MEROMORPHIC FUNCTION SHARING A SMALL FUNCTION WITH A LINEAR DIFFERENTIAL POLYNOMIAL

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Received August 8, 2013
Communicated by Stanisława Kanas

Abstract. The problem of uniqueness of an entire or a meromorphic function when it shares a value or a small function with its derivative became popular among the researchers after the work of Rubel and Yang (1977). Several authors extended the problem to higher order derivatives. Since a linear differential polynomial is a natural extension of a derivative, in the paper we study the uniqueness of a meromorphic function that shares one small function CM with a linear differential polynomial, and prove the following result: Let f be a nonconstant meromorphic function and L a nonconstant linear differential polynomial generated by f. Suppose that $a = a(z)$ ($\not\equiv 0, \infty$) is a small function of f. If $f - a$ and $L - a$ share 0 CM and

$$(k+1)\overline{N}(r,\infty;f) + \overline{N}(r,0;f') + N_k(r,0;f') < \lambda T(r,f') + S(r,f')$$

for some real constant $\lambda \in (0,1)$, then $f - a = (1 + c/a)(L - a)$, where c is a constant and $1 + c/a \not\equiv 0$.

Keywords: meromorphic function; differential polynomial; small function; sharing

MSC 2010: 30D35

1. Introduction, definitions and results

Let f, g be nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f, g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities, and we say that f, g share the value a IM (ignoring multiplicities) if f, g have the same a-points but the multiplicities are not taken into account.

The research of the first author is partially supported by DST-PURSE program, India.

DOI: 10.21136/MB.2016.1

We refer the reader to [6] for the standard notation and definitions of the value distribution theory. However, in the following we explain some notation used in the paper.

Definition 1.1. For a meromorphic function f and for $a \in \mathbb{C} \cup \{\infty\}$ and for a positive integer k

(i) $N_{(k}(r,a;f)$ ($\overline{N}_{(k}(r,a;f)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than k;

(ii) $N_{k)}(r,a;f)$ ($\overline{N}_{k)}(r,a;f)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than k;

(iii) $N_k(r,a;f)$ denotes the sum $\overline{N}(r,a;f) + \sum\limits_{j=2}^{k} \overline{N}_{(j}(r,a;f)$.

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$ and $N_k(r,a;f) \leqslant k\overline{N}(r,a;f)$.

Rubel-Yang [10], Mues-Steinmetz [9], Gundersen [5], Yang [12] and others considered the uniqueness problem of entire functions when their first and kth derivatives share two values CM or IM.

Brück [4] considered the uniqueness problem of an entire function when it shares a single value CM with its first derivative and proved the following theorem.

Theorem A ([4])**.** *Let f be a nonconstant entire function. If f and f' share the value 1 CM and $N(r,0;f') = S(r,f)$, then $f - 1 = c(f' - 1)$, where c is a nonzero constant.*

Yang [11] considered an entire function of finite order and proved the following result.

Theorem B ([11])**.** *Let f be a nonconstant entire function of finite order and let a $(\neq 0)$ be a finite constant. If f, $f^{(k)}$ share the value a CM, then $f - a = c(f^{(k)} - a)$, where c is a nonzero constant and k $(\geqslant 1)$ is an integer.*

Zhang [14] extended Theorem A to meromorphic functions and proved the following results.

Theorem C ([14])**.** *Let f be a nonconstant meromorphic function. If f and f' share 1 CM and if*

$$2\overline{N}(r,\infty;f) + 2N(r,0;f') < \lambda T(r,f') + S(r,f')$$

for some constant $\lambda \in (0,1)$, then $f - 1 = c(f' - 1)$, where c is a nonzero constant.

Theorem D ([14])**.** *Let f be a nonconstant meromorphic function. If f and $f^{(k)}$ share* 1 *CM and if*

$$2\overline{N}(r,\infty;f)+\overline{N}(r,0;f')+N(r,0;f^{(k)})<\lambda T(r,f^{(k)})+S(r,f^{(k)})$$

for some constant $\lambda\in(0,1)$, then $f-1=c(f^{(k)}-1)$, where c is a nonzero constant.

