KANNAN-TYPE CYCLIC CONTRACTION RESULTS IN 2-MENGER SPACE

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Abstract. In this paper we establish Kannan-type cyclic contraction results in probabilistic 2-metric spaces. We use two different types of t-norm in our theorems. In our first theorem we use a Hadzic-type t-norm. We use the minimum t-norm in our second theorem. We prove our second theorem by different arguments than the first theorem. A control function is used in our second theorem. These results generalize some existing results in probabilistic 2-metric spaces. Our results are illustrated with an example.

Keywords: 2-Menger space; Cauchy sequence; fixed point; φ-function; ψ-function; cyclic contraction

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1. Introduction

Banach [1] proved the well-known Banach contraction mapping principle in metric spaces in 1922. This contraction mapping principle is one of the pivotal results of mathematical analysis. Its importance lies in its vast applications in a number of branches of modern mathematics.

The concept of metric space has been extended in various ways. One such extension has been made by Gähler [14], in which a positive real number is assigned to every three elements of the space. Several results of metric fixed point theory have been extended to these spaces. Some of the fixed point results in 2-metric spaces are [23], [28], [30].

In 1972, Sehgal and Bharucha-Reid [35] generalized the Banach contraction mapping principle to probabilistic metric spaces. Probabilistic metric spaces are probabilistic generalization of metric spaces. In these spaces, instead of a non-negative real

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number, every pair of elements is assigned to a distribution function. The inherent flexibility of these spaces allows us to extend the contraction mapping principle in several inequivalent ways. Menger space is a particular type of probabilistic metric space in which the triangular inequality is postulated with the help of a t-norm. The theory of Menger spaces is an important part of stochastic analysis. Schweizer and Sklar have given a comprehensive account of several aspects of such spaces in [34].

Probabilistic 2-metric space is the probabilistic generalization of 2-metric space. Zeng [41] first introduced the concept of probabilistic 2-metric space. References [8], [15], [16] present some fixed point results in probabilistic 2-metric spaces.

In 1984, Khan, Swaleh and Sessa [24] introduced the concept of "altering distance function", which is a control function that alters the distance between two points in a metric space. This concept was further generalized in a number of works. There are several works in metric fixed point theory involving altering distance function, some of these are noted in [32], [33].

Recently, Choudhury and Das have extended the concept of altering distance function to the context of Menger spaces in [5]. They have introduced the Φ-function. With the help of Φ-function Choudhury and Das [5] introduced a new type of contraction mapping in Menger spaces which is known as φ-contraction. The idea of this control function has opened new possibilities of proving more fixed point results in Menger spaces. This concept also applies to coincidence point problems. Some recent results using Φ-function are noted in [3], [9], [10], [12] and [29].

Recently, cyclic contraction and cyclic contractive-type mappings appeared in literature. Kirk, Srinivasan and Veeramani [27] initiated this line of research in metric spaces. Choudhury, Das and Bhandari introduced the concept of cyclic contraction and cyclic contractive-type mappings in both probabilistic metric spaces and probabilistic 2-metric spaces in [7], [9] and [10].

The problems of cyclic contractions are strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems may be found in [22], [39] and [40].

In this paper we define another contraction, namely, a Kannan-type cyclic contraction in 2-Menger spaces, and show that in a 2-Menger space with Hadzic-type t-norm, minimum t-norm, this contraction has a unique fixed point.

Kannan-type mappings are a class of contractive mappings which are different from the Banach contraction. A difference between Banach contraction mappings and Kannan-type mappings is that Banach contraction mappings are always continuous but Kannan-type mappings are not necessarily continuous. After the appearance of Kannan's papers [20], [21], many authors created contractive conditions not requiring the continuity of the mappings and established fixed point results of such mappings. Now this line of research has a vast literature. Another reason for the importance of Kannan-type mappings is that it characterizes metric completeness, which the Banach contraction does not. It has been shown in [37], [38] that the necessary existence of fixed points for Kannan-type mappings implies that the corresponding metric space is complete. The same is not true for Banach contractions. There is an example in [11] of an incomplete metric space where every contraction has a fixed point. Kannan-type mappings, their generalizations and extensions in various spaces have been considered in a large number of works, some of which can be found in [4], [18], [19], [25], [26], [31] and [37]. There are also similarities between Banach and Kannan-type contractions. One is referred to [25] and [26] for similarity between contractions and Kannan-type mappings.

2. Definitions and mathematical preliminaries

In this section we discuss some important definitions and mathematical preliminaries which we use in our main results.

Definition 2.1 (Kannan-type mapping [20], [21]). Let (X, d) be a metric space and f be a self-mapping on X. The mapping f is called a Kannan-type mapping if there exists $0 \leqslant \alpha < 1/2$ such that

$$d(fx, fy) \leqslant \alpha[d(x, fx) + d(y, fy)] \quad \text{for all } x, y \in X. \tag{2.1}$$

Kannan proved the following theorem in 1968.

Theorem 2.1 ([20], [21]). *Let f be a mapping satisfying* (2.1). *Then f has a unique fixed point in X.*

Definition 2.2 (2-metric space [13], [14]). Let X be a nonempty set. A real-valued function d on $X \times X \times X$ is said to be a 2-metric on X if

(i) given distinct elements $x, y \in X$, there exists an element $z \in X$ such that $d(x, y, z) \neq 0$,
(ii) $d(x, y, z) = 0$ when at least two of x, y, z are equal,
(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z \in X$ and
(iv) $d(x, y, z) \leqslant d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

When d is a 2-metric on X, the ordered pair (X, d) is called a 2-metric space.

Definition 2.3 (Probabilistic metric space [17], [34]). A probabilistic metric space (PM-space) is an ordered pair (X, F), where X is a nonempty set and F is a mapping from $X \times X$ into the set of all distribution functions. The function $F_{x,y}$ is assumed to satisfy the following conditions for all $x, y, z \in X$,

(i) $F_{x,y}(0) = 0$,
(ii) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all $t > 0$,
(iv) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(t_1 + t_2) = 1$ for all $t_1, t_2 > 0$,

where $F_{x,y}$ are distribution functions, that is, each $F_{x,y}$, $x, y \in X$ is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F_{x,y}(t) = 0$ and $\sup_{t\in\mathbb{R}} F_{x,y}(t) = 1$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^+$ is the set of non-negative real numbers.

