &= -x_{\mathrm{f}} + \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_{\mathrm{f}}) B u(\tau) \Delta\tau \end{aligned}$$

(use Theorem 2.14 parts (c) and (a)), which implies, again by Theorem 4.1, that the solution x of the state equation of (1.2) with $x(t_0) = 0$ satisfies

$$x(t_{\mathrm{f}}) = \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_{\mathrm{f}}) B u(\tau) \Delta\tau = x_{\mathrm{f}}.$$

Therefore (1.2) is reachable.

Now assume that (1.2) is reachable. Let $x_0 \in \mathbb{R}^n$. Then there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution $\tilde{x}$ of the state equation of (1.2) with $\tilde{x}(t_0) = 0$ satisfies $\tilde{x}(t_{\mathrm{f}}) =$

$-e^T_{A^T}(t_0, t_{\mathrm{f}})x_0$. Then by Theorem 4.1, we have

$$-e^T_{A^T}(t_0, t_{\mathrm{f}})x_0 = \tilde{x}(t_{\mathrm{f}}) = \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_{\mathrm{f}})Bu(\tau)\Delta\tau,$$

which implies, again by Theorem 4.1, that the solution x of the state equation of (1.2) with $x(t_{\mathrm{f}}) = x_0$ satisfies

$$x(t_{\mathrm{f}}) = e^T_{A^T}(t_0, t_{\mathrm{f}})x_0 + \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_{\mathrm{f}})Bu(\tau)\Delta\tau = 0.$$

Hence (1.2) is controllable. $\square$

The remainder of this section features the generalized reachability criterion and the generalized Kalman rank condition for reachability of linear systems (1.2).

Theorem 5.3. *The linear system* (1.2) *is reachable if and only if the reachability Gramian W_{R} is invertible, where*

$$W_{\mathrm{R}} := \int_{t_0}^{t_{\mathrm{f}}} e^T_{A^T}(\tau, t_{\mathrm{f}})BB^T e_{A^T}(\tau, t_{\mathrm{f}})\Delta\tau.$$

Proof. Note that

$$W_{\mathrm{R}} = e^T_{A^T}(t_0, t_{\mathrm{f}})W_{\mathrm{C}}e_{A^T}(t_0, t_{\mathrm{f}}),$$

which implies that W_{R} is invertible if and only if W_{C} is invertible. Then the statement follows from Theorem 4.3 and Theorem 5.2. $\square$

Theorem 5.4. *The linear system* (1.2) *is reachable if and only if the $n \times (nm)$ controllability matrix $\Gamma_{\mathrm{C}}[A, B]$ has full rank n.*

Proof. This follows from Theorem 4.4 and Theorem 5.2. $\square$

Remark 5.5. In the abstract of [14] the authors say: “For discrete-time linear systems, controllability and reachability are not equivalent.” This is shown without assuming regressivity of A. Our Theorem 5.2 shows that in the case $\mathbb{T} = \mathbb{Z}$ (and also for any arbitrary time scale), the two notions are indeed equivalent if regressivity of A is assumed.

Remark 5.6. We note that if we require $t_{\mathrm{f}} > \sigma^{2n}(t_0)$, then the controllability matrix $\Gamma_{\mathrm{C}}[A, B]$ has full rank if and only if for each arbitrary values $x_0, x_{\mathrm{f}} \in \mathbb{R}^n$, there exists $u\colon \mathbb{T} \to \mathbb{R}^m$ such that the solution x of the state equation of (1.2) with $x(t_0) = x_0$ satisfies $x(t_{\mathrm{f}}) = x_{\mathrm{f}}$. This follows from Theorem 4.4 and 5.4.

6. Observability

In this section we consider the linear system (1.1). We need the following variation of parameters result for the state equation of the linear system (1.1).