Let f be a nonconstant meromorphic function in $\mathbb{C}$. A meromorphic function $a=a(z)$, defined in $\mathbb{C}$, is called a small function of f if $T(r,a)=S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f)/T(r,f)\to 0$ as $r\to\infty$, possibly outside a set of finite linear measure.

Yu [13] considered the uniqueness problem of an entire function or a meromorphic function when it shares one small function with its derivative. The next two theorems are the results of Yu [13].

Theorem E ([13])**.** *Let f be a nonconstant entire function and let $a=a(z)$ ($\not\equiv 0,\infty$) be a small function of f. If $f-a$ and $f^{(k)}-a$ share the value* 0 *CM and $\delta(0;f)>3/4$, then $f\equiv f^{(k)}$, where k is a positive integer.*

Theorem F ([13])**.** *Let f be a nonentire meromorphic function and $a=a(z)$ ($\not\equiv 0,\infty$) a small function of f. If*

(i) *f and a have no common pole,*
(ii) *$f-a$ and $f^{(k)}-a$ share the value* 0 *CM,*
(iii) $4\delta(0;f)+2(8+k)\Theta(\infty;f)>19+2k$,

then $f\equiv f^{(k)}$, where k is a positive integer.

In 2004, improving Theorem F, Liu and Gu [8] proved the following theorem.

Theorem G ([8])**.** *Let f be a nonconstant meromorphic function and $a=a(z)$ ($\not\equiv 0,\infty$) a small function of f. If $f-a$ and $f^{(k)}-a$ share the value* 0 *CM, $f^{(k)}$ and $a=a(z)$ do not have any common pole of the same multiplicity and $2\delta(0;f)+4\Theta(\infty;f)>5$, then $f\equiv f^{(k)}$, where k is a positive integer.*

Al-Khaladi [3] observed by considering $f(z)=1+\exp(\mathrm{e}^z)$ and $a(z)=\mathrm{e}^z/(\mathrm{e}^z-1)$ that in Theorem A it is not possible to replace the value 1 by a small function. Instead, he proved the following result.

Theorem H ([3]). *Let f be a nonconstant entire function satisfying $N(r,0;f') = S(r,f)$ and let $a = a(z)$ $(\not\equiv 0,\infty)$ be a small function of f. If $f-a$ and $f'-a$ share 0 CM, then $f-a = (1+c/a)(f'-a)$, where $1+c/a = \mathrm{e}^{\beta}$, c is a constant and β is an entire function.*

In 2005 Al-Khaladi [2] considered the general order derivative of an entire function and proved the following result.

Theorem I ([2]). *Let f be a nonconstant entire function satisfying $\overline{N}(r,0;f^{(k)}) = S(r,f)$ and let $a = a(z)$ $(\not\equiv 0,\infty)$ be a small function of f. If $f-a$ and $f^{(k)}-a$ share 0 CM, then $f-a = (1+P_{k-1}/a)(f^{(k)}-a)$, where $1+P_{k-1}/a = \mathrm{e}^{\beta}$, P_{k-1} is a polynomial of degree at most $k-1$ and β is an entire function.*

Recently Al-Khaladi [1] extended Theorem I to meromorphic functions and proved the following theorem.

Theorem J ([1]). *Let f be a nonconstant meromorphic function and let $a = a(z)$ $(\not\equiv 0,\infty)$ be a small function of f. If $f-a$ and $f^{(k)}-a$ share 0 CM and*

$$(k+1)\overline{N}(r,\infty;f) + (k+1)\overline{N}(r,0;f^{(k)}) < \lambda T(r,f^{(k)}) + S(r,f^{(k)})$$

for some constant $\lambda \in (0,1)$, then $f-a = (1+P_{k-1}/a)(f^{(k)}-a)$, where P_{k-1} is a polynomial of degree at most $k-1$ and $1+P_{k-1}/a \not\equiv 0$.

For a nonconstant meromorphic function f we denote by $L = L(f)$ a linear differential polynomial of the form

$$L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)},$$

where $a_1, a_2, \ldots, a_k$ $(\neq 0)$ are constants.

In the paper we prove the following theorem, which involves the sharing of a small function by f and L.

