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NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING A NON-ZERO POLYNOMIAL WITH FINITE WEIGHT

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Abstract. In the paper, dealing with a question of Lahiri (1999), we study the uniqueness of meromorphic functions in the case when two certain types of nonlinear differential polynomials, which are the derivatives of some typical linear expression, namely $h^n(h-1)^m$ $(h=f,g)$, share a non-zero polynomial with finite weight. The results obtained in the paper improve, extend, supplement and generalize some recent results due to Sahoo (2013), Li and Gao (2010). In particular, we have shown that under a suitable choice of the sharing non-zero polynomial or when the first derivative is taken under consideration, better conclusions can be obtained.

Keywords: uniqueness; meromorphic function; nonlinear differential polynomial

MSC 2010: 30D35

1. Introduction, definitions and results

In this paper, by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, if $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notation of value distribution theory, see [7]. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The symbol $S(r)$ denotes any quantity

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satisfying $S(r) = O(T(r))$ as $r \to \infty$, outside a possible exceptional set of finite linear measure.

Throughout this paper, we need the following definition:

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In 1999, Lahiri [11] asked the following question, which is perhaps the first one concerning the possible relationship between two meromorphic functions related to value sharing of the nonlinear differential polynomials generated by them:

What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 *CM?*

Earlier, in 1997, Yang and Hua [25] already made some contribution in this direction for a specific type of nonlinear differential polynomials, namely differential monomials. Below we recall their result.

Theorem A ([25]). *Let f and g be two non-constant meromorphic functions, $n \geqslant 11$ be a positive integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share a CM, then either $f(z) = c_1 \mathrm{e}^{cz}$, $g(z) = c_2 \mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Fang and Qiu [5] extended the above result as follows:

Theorem B ([5]). *Let f and g be two non-constant meromorphic functions, $n \geqslant 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share* 0 *CM, then either $f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

The introduction of the new idea of scaling between CM and IM, known as weighted sharing of values, by Lahiri [9], [10] in 2001 further encouraged the investigations remarkably in the above direction. To verify the above statement readers are referred to [2]–[5], [13]–[20].

The definition of weighted sharing is given below.

Definition 1.1 ([9], [10]). Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity m ($\leqslant k$) if and only if it is an a-point of g with multiplicity m ($\leqslant k$), and z_0 is an a-point of f with multiplicity m ($> k$) if and only if it is an a-point of g with multiplicity n ($> k$), where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively. If a is a small function we say that f and g share (a, l), which means f and g share a with weight l if $f - a$ and $g - a$ share $(0, l)$.

In 2010, Li and Gao [17] further improved some previous results, e.g., [21] in the following manner:

Theorem C ([17]). *Let f and g be two transcendental meromorphic functions, let $n \geqslant 11$ be a positive integer and let $P \not\equiv 0$ be a polynomial with degree $\gamma_P \leqslant 11$. If $f^n f'$ and $g^n g'$ share (P, ∞), then either $f \equiv tg$ for a constant t such that $t^{n+1} = 1$, or $f(z) = c_1 \mathrm{e}^{cQ}$ and $g(z) = c_2 \mathrm{e}^{-cQ}$, where c_1, c_2 and c are three non-zero constants satisfying $(c_1 c_2)^{n+1}(c)^2 = -1$, $Q(z)$ is a polynomial satisfying $Q = \int_0^z P(\eta)\, \mathrm{d}\eta$.*

Theorem D ([17]). *Let f and g be two transcendental meromorphic functions, n ($\geqslant 15$) be an integer and $P \not\equiv 0$ be a polynomial. If $(f^n(f-1))'$ and $(g^n(g-1))'$ share (P, ∞), and $\Theta(\infty; f) > 2/n$, then $f \equiv g$.*

For the last few years the main trend in the value sharing of nonlinear differential polynomials has somehow been shifted towards the k-th derivative of some linear expression of f and g. Recently Sahoo [22] have extended Theorems C and D for the case of IM sharing, which in turn improved Sahoo's previous result [23]. Sahoo's [22] results are as follows:

Theorem E ([22]). *Let f and g be two transcendental meromorphic functions, let n, k be two positive integers such that $n \geqslant 9k + 15$ and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leqslant n - 1$. Let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share $(P, 0)$. Then*

(i) *if $k = 1$, then either $f \equiv tg$ for a constant t such that $t^n = 1$ or $f(z) = c_1 \mathrm{e}^{cQ}$ and $g(z) = c_2 \mathrm{e}^{-cQ}$, where c_1, c_2 and c are three non-zero constants satisfying $(c_1 c_2)^n (c)^2 = -1$, $Q(z)$ is a polynomial satisfying $Q = \int_0^z P(\eta)\, \mathrm{d}\eta$;*

(ii) *if $k \geqslant 2$, either $(f^n)^{(k)}(g^n)^{(k)} = p^2$ or $f \equiv tg$ for a constant t satisfying $t^n = 1$.*

Theorem F ([22]). *Let f and g be two transcendental meromorphic functions, let n, m, k be three positive integers and let $P \not\equiv 0$ be a polynomial. Let $(f^n(f-1)^m)^{(k)}$ and $(g^n(g-1)^m)^{(k)}$ share $(P, 0)$. Then the following holds:*

(i) *when $m = 1$, $n > 9k + 20$ and $\Theta(\infty; f) + \Theta(\infty; g) > 4/n$, then either $(f^n \times (f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ or $f \equiv g$;*

(ii) *if* $m \geqslant 2$ *and* $n > 9k+4m+16$, *then either* $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ *or* $f \equiv g$ *or* f, g *satisfy the algebraic equation* $R(f,g) = 0$, *where* $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$.
The possibility $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} \equiv P^2$ *does not arise for* $k = 1$.