Theorem 6.1 (see [4, Theorem 5.24]). *Suppose A is a regressive $n \times n$-matrix and B is an $n \times m$-matrix. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Suppose $u\colon \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Then the unique solution of the initial value problem*

$$x^{\Delta}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$

is given by

$$\begin{aligned} x(t) &= e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A(t,\sigma(\tau))Bu(\tau)\Delta\tau \\ &= e_A(t,t_0)\left[x_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))Bu(\tau)\Delta\tau\right]. \end{aligned}$$

We refer to a linear system as being "observable" if given the output y and input u, we can find the state x uniquely. The precise definition reads as follows.

Definition 6.2. The linear system (1.1) is said to be (*completely*) *observable* on $[t_0, t_{\mathrm{f}}]$ if for all $u\colon \mathbb{T} \to \mathbb{R}^m$ and all $y\colon \mathbb{T} \to \mathbb{R}^r$, the linear system (1.1) has at most one solution x on $[t_0, t_{\mathrm{f}}]$.

First we give the generalized observability criterion for linear systems (1.1) as follows.

Theorem 6.3. *The linear system* (1.1) *is observable if and only if the observability Gramian W_{o} is invertible, where*

$$W_{\mathrm{o}} := \int_{t_0}^{t_{\mathrm{f}}} e_A^T(\tau,t_0)C^T C e_A(\tau,t_0)\Delta\tau.$$

Proof. Assume that (1.1) is observable and let $\alpha \in \operatorname{Ker} W_{\mathrm{o}}$. Then

$$\begin{aligned} 0 &= \alpha^T W_{\mathrm{o}}\alpha \\ &= \int_{t_0}^{t_{\mathrm{f}}} \alpha^T e_A^T(\tau,t_0)C^T C e_A(\tau,t_0)\alpha\Delta\tau \\ &= \int_{t_0}^{t_{\mathrm{f}}} \left\| C e_A(\tau,t_0)\alpha\right\|^2 \Delta\tau, \end{aligned}$$

which implies $Ce_A(\tau, t_0)\alpha = 0$ for all $\tau \in [t_0, t_f) \cap \mathbb{T}$. Now let $u = y = 0$ and note that both $\tilde{x} = 0$ and $x = e_A(\cdot, t_0)\alpha$ are solutions of (1.1). Since (1.1) is controllable, we have $x = \tilde{x}$, i.e., $\alpha = 0$ and thus $\operatorname{Ker} W_{\mathrm{o}} = \{0\}$. Hence W_{o} is invertible.

Now suppose that W_{o} is invertible. Let $u\colon \mathbb{T} \to \mathbb{R}^m$ and $y\colon \mathbb{T} \to \mathbb{R}^r$ be given and assume that (1.1) has a solution x. Then by Theorem 6.1,

$$\begin{aligned}
x(t_0) &= W_{\mathrm{o}}^{-1} \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T C e_A(\tau, t_0) x(t_0) \Delta\tau \\
&= W_{\mathrm{o}}^{-1} \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T C \left[x(\tau) - \int_{t_0}^{\tau} e_A(\tau, \sigma(s)) B u(s) \Delta s \right] \Delta\tau \\
&= W_{\mathrm{o}}^{-1} \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T \left[y(\tau) - C \int_{t_0}^{\tau} e_A(\tau, \sigma(s)) B u(s) \Delta s \right] \Delta\tau.
\end{aligned}$$

Then, again by Theorem 6.1,

$$\begin{aligned}
x(t) &= e_A(t, t_0) x(t_0) + \int_{t_0}^{t} e_A(t, \sigma(\tau)) B u(\tau) \Delta\tau \\
&= e_A(t, t_0) W_{\mathrm{o}}^{-1} \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T y(\tau) \Delta\tau \\
&\quad - e_A(t, t_0) W_{\mathrm{o}}^{-1} \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T C \int_{t_0}^{\tau} e_A(\tau, \sigma(s)) B u(s) \Delta s \Delta\tau \\
&\quad + \int_{t_0}^{t} e_A(t, \sigma(\tau)) B u(\tau) \Delta\tau,
\end{aligned}$$

so (1.1) has at most one solution. Therefore the linear system (1.1) is observable. $\square$

Next, we give the generalized Kalman rank condition for observability of linear systems (1.1).