Theorem 1.1. *Let f be a nonconstant meromorphic function such that L is nonconstant. Suppose that $a = a(z)$ $(\not\equiv 0,\infty)$ is a small function of f. If $f-a$ and $L-a$ share 0 CM and*

$$(k+1)\overline{N}(r,\infty;f) + \overline{N}(r,0;f') + N_k(r,0;f') < \lambda T(r,f') + S(r,f')$$

for some real constant $\lambda \in (0,1)$, then $f-a = (1+c/a)(L-a)$, where c is a constant and $1+c/a \not\equiv 0$.

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1 ([6], page 55, Theorem 3.1). *Let f be a nonconstant meromorphic function. Then*

$$T(r,L) \leqslant (k+1)T(r,f') + S(r,f).$$

Lemma 2.2. *Let f be a nonconstant meromorphic function such that L is nonconstant. Suppose that $a = a(z)$ ($\not\equiv 0, \infty$) is a small function of f. If $f - a$ and $L - a$ share 0 IM, then*

$$T(r,f) \leqslant \Big(\frac{1}{k+1} + k + 2\Big)T(r,L) + S(r,f) \leqslant \{(k+1)(k+2)+1\}T(r,f') + S(r,f).$$

P r o o f. By Milloux's basic result [6], page 57, Theorem 3.2, we get

$$T(r,f) \leqslant \overline{N}(r,\infty;f) + N(r,0;f) + \overline{N}(r,1;L) - N_0(r,0;L') + S(r,f),$$

where $N_0(r,0;L')$ is the counting function of those zeros of L' which are not the 1-points of L.

Now $N(r,0;f) - N_0(r,0;L') \leqslant (k+1)\overline{N}(r,0;f)$ and $(k+1)\overline{N}(r,\infty;f) \leqslant N(r,\infty;L) \leqslant T(r,L)$. Therefore

$$\begin{aligned} T(r,f) &\leqslant T(r,L) + \overline{N}(r,1;L) + (k+1)\overline{N}(r,0;f) + S(r,f) \\ &\leqslant \Big(\frac{1}{k+1} + 1\Big)T(r,L) + (k+1)\overline{N}(r,0;f) + S(r,f). \end{aligned} \tag{2.1}$$

Since $L(f-a) = L(f) - \sum\limits_{j=1}^{k} a_j a^{(j)}$, we have $T(r,L(f-a)) = T(r,L) + S(r,f)$.

Now replacing f by $f-a$ in (2.1) and noting that $f-a$ and $L-a$ share 0 IM we get

$$T(r,f-a) \leqslant \Big(\frac{1}{k+1} + 1\Big)T(r,L) + (k+1)\overline{N}(r,0;f-a) + S(r,f)$$

and so

$$T(r,f) \leqslant \Big(\frac{1}{k+1} + k + 2\Big)T(r,L) + S(r,f). \tag{2.2}$$

By Lemma 2.1 we get

$$T(r,L) \leqslant (k+1)T(r,f') + S(r,f). \tag{2.3}$$

Now the lemma follows from (2.2) and (2.3). □

Lemma 2.3 ([6], page 47, Theorem 2.5)**.** *Let f be a nonconstant meromorphic function and a_1, a_2, a_3 three distinct small functions of f. Then*

$$T(r,f) \leqslant \overline{N}(r,0;f-a_1) + \overline{N}(r,0;f-a_2) + \overline{N}(r,0;f-a_3) + S(r,f).$$

Lemma 2.4 ([7])**.** *Let f be a nonconstant meromorphic function and k a positive integer. If f and $f^{(k)}$ share 1 IM and $f^{(k)} = (Af+B)/(Cf+D)$, where A, B, C, D are constants, then $(f^{(k)}-1)/(f-1)$ is a nonzero constant.*

3. Proof of Theorem 1.1

Proof. Let $h = (f-a)/(L-a)$. Then $f - a = h(L-a)$ and differentiating we get

$$f' - a' = (hL)' - (ha)'. \tag{3.1}$$

We now consider the following cases.