Shi, Ren and Wang [36] give the following definition of n-th order t-norm.

Definition 2.4 (n-th order t-norm [36])**.** A mapping $T\colon \prod_{i=1}^{n}[0,1] \to [0,1]$ is called a n-th order t-norm if the following conditions are satisfied:

(i) $T(0,0,\ldots,0) = 0$, $T(a,1,1,\ldots,1) = a$ for all $a \in [0,1]$,
(ii) $T(a_1, a_{2,}, a_3, \ldots, a_n) = T(a_2, a_1, a_3, \ldots, a_n) = T(a_2, a_3, a_1, \ldots, a_n)$
$= \ldots = T(a_2, a_3, a_4, \ldots, a_n, a_1)$,
(iii) $a_i \geqslant b_i,\ i = 1,2,3,\ldots,n$ implies $T(a_1, a_2, a_3, \ldots, a_n) \geqslant T(b_1, b_2, b_3, \ldots, b_n)$,
(iv) $T(T(a_1, a_2, a_3, \ldots, a_n), b_2, b_3, \ldots, b_n) = T(a_1, T(a_2, a_3, \ldots, a_n, b_2), b_3, \ldots, b_n)$
$= T(a_1, a_2, T(a_3, a_4, \ldots, a_n, b_2, b_3), b_4, \ldots, b_n)$
$= \ldots = T(a_1, a_2, \ldots, a_{n-1}, T(a_n, b_2, b_3, \ldots, b_n))$.

When $n = 2$, we have a binary t-norm, which is commonly known as t-norm.

Definition 2.5 (Hadzic-type t-norm [17])**.** A t-norm Δ is said to be Hadzic-type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0,1)$ as

$$\Delta^0(s) = 1, \quad \Delta^{p+1}(s) = \Delta(\Delta^p(s), s) \quad \text{for all integers } p \geqslant 0,$$

is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exist $\eta(\lambda) \in (0,1)$ such that

$$1 \geqslant s > \eta(\lambda) \Rightarrow \Delta^p(s) > 1 - \lambda \quad \text{for all integers } p \geqslant 0.$$

Definition 2.6 (Menger space [17], [34])**.** A Menger space is a triplet (X, F, Δ), where X is a nonempty set, F is a function from $X \times X$ to the set of all distribution functions and Δ is a second order t-norm, such that the following conditions are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,
(ii) $F_{x,y}(s) = 1$ for all $s > 0$ if and only if $x = y$,
(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X,\ s > 0$ and
(iv) $F_{x,y}(u + v) \geqslant \Delta(F_{x,z}(u), F_{z,y}(v))$ for all $u, v \geqslant 0$ and $x, y, z \in X$.

Definition 2.7 (Probabilistic 2-metric space [41]). A probabilistic 2-metric space is an ordered pair (X, F) where X is an arbitrary set and F is a mapping from $X \times X \times X$ into the set of all distribution functions such that the following conditions are satisfied:

(i) $F_{x,y,z}(t) = 0$ for $t \leqslant 0$ and for all $x, y, z \in X$,
(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of x, y, z are equal,
(iii) for distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$,
(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$ for all $x, y, z \in X$ and $t > 0$,
(v) $F_{x,y,w}(t_1) = 1$, $F_{x,w,z}(t_2) = 1$ and $F_{w,y,z}(t_3) = 1$ then $F_{x,y,z}(t_1 + t_2 + t_3) = 1$ for all $x, y, z, w \in X$ and $t_1, t_2, t_3 > 0$.

A special case of the above definition is the following.

Definition 2.8 (2-Menger space [2]). Let X be a nonempty set. A triplet (X, F, Δ) is said to be a 2-Menger space if F is a mapping from $X \times X \times X$ into the set of all distribution functions satisfying the following conditions:

(i) $F_{x,y,z}(0) = 0$,
(ii) $F_{x,y,z}(t) = 1$ for all $t > 0$ if and only if at least two of $x, y, z \in X$ are equal,
(iii) for distinct points $x, y \in X$ there exists a point $z \in X$ such that $F_{x,y,z}(t) \neq 1$ for $t > 0$,
(iv) $F_{x,y,z}(t) = F_{x,z,y}(t) = F_{z,y,x}(t)$ for all $x, y, z \in X$ and $t > 0$,
(v) $F_{x,y,z}(t) \geqslant \Delta(F_{x,y,w}(t_1), F_{x,w,z}(t_2), F_{w,y,z}(t_3))$, where $t_1, t_2, t_3 > 0$, $t_1+t_2+t_3 = t$, $x, y, z, w \in X$ and Δ is a third order t-norm.

Definition 2.9 ([16]). A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be convergent to a limit x if given $\varepsilon > 0$, $0 < \lambda < 1$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that

$$F_{x_n,x,a}(\varepsilon) > 1 - \lambda \tag{2.2}$$

for all $n > N_{\varepsilon,\lambda}$ and for every $a \in X$.

Definition 2.10 ([16]). A sequence $\{x_n\}$ in a 2-Menger space (X, F, Δ) is said to be a Cauchy sequence in X if given $\varepsilon > 0$, $0 < \lambda < 1$ there exists a positive integer $N_{\varepsilon,\lambda}$ such that

$$F_{x_n,x_m,a}(\varepsilon) > 1 - \lambda \tag{2.3}$$

for all $m, n > N_{\varepsilon,\lambda}$ and for every $a \in X$.

Definitions 2.9 and 2.10 can be equivalently written by replacing "$>$" with "$\geqslant$" in (2.2) and (2.3), respectively. More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

Definition 2.11 ([16])**.** A 2-Menger space (X, F, Δ) is said to be complete if every Cauchy sequence is convergent in X.

Recently, Choudhury and Das introduced the following important function.