Theorem 6.4. *The linear system* (1.1) *is observable if and only if the* $(nr) \times n$ *observability matrix* $\Gamma_{\mathrm{o}}[A, C]$ *has full rank* n*, where*

$$\Gamma_{\mathrm{o}}[A, C] := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

P r o o f. First assume that (1.1) is observable. Then W_{o} is invertible by Theorem 6.3. Let $\alpha \in \mathbb{R}^n$. Put $y = Ce_A(\cdot, t_0) W_{\mathrm{o}}^{-1} \alpha$. Then

$$\alpha = \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T C e_A(\tau, t_0) W_{\mathrm{o}}^{-1} \alpha \Delta\tau = \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T y(\tau) \Delta\tau. \tag{6.1}$$

Now as in the proof of Theorem 4.4, there exist s_j for $0 \leqslant j \leqslant n-1$ such that

$$e_A(t,t_0) = \sum_{j=0}^{n-1} s_j(t,t_0)A^j. \tag{6.2}$$

Next, we define G_j for $0 \leqslant j \leqslant n-1$ by

$$G_j = \int_{t_0}^{t_\mathrm{f}} s_j(\tau,t_0)y(\tau)\Delta\tau.$$

Substituting G_j and (6.2) into (6.1), we obtain

$$\begin{aligned}
\alpha &= \int_{t_0}^{t_\mathrm{f}} \left[\sum_{j=0}^{n-1} s_j(\tau,t_\mathrm{f})(A^j)^T\right] C^T y(\tau)\Delta\tau \\
&= \sum_{j=0}^{n-1} (CA^j)^T G_j \\
&= \Gamma_\mathrm{o}^T[A,C] \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_{n-1} \end{bmatrix}.
\end{aligned}$$

Then $\mathbb{R}^n \subset \operatorname{Im}\Gamma_\mathrm{o}^T[A,C] \subset \mathbb{R}^n$, which implies $\operatorname{rank}\Gamma_\mathrm{o}^T[A,C] = n$. Hence $\operatorname{rank}\Gamma_\mathrm{o}[A,C] = n$.

Now assume that $\operatorname{rank}\Gamma_\mathrm{o}[A,C] = n$ and let $\alpha \in \operatorname{Ker} W_\mathrm{o}$. Then

$$\begin{aligned}
0 &= \alpha^T W_\mathrm{o}\alpha \\
&= \int_{t_0}^{t_\mathrm{f}} \alpha^T e_A^T(\tau,t_0)C^T C e_A(\tau,t_0)\alpha\Delta\tau \\
&= \int_{t_0}^{t_\mathrm{f}} \left\| C e_A(\tau,t_0)\alpha \right\|^2 \Delta\tau,
\end{aligned}$$

which implies

$$C e_A(\tau,t_0)\alpha = 0 \quad \text{for all } \tau \in [t_0,t_\mathrm{f}] \cap \mathbb{T}. \tag{6.3}$$

Now differentiating (6.3) m times, where $0 \leqslant m \leqslant n-1$, we have

$$CA^m e_A(\tau,t_0)\alpha = 0 \quad \text{for all } \tau \in [t_0,t_\mathrm{f})^{\kappa^m} \cap \mathbb{T}.$$

Then picking $\tau = t_0 \in [t_0, t_f)^{\kappa^m} \cap \mathbb{T}$ for all $0 \leqslant m \leqslant n-1$ (since $t_f > \sigma^n(t_0)$) and using Theorem 2.14 part (a), we obtain

$$CA^m \alpha = 0 \quad \text{for all } 0 \leqslant m \leqslant n-1,$$

which can be rewritten as $\Gamma_{\mathrm{o}}[A, C]\alpha = 0$. This implies $\alpha \in \operatorname{Ker} \Gamma_{\mathrm{o}}[A, c] = \{0\}$. Thus $\operatorname{Ker} W_{\mathrm{o}} = \{0\}$ and hence W_{o} is invertible. Then by Theorem 6.3, the linear system (1.1) is observable. $\square$

7. Duality principle

The results presented thus far yield the following duality principle theorem.