Case I: Let $a' \not\equiv 0$. We put

$$W = \frac{(hL)'}{hf'} - \frac{(ha)'}{ha'}. \tag{3.2}$$

If z_0 is a zero of $f' - a'$ with $a'(z_0) \neq 0, \infty$, then we get from (3.1) that $W(z_0) = 0$. Let $W \not\equiv 0$. Then

$$\begin{aligned} \overline{N}(r,0;f'-a') &\leqslant N(r,0;W) + S(r,f) \leqslant T(r,W) + S(r,f) \\ &= N(r,W) + m(r,W) + S(r,f) = N(r,W) + S(r,f). \end{aligned} \tag{3.3}$$

From (3.2) we get

$$W = \frac{(hL)'}{hL} \cdot \frac{L}{f'} + \frac{(ha)'}{ha} \cdot \frac{a}{a'}. \tag{3.4}$$

Let z_1 be a pole of f with multiplicity p such that $a(z_1) \neq 0, \infty$ and $a'(z_1) \neq 0$. Then z_1 is a pole of hL with multiplicity p and a pole of L/f' with multiplicity $k-1$. Hence z_1 is a pole of W with multiplicity at most k.

Let z_2 be a zero of f' with multiplicity q such that $a(z_2) \neq 0, \infty$ and $a'(z_2) \neq 0$. If $q \leqslant k-1$ and $L(z_2) \neq 0$, then z_2 is a pole of $(hL)'/(hL) \cdot L/f'$ with multiplicity $q \leqslant k-1$. Also, if $q \leqslant k-1$ and z_2 is a zero of L with multiplicity t ($\geqslant 1$), then z_2 is a pole of $(hL)'/(hL) \cdot L/f'$ with multiplicity $q-(t-1) \leqslant q \leqslant k-1$.

If $q \geqslant k$, then z_2 is a pole of L/f' with multiplicity $k-1$ and a pole of $(hL)'/(hL)$ with multiplicity 1. Hence z_2 is a pole of $(hL)'/(hL) \cdot L/f'$ with multiplicity k.

Therefore from (3.4) we get

$$N(r, W) \leqslant k\overline{N}(r, \infty; f) + N_k(r, 0; f') + S(r, f). \tag{3.5}$$

From (3.3) and (3.5) we obtain

$$\overline{N}(r, 0; f' - a') \leqslant k\overline{N}(r, \infty; f) + N_k(r, 0; f') + S(r, f). \tag{3.6}$$

Since by Lemma 2.1 and Lemma 2.2, $a' = a'(z)$ is a small function of f' and $S(r, f)$ is interchangeable with $S(r, f')$, we get by Lemma 2.3 and (3.6)

$$\begin{aligned} T(r, f') &\leqslant \overline{N}(r, 0; f' - a') + \overline{N}(r, 0; f') + \overline{N}(r, \infty; f') + S(r, f') \\ &\leqslant (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N_k(r, 0; f') + S(r, f'), \end{aligned}$$

which contradicts the hypothesis.

Therefore $W \equiv 0$ and so by (3.1) and (3.2) we get $(f' - a')(ha)' = (f' - a')a'$. Since $f' \not\equiv a'$, we have $(ha)' = a'$ and so $ha = a + c$, where c is a constant. Hence

$$f - a = h(L - a) = \Big(1 + \frac{c}{a}\Big)(L - a),$$

where $1 + c/a \not\equiv 0$.

Case II: Let $a' \equiv 0$ so that a is a constant. We now consider the following subcases.

Subcase (i): Let $k \geqslant 2$. From (3.1) we get

$$f' = (hL)' - ah' = h\Big\{\frac{(hL)'}{h} - a\frac{h'}{h}\Big\}$$

and so

$$\frac{1}{h} = \frac{(hL)'}{hf'} - a\frac{h'}{h} \cdot \frac{1}{f'}.$$

We put $F = f'$, $G = (hL)'/(hf')$ and $b = ah'/h$. Then

$$\frac{1}{h} = G - \frac{b}{F}. \tag{3.7}$$

Differentiating (3.7) we obtain

$$-\frac{1}{h} \cdot \frac{h'}{h} = G' - \frac{b'}{F} + \frac{b}{F} \cdot \frac{F'}{F}. \tag{3.8}$$

Eliminating $1/h$ from (3.7) and (3.8) we get

$$\frac{A}{F} = G' + G\frac{h'}{h}, \tag{3.9}$$

where $A = b \cdot h'/h + b' - b \cdot F'/F$.

First we suppose that $G \equiv 0$. Then $hL = d$, a nonzero constant. Putting $h = (f-a)/(L-a)$ we have $L(f-a) = d(L-a)$. This implies that f is an entire function. Therefore, h is an entire function having no zero. We now put $h = \mathrm{e}^{\alpha}$, where α is an entire function.