The purpose of the paper is to unify all the above theorems into a single one. Our result radically improves the results of Sahoo and Li-Gao by reducing the lower bound of n. We also show that when $P(z) = d =$ constant, better results can be obtained at the cost of assuming f and g share ∞ IM. In short, we shall improve, extend and generalize all the above mentioned theorems in a more convenient and compact manner. The following theorem is the main result of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions, and $P(z)$ ($\not\equiv 0$) be a non-zero polynomial. Also we suppose that $(f^n(f-1)^m)^{(k)}$ and $(g^n(g-1)^m)^{(k)}$ share $(P(z), l)$, where n ($\geqslant 1$), k ($\geqslant 1$), m ($\geqslant 0$) and l ($\geqslant 0$) are integers. When*

(a) $l \geqslant 2$ *and* $n > \max\{3k+8+2\min\{k+2,m\}-m, m+3\}$ *or*
(b) $l = 1$ *and* $n > \max\{4k+9+2\min\{k+2,m\}+\frac{1}{2}\min\{k+1,m\}-m, m+3\}$ *or*
(c) $l = 0$ *and* $n > \max\{9k+14+3\min\{k+1,m\}+2\min\{k+2,m\}-m, m+3\}$,

then the following cases hold:

(I) *when $m = 0$, one of the following two cases holds:*
 (I1) $f \equiv g$ *for some constant* t *such that* $t^n = 1$;
 (I2) $(f^n)^{(k)}(g^n)^{(k)} \equiv P^2$. *In particular, if f and g share ∞ IM, then for* (i) $k = 1$ *and* $\gamma_P \leqslant n-1$, *we have* $f(z) = c_1 \mathrm{e}^{cQ}$ *and* $g(z) = c_2 \mathrm{e}^{-cQ}$, *where* c_1, c_2 *and* c *are three non-zero constants satisfying* $(c_1c_2)^n(c)^2 = -1$, $Q(z)$ *is a polynomial satisfying* $Q = n^{-1}\int_0^z P(\eta)\,\mathrm{d}\eta$; *and for* (ii) $P(z) = d$ *we get* $f(z) = c_1\mathrm{e}^{cz}$ *and* $g(z) = c_2\mathrm{e}^{-cz}$, *where* c_1, c_2 *and* c *are constants satisfying* $(-1)^k(c_1c_2)^n(nc)^{2k} = d^2$;

(II) *when $m \geqslant 1$, one of the following three cases holds:*
 (II1) $f(z) \equiv g(z)$;
 (II2) f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$, *where* $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$, *except for* $m = 1$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 4/n$;
 (II3) $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} \equiv P^2$;
 The possibility (II3) *does not arise for* $k = 1$.

We now present some definitions and notations which are used in the paper.

Definition 1.2 ([16])**.** Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f;\geqslant p)$ ($\overline{N}(r,a;f;\geqslant p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f;\leqslant p)$ ($\overline{N}(r,a;f;\leqslant p)$) denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3 ([8], [27]). For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f;\geqslant 2) + \ldots + \overline{N}(r,a;f;\geqslant p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 1.4. Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r,a;f;\geqslant p; g=b)$ ($\overline{N}(r,a;f;\geqslant p; g\neq b)$) the reduced counting function of those a-points of f with multiplicities $\geqslant p$ which are the b-points (are not the b-points) of g.

Definition 1.5 ([1], [4]). Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p=q\geqslant 2$, each point in these counting functions is counted only once. Similarly we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Definition 1.6 ([1], [4]). Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. We denote by $\overline{N}_{f>k}\ (r,1;g)$ the reduced counting function of those 1-points of f and g where $p>q=k$. The function $\overline{N}_{g>k}\ (r,1;f)$ is defined analogously.

Definition 1.7 ([9], [10]). Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 1.8. Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f;g\neq b_1,b_2,\ldots,b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i=1,2,\ldots,q$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the following function:

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big). \tag{2.1}$$

Lemma 2.1 ([26]). *Let f be a non-constant meromorphic function and let $a_n(z)$ ($\not\equiv 0$), $a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([28]). *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.2}$$

$$N_p(r, 0; f^{(k)}) \leqslant k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.3}$$

Lemma 2.3 ([12]). *If $N(r, 0; f^{(k)}; f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)}; f \neq 0) \leqslant k\overline{N}(r, \infty; f) + N(r, 0; f; < k) + k\overline{N}(r, 0; f; \geqslant k) + S(r, f).$$

Lemma 2.4 ([6]). *Let f be a non-constant entire function, $k \geqslant 2$ be a positive integer. If $ff^{(k)} \neq 0$ then $f = \mathrm{e}^{az+b}$, where $a \neq 0$, b are constants.*

Lemma 2.5 ([26]). *Let f be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$. Then*

$$N(r, 0; f^{(k)}) \leqslant N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.6. *Let f, g be two transcendental meromorphic functions and n, m and k be three positive integers with $n > k + 2 + \min\{k + 1, m\} - m$. Let $P(z)$ ($\not\equiv 0$) be a polynomial. If $(f^n(f-1)^m)^{(k)}$ and $(g^n(g-1)^m)^{(k)}$ share $(P(z), 0)$, then $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$.*

P r o o f. In view of Lemma 2.2 for $p = 1$ and using the second fundamental theorem for small functions [24] we get

$$\begin{aligned}
(n+m)T(r, f) &\leqslant T(r, (f^n(f-1)^m)^{(k)}) - \overline{N}(r, 0; (f^n(f-1)^m)^{(k)}) \\
&\quad + N_{k+1}(r, 0; f^n(f-1)^m) + S(r, f) \\
&\leqslant \overline{N}(r, 0; (f^n(f-1)^m)^{(k)}) + \overline{N}(r, \infty; f) + \overline{N}(r, p; (f^n(f-1)^m)^{(k)}) \\
&\quad - \overline{N}(r, 0; (f^n(f-1)^m)^{(k)}) + N_{k+1}(r, 0; f^n(f-1)^m) + S(r, f)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,p;(f^n(f-1)^m)^{(k)})+(k+1)\overline{N}(r,0;f)\\
&\quad+N_{k+1}(r,0;(f-1)^m)+S(r,f)\\
&\leqslant (k+2+\min\{k+1,m\})T(r,f)+\overline{N}(r,0;(g^n(g-1)^m)^{(k)}-p)+S(r,f)\\
&\leqslant (k+2+\min\{k+1,m\})T(r,f)+(k+1)(n+m)T(r,g)+S(r,f),
\end{aligned}
$$

i.e.,

$$
(n+m-k-2-\min\{k+1,m\})T(r,f)\leqslant (k+1)(n+m)T(r,g)+S(r,f).
$$

Since $n>k+2+\min\{k+1,m\}-m$, we have $T(r,f)=O(T(r,g))$. Similarly we have $T(r,g)=O(T(r,f))$. This completes the proof of the lemma. □