Definition 2.12 (Φ-function [5])**.** A function $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\varphi(t) = 0$ if and only if $t = 0$,
(ii) $\varphi(t)$ is strictly monotone increasing and $\varphi(t) \to \infty$ as $t \to \infty$,
(iii) φ is left continuous in $(0, \infty)$,
(iv) φ is continuous at 0.

The function has been utilized in a number of papers on fixed points in probabilistic metric spaces.

We will make use of the following function in our main theorems.

Definition 2.13. A function $\psi\colon [0,1] \times [0,1] \to [0,1]$ is said to be a Ψ-function if

(i) ψ-is monotone increasing in each variable and continuous,
(ii) $\psi(x, x) > x$ for all $0 < x < 1$,
(iii) $\psi(1,1) = 1,\ \psi(0,0) = 0$.

An example of a Ψ-function is $\psi(x, y) = (\sqrt{x} + \sqrt{y})/2$.

Definition 2.14 ([27])**.** Let A and B be two nonempty sets. A cyclic mapping is a mapping $T\colon A \cup B \to A \cup B$ which satisfies: $TA \subseteq B$ and $TB \subseteq A$.

Kirk, Srinivasan and Veeramani [27], amongst other results, established the following generalization of the contraction mapping principle.

Theorem 2.2 ([27])**.** *Let A and B be two nonempty closed subsets of a complete metric space X and suppose $f\colon X \to X$ satisfies:*

(1) *$fA \subseteq B$ and $fB \subseteq A$,*
(2) *$d(fx, fy) \leqslant kd(x, y)$ for all $x \in A$ and $y \in B$ where $k \in (0, 1)$.*

Then f has a unique fixed point in $A \cap B$.

Recently, Choudhury, Das and Bhandari introduced a φ-contraction in the context of 2-Menger spaces for two mappings in [6]. The following theorem was established.

Theorem 2.3 ([6]). *Let (X, F, Δ) be a complete 2-Menger space, where Δ is the minimum t-norm, T_1, T_2 are two self-maps on X such that for all x, y, a in X and $t > 0$,*

$$F_{T_1x,T_2y,a}(\varphi(t)) \geqslant F_{x,y,a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big), \tag{2.4}$$

where $c \in (0,1)$ and φ is a Φ-function. Then T_1 and T_2 have a unique common fixed point in X.

3. Main results

Lemma 3.1. *Let (X, F, Δ) be a complete 2-Menger space with a Hadzic-type t-norm Δ, whenever $x_n \to x$ and $y_n \to y$, $F_{x_n,y_n,a}(t) \to F_{x,y,a}(t)$ for all $a \in X$. Let there exist two nonempty closed subsets A and B of X and let the mapping $T\colon A \cup B \to A \cup B$ be a cyclic mapping, that is,*

$$TA \subseteq B \quad \text{and} \quad TB \subseteq A \tag{3.1}$$

and such that

$$F_{Tx,Ty,a}(t) \geqslant \psi\Big(F_{x,Tx,a}\Big(\frac{t_1}{\alpha}\Big), F_{y,Ty,a}\Big(\frac{t_2}{\beta}\Big)\Big), \tag{3.2}$$

whenever $x \in A$, $y \in B$ for all $a \in X$, where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $\alpha, \beta > 0$ with $0 < \alpha + \beta < 1$, ψ is a Ψ-function. Then, we have $\lim\limits_{n\to\infty} F_{x_{n+1},x_n,a}(t) = 1$.

Proof. Let x_0 be an arbitrary point of A. Now we construct the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}$ for all positive integers $n \geqslant 1$.

Then, by (3.1), we obtain

$$x_{2n} = Tx_{2n-1} \in A \quad \text{and} \quad x_{2n+1} = Tx_{2n} \in B \quad \text{for all positive integers } n \geqslant 1. \tag{3.3}$$

Now, for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$ and taking n even for all $a \in X$, we have

$$\begin{aligned} F_{x_{n+1},x_n,a}(t) &= F_{Tx_n,Tx_{n-1},a}(t) \\ &\geqslant \psi\Big(F_{x_n,Tx_n,a}\Big(\frac{t_1}{\alpha}\Big), F_{x_{n-1},Tx_{n-1},a}\Big(\frac{t_2}{\beta}\Big)\Big) \quad (\text{since } x_n \in A,\ x_{n-1} \in B) \end{aligned} \tag{3.4}$$

$$= \psi\Big(F_{x_n,x_{n+1},a}\Big(\frac{t_1}{\alpha}\Big), F_{x_{n-1},x_n,a}\Big(\frac{t_2}{\beta}\Big)\Big)$$
$$= \psi\Big(F_{x_{n+1},x_n,a}\Big(\frac{t_1}{\alpha}\Big), F_{x_n,x_{n-1},a}\Big(\frac{t_2}{\beta}\Big)\Big).$$

Let

$$t_1 = \frac{\alpha t}{\alpha+\beta}, \quad t_2 = \frac{\beta t}{\alpha+\beta} \quad \text{and} \quad c = \alpha+\beta, \tag{3.5}$$

then obviously we have $0 < c < 1$.

Then, we have from (3.4),

$$F_{x_{n+1},x_n,a}(t) \geqslant \psi\Big(F_{x_{n+1},x_n,a}\Big(\frac{t}{c}\Big), F_{x_n,x_{n-1},a}\Big(\frac{t}{c}\Big)\Big). \tag{3.6}$$

Again, for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$ and taking n be odd for all $a \in X$, we have

$$\begin{aligned} F_{x_{n+1},x_n,a}(t) &= F_{Tx_n,Tx_{n-1},a}(t) = F_{Tx_{n-1},Tx_n,a}(t) \\ &\geqslant \psi\Big(F_{x_{n-1},Tx_{n-1},a}\Big(\frac{t_1}{\alpha}\Big), F_{x_n,Tx_n,a}\Big(\frac{t_2}{\beta}\Big)\Big) \quad (\text{since } x_{n-1} \in A,\ x_n \in B) \\ &= \psi\Big(F_{x_{n-1},x_n,a}\Big(\frac{t_1}{\alpha}\Big), F_{x_n,x_{n+1},a}\Big(\frac{t_2}{\beta}\Big)\Big). \end{aligned} \tag{3.7}$$