Now $f = a + h(L-a) = a + d - a\mathrm{e}^{\alpha}$ and $L = d\mathrm{e}^{-\alpha}$. Also we see that $L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)} = P(\alpha')\mathrm{e}^{\alpha}$, where $P(\alpha')$ is a differential polynomial in α'. Therefore $P(\alpha')\mathrm{e}^{\alpha} = d\mathrm{e}^{-\alpha}$ and so $P(\alpha')\mathrm{e}^{2\alpha} = d$. This implies $2T(r, \mathrm{e}^{\alpha}) = T(r, P(\alpha')) = S(r, \mathrm{e}^{\alpha})$, a contradiction. Hence $G \not\equiv 0$.

Next we suppose that $A \equiv 0$. Then from (3.9) we get $G'/G + h'/h = 0$. Integrating we obtain $Gh = K$, where K is a nonzero constant. Hence $(hL)' = Kf'$ and again integration yields $hL = Kf + M$, where M is a constant. Since $f - a = hL - ah$, we get

$$(1-K)f = a(1-h) + M. \tag{3.10}$$

If $K = 1$, from (3.10) we see that h is a constant. Hence $f - a = (1 + c/a)(L - a)$, where we put $h = 1 + c/a$ for some constant c such that $1 + c/a \neq 0$.

Let $K \neq 1$. Then from (3.10) we see that h is nonconstant. Since h is entire, (3.10) implies that f is also entire. Therefore $h = (f-a)/(L-a)$ has no zero. So we can put $h = \mathrm{e}^{\beta}$, where β is an entire function. Hence from (3.10) we get

$$f = \frac{a+M}{1-K} - \frac{a\mathrm{e}^{\beta}}{1-K}$$

and so

$$L = K\frac{f}{h} + \frac{M}{h} = \frac{Ka+M}{1-K}\mathrm{e}^{-\beta} - \frac{a}{1-K}. \tag{3.11}$$

Also

$$L = a_1 f^{(1)} + a_2 f^{(2)} + \ldots + a_k f^{(k)} = Q(\beta')\mathrm{e}^{\beta}, \tag{3.12}$$

where $Q(\beta')$ is a differential polynomial in β'.

Since L is nonconstant, we see that $Ka + M \neq 0$. Hence from (3.11) and (3.12) we get

$$Q(\beta')\mathrm{e}^{2\beta} = \frac{Ka+M}{1-K} - \frac{a}{1-K}\mathrm{e}^{\beta}.$$

This implies by the first fundamental theorem

$$2T(r,\mathrm{e}^{\beta}) \leqslant T(r,\mathrm{e}^{\beta}) + T(r,Q(\beta')) + O(1) = T(r,\mathrm{e}^{\beta}) + S(r,\mathrm{e}^{\beta}),$$

a contradiction.

Finally we suppose that $A \not\equiv 0$. Now $m(r,A) \leqslant 2m(r,b) + m(r,b') + m(r,h'/h) + m(r,F'/F) = S(r,f)$. Since $A = a(h'/h)^2 + a(h'/h)' - h'/h \cdot F'/F$, we see that $N(r,\infty;A) \leqslant 2\overline{N}(r,\infty;f) + \overline{N}(r,0;f')$. Hence

$$T(r,A) \leqslant 2\overline{N}(r,\infty;f) + \overline{N}(r,0;f') + S(r,f). \tag{3.13}$$

Now from (3.9) and (3.13) we get

$$\begin{aligned} m\Big(r,\frac{1}{F}\Big) &\leqslant m\Big(r,\frac{1}{A}\Big) + m\Big(r,G' + G\frac{h'}{h}\Big) \\ &\leqslant T(r,A) + S(r,f) \\ &\leqslant 2\overline{N}(r,\infty;f) + \overline{N}(r,0;f') + S(r,f). \end{aligned} \tag{3.14}$$