Lemma 2.7. *Let f, g be two non-constant meromorphic functions sharing ∞ IM. Let n, k be two positive integers such that $n>k$. If $(f^n)^{(k)}(g^n)^{(k)}\equiv d^2$, then $f=c_1\mathrm{e}^{cz}$, $g=c_2\mathrm{e}^{-cz}$, where c_1, c_2 and c are constants such that $(-1)^k(c_1c_2)^n(nc)^{2k}=d^2$.*

Proof. Suppose that

$$
(f^n)^{(k)}(g^n)^{(k)}\equiv d^2. \tag{2.4}
$$

Since f and g share ∞ IM, it follows from (2.4) that both f and g are entire functions. Again, since $n>k$, from (2.4) we get that both f and g have no zeros and we can take f and g as follows:

$$
f=\mathrm{e}^{\alpha},\quad g=\mathrm{e}^{\beta}. \tag{2.5}
$$

Moreover, we see from (2.4) that

$$
N(r,0;(f^n)^{(k)})=0,\quad N(r,0;(g^n)^{(k)})=0. \tag{2.6}
$$

We consider the following cases:

Case 1: Let $k\geqslant 2$. Then from (2.6) and Lemma 2.4 for f^n we have

$$
f(z)=c_1\mathrm{e}^{cz},\quad g(z)=c_2\mathrm{e}^{-cz}, \tag{2.7}
$$

where c, c_1 and c_2 are constants such that $(-1)^k(c_1c_2)^n(nc)^{2k}=1$.

Case 2: Let $k=1$. Suppose that α and β are transcendental. Then from (2.4) we get

$$
AB\alpha'\beta'\mathrm{e}^{n(\alpha+\beta)}\equiv 1, \tag{2.8}
$$

where $AB = n^2$. Let $\alpha + \beta = \gamma$. From (2.8) we know that γ is not a constant since in that case we get a contradiction. Then from (2.8) we get

$$AB\alpha'(\gamma' - \alpha')e^{n\gamma} \equiv 1. \tag{2.9}$$

We have $T(r, \gamma') = m(r, \gamma') = m(r, (e^{n\gamma})'/e^{n\gamma}) = S(r, e^{n\gamma})$. Thus from (2.9) we get

$$\begin{aligned} T(r, e^{n\gamma}) &\leqslant T\Big(r, \frac{1}{\alpha'(\gamma' - \alpha')}\Big) + O(1) \\ &\leqslant T(r, \alpha') + T(r, \gamma' - \alpha') + O(1) \\ &\leqslant 2T(r, \alpha') + S(r, \alpha') + S(r, e^{n\gamma}), \end{aligned}$$

which implies that $T(r, e^{n\gamma}) = O(T(r, \alpha'))$ and so $S(r, e^{n\gamma})$ can be replaced by $S(r, \alpha')$. Thus we get $T(r, \gamma') = S(r, \alpha')$ and so γ' is a small with respect to α'. In view of (2.9), by the second fundamental theorem for small functions we get

$$T(r, \alpha') \leqslant \overline{N}(r, \infty; \alpha') + \overline{N}(r, 0; \alpha') + \overline{N}(r, 0; \alpha' - \gamma') + S(r, \alpha') \leqslant S(r, \alpha'),$$

which shows that α' is a non-zero constant and so α is a polynomial. Similarly we can prove that β is also a polynomial. This contradicts the fact that α and β are transcendental.

Next, suppose without loss of generality that α is a polynomial and β is a transcendental entire function. Then γ is transcendental. So in view of (2.9) we obtain

$$\begin{aligned} nT(r, e^{\gamma}) &\leqslant T\Big(r, \frac{1}{\alpha'(\gamma' - \alpha')}\Big) + O(1) \\ &\leqslant T(r, \alpha') + T(r, \gamma' - \alpha') + S(r, \gamma) \\ &\leqslant T(r, \gamma') + S(r, e^{\gamma}) = S(r, e^{\gamma}), \end{aligned}$$

which leads to a contradiction. Thus α and β are both polynomials. From (2.8) we can conclude that $\alpha(z) + \beta(z) \equiv C$ for a constant C and so $\alpha'(z) + \beta'(z) \equiv 0$. Again from (2.8) we get $n^2 e^{nC}\alpha'\beta' \equiv 1$. By computation we get

$$\alpha' = c, \quad \beta' = -c. \tag{2.10}$$

Hence

$$\alpha = cz + b_1, \quad \beta = -cz + b_2, \tag{2.11}$$

where b_1, b_2 are constants. Finally we take f and g as

$$f(z) = c_1 e^{cz}, \quad g(z) = c_2 e^{-cz},$$

where c_1, c_2 and c are constants such that $(-1)(nc)^2(c_1c_2)^n = 1$. This completes the proof of the lemma. □

Lemma 2.8. *Let f and g be two non-constant meromorphic functions such that*

$$\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n},$$

where n ($\geqslant 3$) is an integer. Then

$$f^n(af + b) \equiv g^n(ag + b)$$

implies $f \equiv g$, where a, b are non-zero constants.

P r o o f. We omit the proof as it can be carried out in the line of the proof of Lemma 6 in [8]. $\square$

Lemma 2.9. *Let f and g be two transcendental meromorphic functions and n ($\geqslant 2$) be an integer. Also let P be a non-constant polynomial with degree $\gamma_P \leqslant n-1$. If $(f^n)'(g^n)' = P^2$, then f and g can be expressed as $f(z) = c_1 e^{cQ}$ and $g(z) = c_2 e^{-cQ}$, where c_1, c_2 and c are three non-zero constants satisfying $(c_1c_2)^{n+1}(c)^2 = -1$, $Q(z)$ is a polynomial satisfying $Q = \int_0^z P(\eta)\,d\eta$.*

P r o o f. Suppose

$$(f^n)'(g^n)' \equiv P^2. \tag{2.12}$$

Following the same arguments as in Lemma 2.7, one can easily get

$$f = h_1 e^{\alpha}, \quad g = h_2 e^{\beta}, \tag{2.13}$$

where h_1 and h_2 are two non-zero polynomials. From (2.12) we get

$$f^{n-1} f' g^{n-1} g' \equiv P_1^2, \tag{2.14}$$

where $P_1^2 = P^2/n^2$.