Taking t_1, t_2 and c as in (3.5), we have from (3.7),

$$F_{x_{n+1},x_n,a}(t) \geqslant \psi\Big(F_{x_n,x_{n-1},a}\Big(\frac{t}{c}\Big), F_{x_{n+1},x_n,a}\Big(\frac{t}{c}\Big)\Big). \tag{3.8}$$

We now claim that for all $t > 0$ and for all $a \in X$,

$$F_{x_{n+1},x_n,a}\Big(\frac{t}{c}\Big) \geqslant F_{x_n,x_{n-1},a}\Big(\frac{t}{c}\Big). \tag{3.9}$$

If possible, let for some $s > 0$ and some $p \in X$,

$$F_{x_{n+1},x_n,p}\Big(\frac{s}{c}\Big) < F_{x_n,x_{n-1},p}\Big(\frac{s}{c}\Big).$$

Then, we have from (3.6), (3.8) and by the properties of Ψ-function,

$$\begin{aligned} F_{x_{n+1},x_n,p}(s) &\geqslant \psi\Big(F_{x_{n+1},x_n,p}\Big(\frac{s}{c}\Big), F_{x_{n+1},x_n,p}\Big(\frac{s}{c}\Big)\Big) \\ &> F_{x_{n+1},x_n,p}\Big(\frac{s}{c}\Big) \geqslant F_{x_{n+1},x_n,p}(s), \end{aligned}$$

which is a contradiction, since $0 < c < 1$ and F is nondecreasing. Therefore, for all $t > 0$, $n \geqslant 1$ and for all $a \in X$, (3.9) holds.

Now, using (3.9), we have from (3.6), (3.8) for all $t > 0$ and for all $a \in X$,

$$
\begin{aligned}
F_{x_{n+1},x_n,a}(t) &\geqslant \psi\Big(F_{x_{n-1},x_n,a}\Big(\frac{t}{c}\Big), F_{x_{n-1},x_n,a}\Big(\frac{t}{c}\Big)\Big) \\
&= \psi\Big(F_{x_n,x_{n-1},a}\Big(\frac{t}{c}\Big), F_{x_n,x_{n-1},a}\Big(\frac{t}{c}\Big)\Big) \\
&> F_{x_n,x_{n-1},a}\Big(\frac{t}{c}\Big).
\end{aligned} \tag{3.10}
$$

By repeated applications of (3.10), after n steps for all $t > 0$, $n \geqslant 1$ and for all $a \in X$, we obtain

$$
F_{x_n,x_{n+1},a}(t) > F_{x_0,x_1,a}\Big(\frac{t}{c^n}\Big). \tag{3.11}
$$

Taking limit as $n \to \infty$ on both sides for all $t > 0$ and $a \in X$, we have

$$
\lim_{n\to\infty} F_{x_{n+1},x_n,a}(t) = 1. \tag{3.12}
$$

□

Theorem 3.1. *Let (X, F, Δ) be a complete 2-Menger space with a Hadzic-type t-norm Δ, whenever $x_n \to x$ and $y_n \to y$, $F_{x_n,y_n,a}(t) \to F_{x,y,a}(t)$ for all $a \in X$. Let there exist two nonempty closed subsets A and B of X and let the mapping $T\colon A \cup B \to A \cup B$ be a cyclic mapping that satisfies the conditions* (3.1) *and* (3.2)*, whenever $x \in A$, $y \in B$ for all $a \in X$, where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $\alpha, \beta > 0$ with $0 < \alpha + \beta < 1$, ψ is a Ψ-function. Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$.*

Proof. By an application of Lemma 3.1 we arrive at (3.12), that is,

$$
\lim_{n\to\infty} F_{x_{n+1},x_n,a}(t) = 1.
$$

Again, by repeated applications of (3.10), it follows that for all $a \in X$, $t > 0$, $n \geqslant 0$ and each $i \geqslant 1$,

$$
F_{x_{n+i},x_{n+i+1},a}(t) > F_{x_n,x_{n+1},a}\Big(\frac{t}{c^i}\Big). \tag{3.13}
$$

We next prove that $\{x_n\}$ is a Cauchy sequence (Definition 2.10), that is, we prove that for arbitrary $\varepsilon > 0$ and $0 < \lambda < 1$, there exists $N(\varepsilon, \lambda)$ such that for all $a \in X$,

$$
F_{x_n,x_m,a}(\varepsilon) > 1 - \lambda \quad \text{for all } n, m \geqslant N(\varepsilon, \lambda).
$$

Without loss of generality we can assume that $m > n$.

Now,

$$\varepsilon = \varepsilon\frac{1-c}{1-c} > \varepsilon(1-c)(1+c+c^2+\ldots+c^{m-n-1}).$$

Then, by the monotone increasing property of F, and for all $a \in X$, we have

$$F_{x_n,x_m,a}(\varepsilon) \geqslant F_{x_n,x_m,a}(\varepsilon(1-c)(1+c+c^2+\ldots+c^{m-n-1})),$$

that is,

$$\tag{3.14} \begin{aligned} F_{x_n,x_m,a}(\varepsilon) \geqslant \Delta\big(F_{x_n,x_{n+1},a}(\varepsilon(1-c)), \Delta\big(F_{x_{n+1},x_{n+2},a}(\varepsilon c(1-c)), \ldots,\\ \Delta(\ldots, \Delta(F_{x_{m-2},x_{m-1},a}(\varepsilon c^{m-n-2}(1-c)),\\ F_{x_{m-1},x_m,a}(\varepsilon c^{m-n-1}(1-c)))\ldots)\big)\big). \end{aligned}$$

Putting $t = (1-c)\varepsilon c^i$ in (3.13) for all $a \in X$, we get

$$F_{x_{n+i},x_{n+i+1},a}((1-c)\varepsilon c^i) > F_{x_n,x_{n+1},a}((1-c)\varepsilon).$$