Theorem 7.1. *The linear system*

$$\left\{\begin{aligned} x^{\Delta}(t) &= -Ax^{\sigma}(t) + Bu(t), \\ y(t) &= Cx^{\sigma}(t) \end{aligned}\right. \tag{7.1}$$

is controllable if and only if the linear system

$$\left\{\begin{aligned} x^{\Delta}(t) &= A^T x(t) + C^T u(t), \\ y(t) &= B^T x(t) \end{aligned}\right. \tag{7.2}$$

is observable.

Proof. By Theorem 4.4, the linear system (7.1) is controllable if and only if

$$\Gamma_{\mathrm{C}}[A, B] = [\, B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B \,]$$

has full rank n. Clearly this is true if and only if

$$\Gamma_{\mathrm{C}}^T[A, B] = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix}$$

has full rank n. Since $\Gamma_{\mathrm{C}}^T[A, B] = \Gamma_{\mathrm{o}}[A^T, B^T]$, Theorem 6.4 yields that this is true if and only if the linear system (7.2) is observable. $\square$

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References

[1] *B. Aulbach, S. Hilger*: A unified approach to continuous and discrete dynamics. Colloq. Math. Soc. János Bolyai. *53* (1990), 37–56. zbl

[2] *Z. Bartosiewicz, E. Pawłuszewicz*: Linear control systems on time scales: unification of continuous and discrete. Proc. of 10th IEEE Int. Conference MMAR. (2004), 263–266.

[3] *Z. Bartosiewicz, E. Pawłuszewicz*: Realizations of linear control systems on time scales. Control Cybernet. *35* (2006), 769–786. zbl

[4] *M. Bohner, A. Peterson*: Dynamic Equations on Time Scales. Birkhäuser, Basel, 2001. zbl

[5] *M. Bohner, A. Peterson, eds.*: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston, MA, 2003. zbl

[6] *J. DaCunha*: Transition matrix and generalized matrix exponential via the Peano-Baker series. J. Difference Equ. Appl. *11* (2005), 1245–1264. zbl

[7] *J. M. Davis, I. A. Gravagne, B. J. Jackson, R. J. Marks II*: Controllability, observability, realizability, and stability of dynamic linear systems. Electron. J. Diff. Equ. *37* (2009), 1–32. zbl

[8] *L. V. Fausett, K. N. Murty*: Controllability, observability and realizability criteria on time scale dynamical systems. Nonlinear Stud. *11* (2004), 627–638. zbl

[9] *R. Hilscher, V. Zeidan*: Weak maximum principle and accessory problem for control problems on time scales. Nonlinear Anal. *70* (2009), 3209–3226. zbl

[10] *R. E. Kalman*: Contributions to the theory of optimal control. Bol. Soc. Mat. Mexicana. *2* (1960), 102–119. zbl

[11] *R. E. Kalman*: On the general theory of control systems. Proc. 1st IFAC Congress Automatic Control. *1* (1960), 481–492.

[12] *R. E. Kalman*: Mathematical description of linear dynamical systems. J. SIAM Control Ser. A Control. *1* (1963), 152–192. zbl

[13] *R. E. Kalman, Y. C. Ho, K. S. Narendra*: Controllability of linear dynamical systems. Contrib. Differ. Equ. *1* (1963), 189–213. zbl

[14] *S. Molnar, F. Szigeti*: Controllability and reachability of dynamic discrete-time linear systems. Proceedings of the 4th International Conference on Control and Automation (2003), 350–354.

[15] *N. Wintz*: The Kalman filter on time scales. PhD Thesis, Missouri University of Science and Technology, Rolla, Missouri, USA (2009).

[16] *A. Zafer*: The matrix exponential on time scales. ANZIAM J. *48* (2006), 99–106. zbl

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