First, we suppose both α and β are transcendental entire functions and let $h = fg$. If h is a polynomial, then we get a contradiction from (2.13) and (2.14). Next, we suppose h is a transcendental entire function. Now from (2.14) we get

$$\Big(\frac{g'}{g} - \frac{1}{2}\frac{h'}{h}\Big)^2 \equiv \frac{1}{4}\Big(\frac{h'}{h}\Big)^2 - h^{-n} P_1^2. \tag{2.15}$$

Let

$$\alpha_2 = \frac{g'}{g} - \frac{1}{2}\frac{h'}{h}.$$

From (2.15) we get

$$\alpha_2^2 \equiv \frac{1}{4}\Big(\frac{h'}{h}\Big)^2 - h^{-n}P_1^2. \tag{2.16}$$

If we suppose $\alpha_2 \equiv 0$, then we get $h^{-n}P_1^2 \equiv \frac{1}{4}(h'/h)^2$ and so $T(r,h) = S(r,h)$, which is impossible. Hence we suppose that $\alpha_2 \not\equiv 0$. Differentiating (2.16) we get

$$2\alpha_2\alpha_2' \equiv \frac{1}{2}\frac{h'}{h}\Big(\frac{h'}{h}\Big)' + nh'h^{-n-1}P_1^2 - 2h^{-n}P_1P_1'.$$

Applying (2.16) we obtain

$$h^{-n}\bigg(-n\frac{h'}{h}P_1^2 + 2P_1P_1' - 2\frac{\alpha_2'}{\alpha_2}P_1^2\bigg) \equiv \frac{1}{2}\frac{h'}{h}\bigg(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\bigg). \tag{2.17}$$

If we assume

$$-n\frac{h'}{h}P_1^2 + 2P_1P_1' - 2\frac{\alpha_2'}{\alpha_2}P_1^2 \equiv 0,$$

then there exists a non-zero constant c such that $\alpha_2^2 \equiv ch^{-n}P_1^2$ and so from (2.16) we get

$$(c+1)h^{-n}P_1^2 \equiv \frac{1}{4}\Big(\frac{h'}{h}\Big)^2.$$

If $c = -1$, then h is a constant, which is impossible. On the other hand, if $c \neq -1$, then we have $T(r,h) = S(r,h)$, which is also impossible. So we must have

$$-n\frac{h'}{h}P_1^2 + 2P_1P_1' - 2\frac{\alpha_2'}{\alpha_2}P_1^2 \not\equiv 0.$$

Then by (2.17) we have

$$\begin{aligned} nT(r,h) &= nm(r,h) \\ &\leqslant m\bigg(r, h^n\frac{1}{2}\frac{h'}{h}\bigg(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\bigg)\bigg) \\ &\quad + m\Bigg(r, \bigg(\frac{1}{2}\frac{h'}{h}\bigg(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\bigg)\bigg)^{-1}\Bigg) + O(1) \\ &\leqslant T\bigg(r, \frac{1}{2}\frac{h'}{h}\bigg(\Big(\frac{h'}{h}\Big)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\bigg)\bigg) + m\bigg(r, n\frac{h'}{h}P_1^2 - 2P_1P_1' + 2\frac{\alpha_2'}{\alpha_2}P_1^2\bigg) \\ &\leqslant \overline{N}(r,0;\alpha_2) + S(r,h) + S(r,\alpha_2). \end{aligned} \tag{2.18}$$

From (2.16) we get

$$T(r,\alpha_2) \leqslant \frac{1}{2}nT(r,h) + S(r,h).$$

In view of (2.18) we get

$$\frac{1}{2}nT(r,h) \leqslant S(r,h),$$

which is impossible. Thus both α and β are polynomials.

From (2.12) we can conclude that $\alpha(z) + \beta(z) \equiv C$ for a constant C and so $\alpha'(z) + \beta'(z) \equiv 0$. Hence we can deduce from (2.12) that

$$(f^n)' \equiv n(h_1^n \alpha' + h_1^{n-1} h_1')\mathrm{e}^{n\alpha} \equiv P(z)\mathrm{e}^{n\alpha}, \tag{2.19}$$

and

$$(g^n)' = n(h_2^n \beta' + h_2^{n-1} h_2')\mathrm{e}^{n\beta} \equiv P(z)\mathrm{e}^{n\beta}. \tag{2.20}$$

By virtue of the polynomial P, from (2.19) and (2.20) we conclude that both h_1 and h_2 are non-zero constants.

So we can rewrite f and g as follows:

$$f = \mathrm{e}^{\gamma_3}, \quad g = \mathrm{e}^{\delta_3}. \tag{2.21}$$

Now from (2.12) we get

$$n^2 \gamma_3' \delta_3' \mathrm{e}^{n(\gamma_3 + \delta_3)} \equiv P^2. \tag{2.22}$$

From (2.22) we can conclude that $\gamma_3(z) + \delta_3(z) \equiv C$ for a constant C and so $\gamma_3'(z) + \delta_3'(z) \equiv 0$. Thus from (2.22) we get $n^2 \mathrm{e}^{nC} \gamma_3' \delta_3' \equiv P^2(z)$. By computation we get

$$\gamma_3' = cP(z), \quad \delta_3' = -cP(z). \tag{2.23}$$

Hence

$$\gamma_3 = cQ(z) + b_1, \quad \delta_3 = -cQ(z) + b_2, \tag{2.24}$$

where $Q(z) = \int_0^z P(z)\,\mathrm{d}z$ and b_1, b_2 are constants. Finally, we take f and g as

$$f(z) = c_1 \mathrm{e}^{cQ(z)}, \quad g(z) = c_2 \mathrm{e}^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$. □

Lemma 2.10 ([22])**.** *Let f and g be two transcendental meromorphic functions, n, m be two positive integers and P be a non-constant polynomial. If $m = 1$, $n \geqslant 6$ or if $m \geqslant 2$, $n \geqslant m + 3$, then*

$$(f^n(f-1)^m)'(g^n(g-1)^m)' \not\equiv P^2.$$

Lemma 2.11. *Let f and g be two non-constant meromorphic functions and k, m ($n > 3k + 2\min\{k, m\} - m$) be three positive integers. If $(f^n(f-1)^m)^{(k)} \equiv (g^n(g-1)^m)^{(k)}$, then $f^n(f-1)^m \equiv g^n(g-1)^m$.*

P r o o f. We have $(f^n(f-1)^m)^{(k)} \equiv (g^n(g-1)^m)^{(k)}$.