Then, by (3.14), for all $a \in X$, we have

$$\begin{aligned} F_{x_n,x_m,a}(\varepsilon) \geqslant \Delta(F_{x_n,x_{n+1},a}(\varepsilon(1-c)), \Delta(F_{x_n,x_{n+1},a}(\varepsilon(1-c)),\\ \Delta(\ldots, \Delta(F_{x_n,x_{n+1},a}(\varepsilon(1-c)), F_{x_n,x_{n+1},a}(\varepsilon(1-c)))\ldots))), \end{aligned}$$

that is,

$$\tag{3.15} F_{x_n,x_m,a}(\varepsilon) \geqslant \Delta^{(m-n)}F_{x_n,x_{n+1},a}(\varepsilon(1-c)).$$

Since the t-norm Δ is a Hadzic-type t-norm, the family $\{\Delta^p\}$ of its iterates is equi-continuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\tag{3.16} \Delta^{(m-n)}(s) > 1-\lambda \quad \text{whenever } \eta(\lambda) < s \leqslant 1.$$

Since $F_{x_0,x_1,a}(t) \to 1$ as $t \to \infty$ and $0 < c < 1$, there exists a positive integer $N(\varepsilon,\lambda)$ such that for all $a \in X$,

$$\tag{3.17} F_{x_0,x_1,a}\Big(\frac{(1-c)\varepsilon}{c^n}\Big) > \eta(\lambda) \quad \text{for all } n \geqslant N(\varepsilon,\lambda).$$

From (3.17) and (3.13), with $n = 0$, $i = n$ and $t = (1-c)\varepsilon$ for all $a \in X$, we get

$$F_{x_n,x_{n+1},a}(\varepsilon(1-c)) > F_{x_0,x_1,a}\Big(\frac{(1-c)\varepsilon}{c^n}\Big) > \eta(\lambda) \quad \text{for all } n \geqslant N(\varepsilon,\lambda).$$

Then, from (3.16) with $s = F_{x_n,x_{n+1},a}(\varepsilon(1-c))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1},a}(\varepsilon(1-c))) > 1-\lambda.$$

It then follows from (3.15) that for all $a \in X$,

$$F_{x_n,x_m,a}(\varepsilon) > 1-\lambda \quad \text{for all } m,n \geqslant N(\varepsilon,\lambda).$$

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have $x_n \to z$ in X for $n \to \infty$. The subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of $\{x_n\}$ also converge to z. Now $\{x_{2n}\} \subset A$ and A is closed. Therefore $z \in A$. Similarly, $\{x_{2n-1}\} \subset B$ and B is closed. Therefore $z \in B$. Thus we have $z \in A \cap B$.

Now, we prove that $Tz = z$.

For that we get the following two possible cases.

Case I: Let n be even. Then $x_n \in A$ and $z \in A \cap B \Rightarrow z \in B$.

Now, using (3.2) and (3.3), we have

$$F_{Tx_n,Tz,a}(t) \geqslant \psi\Big(F_{x_n,Tx_n,a}\Big(\frac{t_1}{\alpha}\Big), F_{z,Tz,a}\Big(\frac{t_2}{\beta}\Big)\Big),$$

that is,

$$F_{x_{n+1},Tz,a}(t) \geqslant \psi\Big(F_{x_n,x_{n+1},a}\Big(\frac{t_1}{\alpha}\Big), F_{z,Tz,a}\Big(\frac{t_2}{\beta}\Big)\Big).$$

Taking limit as $n \to \infty$ on both sides, we have

$$\begin{aligned}
F_{z,Tz,a}(t) &\geqslant \psi\Big(F_{z,z,a}\Big(\frac{t_1}{\alpha}\Big), F_{z,Tz,a}\Big(\frac{t_2}{\beta}\Big)\Big) \quad \text{(since by our assumption,} \\
&\qquad\qquad x_n \to x,\ y_n \to y \text{ implies } F_{x_n,y_n,a}(t) \to F_{x,y,a}(t)) \\
&= \psi\Big(1, F_{z,Tz,a}\Big(\frac{t}{c}\Big)\Big) \quad \text{(by (3.5))} \\
&\geqslant \psi\Big(F_{z,Tz,a}\Big(\frac{t}{c}\Big), F_{z,Tz,a}\Big(\frac{t}{c}\Big)\Big) \quad \text{(by the properties of } \psi\text{-function)} \\
&> F_{z,Tz,a}\Big(\frac{t}{c}\Big) > F_{z,Tz,a}\Big(\frac{t}{c^2}\Big).
\end{aligned}$$

Continuing this process n times we obtain

$$F_{z,Tz,a}(t) > F_{z,Tz,a}\Big(\frac{t}{c^n}\Big).$$

Again, taking limit as $n \to \infty$ on both sides, we obtain

$$\lim_{n\to\infty} F_{z,Tz,a}(t) \geqslant \lim_{n\to\infty} F_{z,Tz,a}\Big(\frac{t}{c^n}\Big) = 1 \quad \text{for all } a \in X.$$

Case II: Let n be odd. Then $x_n \in B$ and $z \in A \cap B \Rightarrow z \in A$.
Now, using (3.2) and (3.3), we have

$$F_{Tz,Tx_n,a}(t) \geqslant \psi\Big(F_{z,Tz,a}\Big(\frac{t_1}{\alpha}\Big), F_{x_n,Tx_n,a}\Big(\frac{t_2}{\beta}\Big)\Big),$$

that is,

$$F_{Tz,x_{n+1},a}(t) \geqslant \psi\Big(F_{z,Tz,a}\Big(\frac{t_1}{\alpha}\Big), F_{x_n,x_{n+1},a}\Big(\frac{t_2}{\beta}\Big)\Big).$$