Since $A \not\equiv 0$, it is clear that $b \not\equiv 0$. Let z_3 be a zero of F with multiplicity q $(\geqslant k+1)$. Then z_3 is a zero of $b = af'/(f-a) - aL'/(L-a)$ with multiplicity at least $q-k$. Hence

$$N_{(k+1}\Big(r,\frac{1}{F}\Big) - k\overline{N}_{(k+1}\Big(r,\frac{1}{F}\Big) \leqslant N(r,0;b)$$

and so

$$\begin{aligned} N_{(k+1}\Big(r,\frac{1}{F}\Big) &\leqslant k\overline{N}_{(k+1}\Big(r,\frac{1}{F}\Big) + N(r,0;b) \\ &\leqslant k\overline{N}_{(k+1}\Big(r,\frac{1}{F}\Big) + T(r,b) + O(1) \\ &= k\overline{N}_{(k+1}\Big(r,\frac{1}{F}\Big) + N(r,b) + S(r,f) \\ &\leqslant k\overline{N}_{(k+1}\Big(r,\frac{1}{F}\Big) + \overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

So

$$\begin{aligned} N\Big(r,\frac{1}{F}\Big) &= N_{k)}\Big(r,\frac{1}{F}\Big) + N_{(k+1}\Big(r,\frac{1}{F}\Big) \\ &\leqslant N_k(r,0;f') + \overline{N}(r,\infty;f) + S(r,f). \end{aligned} \tag{3.15}$$

Adding (3.14) and (3.15) and using the first fundamental theorem we get

$$T(r,f') \leqslant 3\overline{N}(r,\infty;f) + N_k(r,0;f') + \overline{N}(r,0;f') + S(r,f),$$

which is a contradiction with the hypothesis for $k \geqslant 2$.

Subcase (ii): Let $k = 1$. We put $g = f/a$ and $R = L/a$. Then g and R share the value 1 CM. Let

$$H = \Big(\frac{g''}{g'} - \frac{2g'}{g-1}\Big) - \Big(\frac{R''}{R'} - \frac{2R'}{R-1}\Big).$$

We first suppose that $H \not\equiv 0$. Since g and R share 1 CM, we get

$$N(r,H) = \overline{N}(r,H) \leqslant \overline{N}(r,\infty;f) + \overline{N}(r,0;f') - \overline{N}_{(2}(r,a;f) + \overline{N}_*(r,0;f^{(2)}),$$

where $\overline{N}_*(r,0;f^{(2)})$ denotes the reduced counting function of those zeros of $f^{(2)}$ which are not the zeros of $(f-a)f'$.

Since g and R share the value 1 CM, it is easy to see that

$$\begin{aligned} N_{1)}(r,a;f) &= N_{1)}(r,1;g) \leqslant N(r,0;H) \leqslant T(r,H) + O(1) = N(r,H) + S(r,f) \\ &\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,0;f') - \overline{N}_{(2}(r,a;f) + \overline{N}_*(r,0;f^{(2)}) + S(r,f) \end{aligned}$$

and so

$$\begin{aligned} \overline{N}(r,a;f) &= N_{1)}(r,a;f) + \overline{N}_{(2}(r,a;f) \\ &\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,0;f') + \overline{N}_*(r,0;f^{(2)}) + S(r,f). \end{aligned} \tag{3.16}$$

Now by the second fundamental theorem and (3.16) we get in view of the fact that $L-a$ and $f-a$ share 0 CM:

$$\begin{aligned} T(r,f') &= T(r,L) + O(1) \\ &\leqslant \overline{N}(r,\infty;L) + \overline{N}(r,0;L) + \overline{N}(r,a;L) - \overline{N}_*(r,0;f^{(2)}) + S(r,L) \\ &= \overline{N}(r,\infty;f) + \overline{N}(r,0;f') + \overline{N}(r,a;f) - \overline{N}_*(r,0;f^{(2)}) + S(r,f') \\ &\leqslant 2\overline{N}(r,\infty;f) + \overline{N}(r,0;f') + N_1(r,0;f') + S(r,f'), \end{aligned}$$

a contradiction with the hypothesis.

Therefore $H \equiv 0$ and so integration yields $R = (Ag+B)/(Cg+D)$, where A, B, C, D are constants. Hence by Lemma 2.4 we get $(g-1)/(R-1)$ is a nonzero constant. So we can put $f-a = (1+c/a)(L-a)$, where c is a constant and $1+c/a \neq 0$. This proves the theorem. □

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