When $k \geqslant 2$, integrating we get

$$(f^n(f-1)^m)^{(k-1)} \equiv (g^n(g-1)^m)^{(k-1)} + c_{k-1}.$$

If possible, suppose $c_{k-1} \neq 0$. In view of Lemma 2.2 with $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,f) &\leqslant T(r,(f^n(f-1)^m)^{(k-1)}) - \overline{N}(r,0;(f^n(f-1)^m)^{(k-1)}) \\
&\quad + N_k(r,0;f^n(f-1)^m) + S(r,f) \\
&\leqslant \overline{N}(r,0;(f^n(f-1)^m)^{(k-1)}) + \overline{N}(r,\infty;f) \\
&\quad + \overline{N}(r,c_{k-1};(f^n(f-1)^m)^{(k-1)}) - \overline{N}(r,0;(f^n(f-1)^m)^{(k-1)}) \\
&\quad + N_k(r,0;f^n(f-1)^m) + S(r,f) \\
&\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,0;(g^n(g-1)^m)^{(k-1)}) \\
&\quad + k\overline{N}(r,0;f) + N_k(r,0;(f-1)^m) + S(r,f) \\
&\leqslant (k+1+\min\{k,m\})T(r,f) + (k-1)\overline{N}(r,\infty;g) \\
&\quad + N_k(r,0;g^n(g-1)^m) + S(r,f) \\
&\leqslant (k+1+\min\{k,m\})T(r,f) + (k-1)\overline{N}(r,\infty;g) \\
&\quad + k\overline{N}(r,0;g) + N_k(r,0;(g-1)^m) + S(r,f) \\
&\leqslant (k+1+\min\{k,m\})T(r,f) + (2k-1+\min\{k,m\})T(r,g) \\
&\quad + S(r,f) + S(r,g) \\
&\leqslant (3k+2\min\{k,m\})T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)T(r,g) \leqslant (3k+2\min\{k,m\})T(r) + S(r).$$

Combining the above two inequalities we get

$$(n+m-3k-2\min\{k,m\})T(r) \leqslant S(r),$$

which is a contradiction since $n > 3k + 2\min\{k,m\} - m$.

Therefore $c_{k-1} = 0$ and so $(f^n(f-1)^m)^{(k-1)} \equiv (g^n(g-1)^m)^{(k-1)}$. Repeating $k-1$ times, we obtain

$$f^n(f-1)^m \equiv g^n(g-1)^m + c_0.$$

If $k = 1$, clearly, integrating once we obtain the above expression. If possible, suppose $c_0 \neq 0$.

Now using the second fundamental theorem we get

$$\begin{aligned}(n+m)T(r,f) &\leqslant \overline{N}(r,0;f^n(f-1)^m) + \overline{N}(r,\infty;f^n(f-1)^m) \\ &\quad + \overline{N}(r,c_0;f^n(f-1)^m) + S(r,f) \\ &\leqslant \overline{N}(r,0;f) + T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g^n(g-1)^m) \\ &\leqslant 3T(r,f) + \overline{N}(r,0;g) + T(r,g) + S(r,f) \\ &\leqslant 3T(r,f) + 2T(r,g) + S(r,f) + S(r,g) \\ &\leqslant 5T(r) + S(r).\end{aligned}$$

Similarly we get

$$(n+m)T(r,g) \leqslant 5T(r) + S(r).$$

Combining these we get

$$(n+m-5)T(r) \leqslant S(r),$$

which is a contradiction since $n+m > 5$.

Therefore $c_0 = 0$ and so

$$f^n(f-1)^m \equiv g^n(g-1)^m.$$

This completes the proof. □

Lemma 2.12. *Let f, g be two transcendental meromorphic functions and $F = (f^n(f-1)^m)^{(k)}/P$, $G = (g^n(g-1)^m)^{(k)}/P$, where $P(z)$ ($\not\equiv 0$) is a polynomial, n ($\geqslant 1$), k ($\geqslant 1$), m ($\geqslant 0$) are positive integers such that $n > \max\{3k+3+2\min\{k+1,m\}-m, m+3\}$. If $H \equiv 0$ then*

- (I) *for $m = 0$, one of the following two cases holds:*
  - (I1) *$f \equiv g$ for some constant t such that $t^n = 1$;*
  - (I2) *$(f^n)^{(k)}(g^n)^{(k)} \equiv P^2$. In particular, if f and g share ∞ IM, then for* (i) *$k = 1$ and $\gamma_P \leqslant n-1$, we have $f(z) = c_1 \mathrm{e}^{cQ}$ and $g(z) = c_2 \mathrm{e}^{-cQ}$, where c_1, c_2 and c are three non-zero constants satisfying $(c_1c_2)^n(c)^2 = -1$, $Q(z)$ is a polynomial satisfying $Q = n^{-1}\int_0^z P(\eta)\,\mathrm{d}\eta$; and for* (ii) *$P(z) = d$, we get $f(z) = c_1\mathrm{e}^{cz}$ and $g(z) = c_2\mathrm{e}^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = d^2$.*
- (II) *for $m \geqslant 1$, one of the following three cases holds:*
  - (II1) *$f(z) \equiv g(z)$;*
  - (II2) *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(x,y) = x^n \times (x-1)^m - y^n(y-1)^m$, except for $m = 1$ and $\Theta(\infty;f) + \Theta(\infty;g) > 4/n$;*

(II3) $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} \equiv P^2$.

The possibility (II3) *does not arise for* $k=1$.

P r o o f. Since $H \equiv 0$, by integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{2.25}$$

where a, b are constants and $a \neq 0$. We now consider the following cases:

Case 1: Let $b \neq 0$ and $a \neq b$.