Taking limit as $n \to \infty$ on both sides, we have

$$\begin{aligned}
F_{Tz,z,a}(t) &\geqslant \psi\Big(F_{z,Tz,a}\Big(\frac{t_1}{\alpha}\Big), F_{z,z,a}\Big(\frac{t_2}{\beta}\Big)\Big) \quad \text{(since by our assumption,} \\
&\qquad\qquad x_n \to x, y_n \to y \text{ implies } F_{x_n,y_n,a}(t) \to F_{x,y,a}(t)) \\
&= \psi\Big(F_{z,Tz,a}\Big(\frac{t}{c}\Big), 1\Big) \quad \text{(by (3.5))} \\
&\geqslant \psi\Big(F_{z,Tz,a}\Big(\frac{t}{c}\Big), F_{z,Tz,a}\Big(\frac{t}{c}\Big)\Big) \quad \text{(by the properties of } \Psi\text{-function)} \\
&> F_{z,Tz,a}\Big(\frac{t}{c}\Big) > F_{z,Tz,a}\Big(\frac{t}{c^2}\Big).
\end{aligned}$$

Continuing this process n times we obtain

$$F_{z,Tz,a}(t) > F_{z,Tz,a}\Big(\frac{t}{c^n}\Big).$$

Again, taking limit as $n \to \infty$ on both sides, we obtain

$$\lim_{n\to\infty} F_{z,Tz,a}(t) \geqslant \lim_{n\to\infty} F_{z,Tz,a}\Big(\frac{t}{c^n}\Big) = 1 \quad \text{for all } a \in X.$$

Combining both cases we can conclude that $z = Tz$.
To prove the uniqueness of the fixed point, let u be another fixed point of T, that is, $Tu = u$ in $A \cap B$. Let $a \in X$ be any element different from z and u.
Then, for all $t > 0$,

$$\begin{aligned}
F_{z,u,a}(t) &= F_{Tz,Tu,a}(t) \\
&\geqslant \psi\Big(F_{z,Tz,a}\Big(\frac{t_1}{\alpha}\Big), F_{u,Tu,a}\Big(\frac{t_2}{\beta}\Big)\Big) \quad \text{(for } t_1, t_2 > 0 \text{ and } t_1 + t_2 = t) \\
&\qquad\qquad \text{(since we can take } z \in A \text{ and } u \in B) \\
&= \psi\Big(F_{z,z,a}\Big(\frac{t_1}{\alpha}\Big), F_{u,u,a}\Big(\frac{t_2}{\beta}\Big)\Big) = \psi(1,1) = 1.
\end{aligned}$$

Therefore, $z = u$.
This completes the proof of our theorem. □

In our next theorem we use the control function φ (Definition 2.12) in the inequality (3.2). Here we also use the minimum t-norm. We prove our next theorem by different arguments than the first theorem.

First we prove the following lemma.

Lemma 3.2. *Let (X, F, Δ) be a complete 2-Menger space with a third-order minimum t-norm Δ. Let there exist two nonempty closed subsets A and B of X and let the mapping $T\colon A \cup B \to A \cup B$ be a cyclic mapping, that is,*

$$TA \subseteq B \quad \text{and} \quad TB \subseteq A \tag{3.18}$$

and such that

$$F_{Tx,Ty,a}(\varphi(t)) \geqslant \psi\Big(F_{x,Tx,a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{y,Ty,a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big) \tag{3.19}$$

whenever $x \in A$, $y \in B$ for all $a \in X$, where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $\alpha, \beta > 0$ with $0 < \alpha + \beta < 1$, φ is a Φ-function, ψ is a Ψ-function. Then, we have

$$\lim_{n\to\infty} F_{x_{n+1},x_n,a}(\varphi(t)) = 1.$$

Proof. Let x_0 be an arbitrary point of A. Now we construct the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1}$ for all positive integers $n \geqslant 1$.

Then, by (3.18), we obtain

$$x_{2n} = Tx_{2n-1} \in A \ \text{ and } \ x_{2n+1} = Tx_{2n} \in B \ \text{ for all positive integers } n \geqslant 1. \tag{3.20}$$

Now, for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$ and taking n even for all $a \in X$, we have

$$\begin{aligned}
F_{x_{n+1},x_n,a}(\varphi(t)) &= F_{Tx_n,Tx_{n-1},a}(\varphi(t)) \\
&\geqslant \psi\Big(F_{x_n,Tx_n,a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{x_{n-1},Tx_{n-1},a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big) \\
&\qquad\qquad\qquad\qquad\qquad \text{(since } x_n \in A,\ x_{n-1} \in B) \\
&= \psi\Big(F_{x_n,x_{n+1},a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{x_{n-1},x_n,a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big) \\
&= \psi\Big(F_{x_{n+1},x_n,a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{x_n,x_{n-1},a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big).
\end{aligned}$$

Let

$$t_1 = \frac{\alpha t}{\alpha + \beta}, \quad t_2 = \frac{\beta t}{\alpha + \beta} \quad \text{and} \quad c = \alpha + \beta. \tag{3.21}$$

Then obviously we have $0 < c < 1$.

Then, we have from (3.21),

$$F_{x_{n+1},x_n,a}(\varphi(t)) \geqslant \psi\Big(F_{x_{n+1},x_n,a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big), F_{x_n,x_{n-1},a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big)\Big). \tag{3.22}$$

Again, for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$ and taking n odd for all $a \in X$, we have

$$\begin{aligned} F_{x_{n+1},x_n,a}(\varphi(t)) &= F_{Tx_n,Tx_{n-1},a}(\varphi(t)) = F_{Tx_{n-1},Tx_n,a}(\varphi(t)) \\ &\geqslant \psi\Big(F_{x_{n-1},Tx_{n-1},a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{x_n,Tx_n,a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big) \\ &\qquad\qquad (\text{since } x_{n-1} \in A,\ x_n \in B) \\ &= \psi\Big(F_{x_{n-1},x_n,a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{x_n,x_{n+1},a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big). \end{aligned} \tag{3.23}$$

Taking t_1, t_2 and c as in (3.22), we have from (3.24),

$$F_{x_{n+1},x_n,a}(\varphi(t)) \geqslant \psi\Big(F_{x_n,x_{n-1},a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big), F_{x_{n+1},x_n,a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big)\Big). \tag{3.24}$$

We now claim that for all $t > 0$ and for all $a \in X$,

$$F_{x_{n+1},x_n,a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big) \geqslant F_{x_n,x_{n-1},a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big). \tag{3.25}$$

If possible, let for some $s > 0$ and some $p \in X$,

$$F_{x_{n+1},x_n,p}\Big(\varphi\Big(\frac{s}{c}\Big)\Big) < F_{x_n,x_{n-1},p}\Big(\varphi\Big(\frac{s}{c}\Big)\Big).$$

Then, we have from (3.23), (3.25) and by the properties of Ψ-function,

$$\begin{aligned} F_{x_{n+1},x_n,p}(\varphi(s)) &\geqslant \psi\Big(F_{x_{n+1},x_n,p}\Big(\varphi\Big(\frac{s}{c}\Big)\Big), F_{x_{n+1},x_n,p}\Big(\varphi\Big(\frac{s}{c}\Big)\Big)\Big) \\ &> F_{x_{n+1},x_n,p}\Big(\varphi\Big(\frac{s}{c}\Big)\Big) \geqslant F_{x_{n+1},x_n,p}(\varphi(s)), \end{aligned}$$

which is a contradiction, since $0 < c < 1$ and F is nondecreasing.

Therefore, for all $t > 0$, $n \geqslant 1$ and for all $a \in X$, (3.26) holds.

Now, using (3.26), we have from (3.23), (3.25) for all $t > 0$ and for all $a \in X$,

$$\begin{aligned} F_{x_{n+1},x_n,a}(\varphi(t)) &\geqslant \psi\Big(F_{x_{n-1},x_n,a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big), F_{x_{n-1},x_n,a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big)\Big) \\ &= \psi\Big(F_{x_n,x_{n-1},a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big), F_{x_n,x_{n-1},a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big)\Big) \\ &> F_{x_n,x_{n-1},a}\Big(\varphi\Big(\frac{t}{c}\Big)\Big). \end{aligned} \tag{3.26}$$

By repeated applications of (3.27), after n steps for all $t > 0$, $n \geqslant 1$ and for all $a \in X$, we obtain

$$F_{x_n,x_{n+1},a}(\varphi(t)) > F_{x_0,x_1,a}\Big(\varphi\Big(\frac{t}{c^n}\Big)\Big). \tag{3.27}$$

Taking limit as $n \to \infty$ on both sides for all $t > 0$ and $a \in X$, we have

$$\lim_{n\to\infty} F_{x_{n+1},x_n,a}(\varphi(t)) = 1. \tag{3.28}$$

By virtue of the properties of φ and F we can choose $s > 0$ such that $s > \varphi(t)$. Then for all $a \in X$ and $t > 0$ we have

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(s) = 1. \tag{3.29}$$

$\square$

Theorem 3.2. *Let (X, F, Δ) be a complete 2-Menger space with a third-order minimum t-norm Δ. Let there exist two nonempty closed subsets A and B of X and let the mapping $T\colon A \cup B \to A \cup B$ be a cyclic mapping, that is, the mapping T satisfies the conditions (3.18) and (3.19), whenever $x \in A$, $y \in B$ for all $a \in X$, where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $\alpha, \beta > 0$ with $0 < \alpha + \beta < 1$, φ is a Φ-function, ψ is a Ψ-function. Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$.*

Proof. By an application of Lemma 3.2 we arrive at (3.30), that is,

$$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(s) = 1.$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\varepsilon) \leqslant 1 - \lambda. \tag{3.30}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (3.31), so that

$$F_{x_{m(k)},x_{n(k)-1},a}(\varepsilon) > 1 - \lambda. \tag{3.31}$$

If $\varepsilon_1 < \varepsilon$, then we have

$$F_{x_{m(k)},x_{n(k)},a}(\varepsilon_1) \leqslant F_{x_{m(k)},x_{n(k)},a}(\varepsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (3.31), (3.32), whenever ε is replaced by a smaller positive value. As φ is continuous at 0 and strictly monotone increasing with $\varphi(0) = 0$, it is possible to obtain $\varepsilon_2 > 0$ such that $\varphi(\varepsilon_2) < \varepsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)},a}(\varphi(\varepsilon_2)) \leqslant 1-\lambda, \tag{3.32}$$

and

$$F_{x_{m(k)},x_{n(k)-1},a}(\varphi(\varepsilon_2)) > 1-\lambda. \tag{3.33}$$

Now, we have the following possible cases.

Case I: The integer $m(k)$ is odd and $n(k)$ is even for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$ is odd and $n(l)$ is even for all l with $n(l) > m(l) > l$ such that for some $a \in X$,

$$F_{x_{m(l)},x_{n(l)},a}(\varphi(\varepsilon_2)) \leqslant 1-\lambda \tag{3.34}$$

and

$$F_{x_{m(l)},x_{n(l)-1},a}(\varphi(\varepsilon_2)) > 1-\lambda. \tag{3.35}$$

Now, from (3.35), for some $a \in X$ and for $\varepsilon_2 > 0$, we have

$$\begin{aligned}
1-\lambda &\geqslant F_{x_{m(l)},x_{n(l)},a}(\varphi(\varepsilon_2)) = F_{Tx_{m(l)-1},Tx_{n(l)-1},a}(\varphi(\varepsilon_2))\\
&\geqslant \psi\Big(F_{x_{m(l)-1},Tx_{m(l)-1},a}\Big(\varphi\Big(\frac{\varepsilon_2'}{\alpha}\Big)\Big), F_{x_{n(l)-1},Tx_{n(l)-1},a}\Big(\varphi\Big(\frac{\varepsilon_2''}{\beta}\Big)\Big)\Big)\\
&\qquad (x_{m(l)-1} \in A,\ x_{n(l)-1} \in B \text{ where } \varepsilon_2 = \varepsilon_2' + \varepsilon_2'' \text{ and } \varepsilon_2', \varepsilon_2'' > 0)\\
&= \psi\Big(F_{x_{m(l)-1},x_{m(l)},a}\Big(\varphi\Big(\frac{\varepsilon_2'}{\alpha}\Big)\Big), F_{x_{n(l)-1},x_{n(l)},a}\Big(\varphi\Big(\frac{\varepsilon_2''}{\beta}\Big)\Big)\Big)\\
&\qquad \text{(by the properties of } \psi \text{ and (3.30))}\\
&\geqslant \psi(1-\lambda, 1-\lambda) > 1-\lambda,
\end{aligned}$$

which is a contradiction.