If $b=-1$, then from (2.25) we have

$$F \equiv \frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r,a+1;G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + O(\log r).$$

So in view of Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned}
(n+m)T(r,g) &\leqslant T(r,G) + N_{k+1}(r,0;g^n(g-1)^m) - \overline{N}(r,0;G) + O(\log r)\\
&\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G)\\
&\quad + N_{k+1}(r,0;g^n(g-1)^m) - \overline{N}(r,0;G) + S(r,g)\\
&\leqslant \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n(g-1)^m) + \overline{N}(r,\infty;f) + S(r,g)\\
&\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g)\\
&\quad + N_{k+1}(r,0;(g-1)^m) + S(r,g)\\
&\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g)\\
&\quad + \min\{k+1,m\}T(r,g) + S(r,g)\\
&\leqslant T(r,f) + \{k+2+\min\{k+1,m\}\}T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I of infinite measure such that $T(r,f) \leqslant T(r,g)$ for $r \in I$.

So for $r \in I$ we have

$$\{n+m-k-3-\min\{k+1,m\}\}T(r,g) \leqslant S(r,g),$$

which is a contradiction since $n > \max\{3k+3+2\min\{k+1,m\}-m, m+3\}$.

If $b \neq -1$, from (2.25) we obtain that

$$F - \left(1+\frac{1}{b}\right) \equiv \frac{-a}{b^2(G+(a-b)/b)}.$$

So

$$\overline{N}\Big(r, \frac{b-a}{b}; G\Big) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

Using Lemma 2.2 and the same argument as used for $b = -1$ we get a contradiction.

Case 2: Let $b \neq 0$ and $a = b$.

If $b = -1$, then from (2.25) we have

$$FG \equiv P^2,$$

that is,

$$(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} \equiv P^2.$$

In particular, when $m = 0$ and $k = 1$, then from above we get $f^{n-1}f'g^{n-1}g' = P^2/n^2$. Applying Lemma 2.9 we get $f(z) = c_1\mathrm{e}^{cQ}$ and $g(z) = c_2\mathrm{e}^{-cQ}$, where c_1, c_2 and c are three non-zero constants satisfying $(c_1c_2)^n(c)^2 = -1$, $Q(z)$ is a polynomial satisfying $Q = n^{-1}\int_0^z P(\eta)\,\mathrm{d}\eta$. On the other hand, when $m = 0$ and $P(z) = d =$ constant, then since $n > \max\{3k+3+2\min\{k+1, m\} - m, m+3\} = 3k+3$ always implies $n > k$, we have by Lemma 2.7 that $f(z) = c_1\mathrm{e}^{cz}$ and $g(z) = c_2\mathrm{e}^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = d^2$.

Also when $m \geqslant 1$ and $k = 1$, then by Lemma 2.10 we know $(f^n(f-1)^m)' \times (g^n(g-1)^m)' \not\equiv P^2$.

If $b \neq -1$, from (2.25) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G - 1}.$$

Therefore

$$\overline{N}\Big(r, \frac{1}{1+b}; G\Big) = \overline{N}(r, 0; F).$$

So in view of Lemma 2.2 and the second fundamental theorem we get

$$\begin{aligned}(n+m)T(r,g) &\leqslant T(r,G) + N_{k+1}(r,0;g^n(g-1)^m) - \overline{N}(r,0;G) + S(r,g)\\ &\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r, \frac{1}{1+b}; G\Big)\\ &\quad + N_{k+1}(r,0;g^n(g-1)^m) - \overline{N}(r,0;G) + S(r,g)\\ &\leqslant \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + N_{k+1}(r,0;(g-1)^m) + (k+1)\\ &\quad \times \overline{N}(r,0;f) + N_{k+1}(r,(f-1)^m) + k\overline{N}(r,\infty;f) + S(r,f) + S(r,g)\\ &\leqslant \{k+2+\min\{k+1,m\}\}T(r,g)\\ &\quad + \{2k+1+\min\{k+1,m\}\}T(r,f) + S(r,f) + S(r,g).\end{aligned}$$

So for $r \in I$ we have

$$(n+m-3k-3-2\min\{k+1,m\})T(r,g) \leqslant S(r,g),$$

which is a contradiction since $n > \max\{3k+3+2\min\{k+1,m\}-m, m+3\}$.

Case 3: Let $b=0$. From (2.25) we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{2.26}$$

If $a \neq 1$ then from (2.26) we obtain

$$\overline{N}(r,1-a;G) = \overline{N}(r,0;F).$$

We can deduce a contradiction similarly as in Case 2. Therefore $a=1$ and from (2.26) we obtain $F \equiv G$, i.e.,

$$(f^n(f-1)^m)^{(k)} \equiv (g^n(g-1)^m)^{(k)}.$$

So by Lemma 2.11 we have

$$f^n(f-1)^m \equiv g^n(g-1)^m. \tag{2.27}$$

When $m=0$ we have from (2.27) that $f = tg$, where $t^n = 1$.

When $m=1$ and $\Theta(\infty;f)+\Theta(\infty;g) > 4/n$, we then by Lemma 2.8 can prove that $f \equiv g$.

When $m \geqslant 2$, then proceeding in the same way as in the proof of Theorem 2 in [22] we can show that either $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g)=0$, where $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$. $\square$

Lemma 2.13 ([1]). *Let f, g be two non-constant meromorphic functions which share $(1,1)$. Then*

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \leqslant N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 2.14 ([4]). *Let f, g share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \leqslant \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f),$$

where $N_0(r,0;f')$ is the counting function of those zeros of f' which are not zeros of $f(f-1)$.

Lemma 2.15 ([4]). *Let f and g be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+\overline{N}_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)$$
$$\leqslant N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.16 ([4]). *Let f, g share $(1,0)$. Then*

$$\overline{N}_L(r,1;f)\leqslant \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.17 ([4]). *Let f, g share $(1,0)$. Then*

(i) $\overline{N}_{f>1}(r,1;g)\leqslant \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_0(r,0;f')+S(r,f)$,
(ii) $\overline{N}_{g>1}(r,1;f)\leqslant \overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_0(r,0;g')+S(r,g)$.

3. Proof of the theorem

Proof of Theorem 1.1. Let $F=(f^n(f-1)^m)^{(k)}/P(z)$ and $G=(g^n(g-1)^m)^{(k)}/P(z)$. It follows that F and G share $(1,l)$ except the zeros of P.

Case 1: Let $H\not\equiv 0$.

Subcase 1: $l\geqslant 1$. Let z' be a pole of H such that $P(z')\neq 0$. From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1-points of F and G whose multiplicities are different, (iii) poles of F and G, (iv) zeros of $F'(G')$ which are not zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned}(3.1)\quad N(r,\infty;H)\leqslant{}& \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F;\geqslant 2)\\ &+\overline{N}(r,0;G;\geqslant 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+O(\log r),\end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined.

Again, let z_0 be a simple zero of $F-1$ but $P(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$(3.2)\qquad N(r,1;F;=1)\leqslant N(r,0;H)\leqslant N(r,\infty;H)+S(r,f)+S(r,g).$$

While $l \geqslant 2$, using (3.1) and (3.2) we get

$$\begin{aligned}\overline{N}(r,1;F) &\leqslant N(r,1;F;=1)+\overline{N}(r,1;F;\geqslant 2)\\ &\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)\\ &\quad+\overline{N}(r,0;G;\geqslant 2)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F;\geqslant 2)\\ &\quad+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).\end{aligned}\tag{3.3}$$

Now in the view of Lemma 2.3 we get

$$\begin{aligned}&\overline{N}_0(r,0;G')+\overline{N}(r,1;F;\geqslant 2)+\overline{N}_*(r,1;F,G)\\ &\leqslant \overline{N}_0(r,0;G')+\overline{N}(r,1;F;\geqslant 2)+\overline{N}(r,1;F;\geqslant 3)\\ &=\overline{N}_0(r,0;G')+\overline{N}(r,1;G;\geqslant 2)+\overline{N}(r,1;G;\geqslant 3)\\ &\leqslant N(r,0;G';G\neq 0)+O(\log r)\\ &\leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;g)+S(r,g),\end{aligned}\tag{3.4}$$

Hence using (3.3), (3.4), Lemmas 2.1 and 2.2 we get from second fundamental theorem that

$$\begin{aligned}&(n+m)T(r,f)\\ &\leqslant T(r,F)+N_{k+2}(r,0;f^n(f-1)^m)-N_2(r,0;F)+S(r,f)\\ &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^n(f-1)^m)\\ &\quad-N_2(r,0;F)-N_0(r,0;F')+S(r,f)\\ &\leqslant 2\overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^n(f-1)^m)\\ &\quad+\overline{N}(r,0;F;\geqslant 2)+\overline{N}(r,0;G;\geqslant 2)+\overline{N}(r,1;F;\geqslant 2)+\overline{N}_*(r,1;F,G)\\ &\quad+\overline{N}_0(r,0;G')-N_2(r,0;F)+S(r,f)+S(r,g)\\ &\leqslant 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^n(f-1)^m)\\ &\quad+N_2(r,0;G)+S(r,f)+S(r,g)\\ &\leqslant 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^n(f-1)^m)+k\overline{N}(r,\infty;g)\\ &\quad+N_{k+2}(r,0;g^n(g-1)^m)+S(r,f)+S(r,g)\\ &\leqslant 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+(k+2)\overline{N}(r,0;f)+\min\{k+2,m\}T(r,f)\\ &\quad+(k+2)\overline{N}(r,0;g)+\min\{k+2,m\}T(r,g)\\ &\quad+k\overline{N}(r,\infty;g)+S(r,f)+S(r,g)\\ &\leqslant (k+4+\min\{k+2,m\})T(r,f)\\ &\quad+(2k+4+\min\{k+2,m\})T(r,g)+S(r,f)+S(r,g)\\ &\leqslant (3k+8+2\min\{k+2,m\})T(r)+S(r).\end{aligned}\tag{3.5}$$

In a similar way we can obtain

$$(n+m)T(r,g) \leqslant (3k+8+2\min\{k+2,m\})T(r)+S(r). \tag{3.6}$$

Combining (3.5) and (3.6) we see that

$$(n+m)T(r) \leqslant (3k+8+2\min\{k+2,m\})T(r)+S(r),$$

i.e.,

$$(n+m-3k-8-2\min\{k+2,m\})T(r) \leqslant S(r). \tag{3.7}$$

Clearly, (3.7) leads to a contradiction.

While $l=1$, using Lemmas 2.3, 2.13, 2.14, (3.1) and (3.2) we get

$$\begin{aligned}
\overline{N}(r,1;F) &\leqslant N(r,1;F;=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\overline{N}(r,0;G;\geqslant 2) \\
&\quad +\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \\
&\quad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g) \\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\overline{N}(r,0;G;\geqslant 2) \\
&\quad +2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \\
&\quad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g) \\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2) \\
&\quad +\overline{N}(r,0;G;\geqslant 2)+\overline{N}_{F>2}(r,1;G)+N(r,1;G)-\overline{N}(r,1;G) \\
&\quad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g) \\
&\leqslant \frac{3}{2}\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\frac{1}{2}\overline{N}(r,0;F) \\
&\quad +\overline{N}(r,0;G;\geqslant 2)+N(r,1;G)-\overline{N}(r,1;G) \\
&\quad +\overline{N}_0(r,0;G')+\overline{N}_0(r,0;F')+S(r,f)+S(r,g) \\
&\leqslant \frac{3}{2}\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\frac{1}{2}\overline{N}(r,0;F) \\
&\quad +\overline{N}(r,0;G;\geqslant 2)+N(r,0;G';G\neq 0) \\
&\quad +\overline{N}_0(r,0;F')+S(r,f)+S(r,g) \\
&\leqslant \frac{3}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\frac{1}{2}\overline{N}(r,0;F) \\
&\quad +N_2(r,0;G)+\overline{N}_0(r,0;F')+S(r,f)+S(r,g).
\end{aligned} \tag{3.8}$$

Hence using (3.8), Lemmas 2.1 and 2.2 we get from second fundamental theorem that

$$
\begin{aligned}
(3.9)\quad & (n+m)T(r,f)\\
&\leqslant T(r,F)+N_{k+2}(r,0;f^n(f-1)^m)-N_2(r,0;F)+S(r,f)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^n(f-1)^m)\\
&\quad -N_2(r,0;F)-N_0(r,0;F')\\
&\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\frac{1}{2}\overline{N}(r,0;F)\\
&\quad +N_{k+2}(r,0;f^n(f-1)^m)+N_2(r,0;G)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^n(f-1)^m)\\
&\quad +\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;G)+S(r,f)+S(r,g)\\
&\leqslant \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^n(f-1)^m)+k\overline{N}(r,\infty;g)\\
&\quad +N_{k+2}(r,0;g^n(g-1)^m)+\frac{1}{2}\{k\overline{N}(r,\infty;f)\\
&\quad +N_{k+1}(r,0;f^n(f-1)^m)\}+S(r,f)+S(r,g)\\
&\leqslant \frac{k+5}{2}\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+\frac{3k+5}{2}\overline{N}(r,0;f)\\
&\quad +\Big(\frac{1}{2}\min\{k+1,m\}+\min\{k+2,m\}\Big)T(r,f)+\min\{k+2,m\}T(r,g)\\
&\quad +(k+2)\overline{N}(r,0;g)+S(r,f)+S(r,g)\\
&\leqslant \Big(2k+5+\frac{1}{2}\min\{k+1,m\}+\min\{k+2,m\}\Big)T(r,f)\\
&\quad +(2k+4+\min\{k+2,m\})T(r,g)+S(r,f)+S(r,g)\\
&\leqslant \Big(4k+9+2\min\{k+2,m\}+\frac{1}{2}\min\{k+1,m\}\Big)T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$(3.10)\quad (n+m)T(r,g)\leqslant \Big(4k+9+2\min\{k+2,m\}+\frac{1}{2}\min\{k+1,m\}\Big)T(r)+S(r).$$

Combining (3.9) and (3.10) we see that

$$(n+m)T(r)\leqslant \Big(4k+9+2\min\{k+2,m\}+\frac{1}{2}\min\{k+1,m\}\Big)T(r)+S(r),$$

i.e.,

$$(3.11)\qquad \Big(n+m-4k-9-2\min\{k+2,m\}-\frac{1}{2}\min\{k+1,m\}\Big)T(r)\leqslant S(r).$$

Since $n > \max\{4k+9+2\min\{k+2,m\}+1/2\min\{k+1,m\}-m, m+3\}$, (3.11) leads to a contradiction.

Subcase 2: $l=0$. Here (3.2) changes to

$$N_E^{1)}(r,1;F;=1) \leqslant N(r,0;H) \leqslant N(r,\infty;H)+S(r,F)+S(r,G). \tag{3.12}$$

Using Lemmas 2.3, 2.15, 2.16, 2.17, (3.1) and (3.12) we get

$$\begin{aligned}
(3.13)\quad & \overline{N}(r,1;F) \\
&\leqslant N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\overline{N}(r,0;G;\geqslant 2) \\
&\quad +\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \\
&\quad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g) \\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\overline{N}(r,0;G;\geqslant 2) \\
&\quad +2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F') \\
&\quad +\overline{N}_0(r,0;G')+S(r,f)+S(r,g) \\
&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F;\geqslant 2)+\overline{N}(r,0;G;\geqslant 2) \\
&\quad +\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F)+\overline{N}_L(r,1;F)+N(r,1;G) \\
&\quad -\overline{N}(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g) \\
&\leqslant 3\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F) \\
&\quad +N_2(r,0;G)+N(r,1;G)-\overline{N}(r,1;G) \\
&\quad +\overline{N}_0(r,0;G')+\overline{N}_0(r,0;F')+S(r,f)+S(r,g) \\
&\leqslant 3\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G) \\
&\quad +N(r,0;G';G\neq 0)+\overline{N}_0(r,0;F')+S(r,f)+S(r,g) \\
&\leqslant 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G) \\
&\quad +\overline{N}(r,0;G)+\overline{N}_0(r,0;F')+S(r,f)+S(r,g).
\end{aligned}$$

Hence using (3.13), Lemmas 2.1 and 2.2 we get from second fundamental theorem that

$$\begin{aligned}
(3.14)\quad & (n+m)T(r,f) \\
&\leqslant T(r,F)+N_{k+2}(r,0;f^n(f-1)^m)-N_2(r,0;F)+S(r,f) \\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F) \\
&\quad +N_{k+2}(r,0;f^n(f-1)^m)-N_2(r,0;F)-N_0(r,0;F')
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+2\overline{N}(r,0;F)\\
&\quad+N_{k+2}(r,0;f^n(f-1)^m)+N_2(r,0;G)\\
&\quad+\overline{N}(r,0;G)-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leqslant 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^n(f-1)^m)+2\overline{N}(r,0;F)\\
&\quad+N_2(r,0;G)+\overline{N}(r,0;G)+S(r,f)+S(r,g)\\
&\leqslant 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^n(f-1)^m)+2k\overline{N}(r,\infty;f)\\
&\quad+2N_{k+1}(r,0;f^n(f-1)^m)+k\overline{N}(r,\infty;g)+N_{k+2}(r,0;g^n(g-1)^m)\\
&\quad+k\overline{N}(r,\infty;g)+N_{k+1}(r,0;g^n(g-1)^m)+S(r,f)+S(r,g)\\
&\leqslant (2k+4)\overline{N}(r,\infty;f)+(2k+3)\overline{N}(r,\infty;g)+(3k+4)\overline{N}(r,0;f)\\
&\quad+(2k+3)\overline{N}(r,0;g)+(\min\{k+1,m\}+\min\{k+2,m\})(T(r,f)\\
&\quad+T(r,g))+\min\{k+1,m\}T(r,f)+S(r,f)+S(r,g)\\
&\leqslant (5k+8+2\min\{k+1,m\}+\min\{k+2,m\})T(r,f)+(4k+6\\
&\quad+\min\{k+1,m\}+\min\{k+2,m\})T(r,g)+S(r,f)+S(r,g)\\
&\leqslant (9k+14+3\min\{k+1,m\}+2\min\{k+2,m\})T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$(n+m)T(r,g)\leqslant(9k+14+3\min\{k+1,m\}+2\min\{k+2,m\})T(r)+S(r). \tag{3.15}$$

Combining (3.14) and (3.15) we see that

$$(n+m)T(r)\leqslant(9k+14+3\min\{k+1,m\}+2\min\{k+2,m\})T(r)+S(r),$$

i.e.,

$$(n+m-9k-14-3\min\{k+1,m\}-2\min\{k+2,m\})T(r)\leqslant S(r). \tag{3.16}$$

Since $n>\max\{9k+14+3\min\{k+1,m\}+2\min\{k+2,m\}-m,m+3\}$, (3.16) leads to a contradiction.

Case 2: Let $H\equiv 0$. Then the theorem follows from Lemma 2.12. □

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