Case II: The integer $m(k)$ is even and $n(k)$ is odd for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$ is even and $n(l)$ is odd for all l with $n(l) > m(l) > l$ such that for some $a \in X$, (3.35), (3.36) hold.

Then, we arrive at a contradiction exactly as in Case I above.

Case III: The integers $m(k)$ and $n(k)$ are both even for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$ and $n(l)$ are both even for all l with $n(l) > m(l) > l$ such that for some $a \in X$, (3.35), (3.36) hold.

By the properties of φ, we can choose $\eta_1, \eta_2 > 0$ such that $\varphi(\varepsilon_2) > \eta_1 + \eta_2$.

Now, from (3.35), for some $a \in X$ and for $\varepsilon_2 > 0$, we have

$$\begin{aligned}
(3.36)\quad 1-\lambda &\geqslant F_{x_{m(l)},x_{n(l)},a}(\varphi(\varepsilon_2)) \\
&\geqslant \Delta(F_{x_{m(l)},x_{n(l)},x_{m(l)+1}}(\eta_1), F_{x_{m(l)},x_{m(l)+1},a}(\eta_2), \\
&\qquad F_{x_{m(l)+1},x_{n(l)},a}(\varphi(\varepsilon_2)-\eta_1-\eta_2)) \\
&= \Delta(F_{x_{m(l)},x_{n(l)},x_{m(l)+1}}(\eta_1), F_{x_{m(l)},x_{m(l)+1},a}(\eta_2), F_{x_{m(l)+1},x_{n(l)},a}(\varphi(\xi)))
\end{aligned}$$

(by the properties of φ, we can take $\varphi(\xi) = \varphi(\varepsilon_2) - \eta_1 - \eta_2$ where $\xi > 0$).

Now, by (3.30) for sufficiently large l, we have

$$F_{x_{m(l)},x_{n(l)},x_{m(l)+1}}(\eta_1) > 1-\lambda \tag{3.37}$$

and

$$F_{x_{m(l)},x_{m(l)+1},a}(\eta_2) > 1-\lambda. \tag{3.38}$$

$$\begin{aligned}
(3.39)\quad F_{x_{m(l)+1},x_{n(l)},a}(\varphi(\xi)) &= F_{Tx_{m(l)},Tx_{n(l)-1},a}(\varphi(\xi)) \\
&\geqslant \psi\Big(F_{x_{m(l)},Tx_{m(l)},a}\Big(\varphi\Big(\frac{\xi_1}{\alpha}\Big)\Big), F_{x_{n(l)-1},Tx_{n(l)-1},a}\Big(\varphi\Big(\frac{\xi_2}{\beta}\Big)\Big)\Big) \\
&\qquad (x_{m(l)} \in A,\ x_{n(l)-1} \in B \text{ where } \xi = \xi_1 + \xi_2 \text{ and } \xi_1, \xi_2 > 0) \\
&= \psi\Big(F_{x_{m(l)},x_{m(l)+1},a}\Big(\varphi\Big(\frac{\xi_1}{\alpha}\Big)\Big), F_{x_{n(l)-1},x_{n(l)},a}\Big(\varphi\Big(\frac{\xi_2}{\beta}\Big)\Big)\Big) \\
&\geqslant \psi(1-\lambda, 1-\lambda) \quad \text{(by (3.30))} \\
&> 1-\lambda.
\end{aligned}$$

Now, using (3.38), (3.39) and (3.40) in (3.37), we have

$$1-\lambda > 1-\lambda,$$

which is a contradiction.

Case IV: The integers $m(k)$ and $n(k)$ are both odd for an infinite number of values of k. Then, there exist $\{m(l)\} \subset \{m(k)\}$ and $\{n(l)\} \subset \{n(k)\}$ where $m(l)$ and $n(l)$

are both odd for all l with $n(l) > m(l) > l$ such that for some $a \in X$, (3.35), (3.36) hold.

Then, we arrive at a contradiction exactly as in Case III above.

Combining all the above four cases we can conclude that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have

$$x_n \to z \quad \text{in } X \text{ for } n \to \infty. \tag{3.40}$$

The subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of $\{x_n\}$ also converge to z. Now $\{x_{2n}\} \subset A$ and A is closed. Therefore $z \in A$. Similarly, $\{x_{2n-1}\} \subset B$ and B is closed. Therefore $z \in B$. Thus we have $z \in A \cap B$.

We now show that $Tz = z$.

If possible, let $0 < F_{z,Tz,a}(\varphi(t)) < 1$ for some $t > 0$.

By the properties of φ we can choose $\xi_1, \xi_2, t_1, t_2 > 0$ such that $\varphi(t) = \xi_1 + \xi_2 + \varphi(t_1 + t_2)$.

Now, we consider the sequence $\{x_{n(k)}\} \subset \{x_n\}$ for which integers $n(k)$ are even or odd for an infinite number of values of k.

Then, we get the following two possible cases.

Case Ia: Let $n(k)$ be even. Then $x_{n(k)} \in A$ and $z \in A \cap B \Rightarrow z \in B$.

Again, since $0 < \beta < 1$, we can get $\varphi(t_2/\beta) > \varphi(t)$.

Then, we have

$$\begin{aligned} F_{z,Tz,a}(\varphi(t)) &\geqslant \Delta(F_{z,Tz,x_{n(k)+1}}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), F_{x_{n(k)+1},Tz,a}(\varphi(t_1+t_2))) \\ &= \Delta(F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), F_{Tx_{n(k)},Tz,a}(\varphi(t_1+t_2))) \\ &\geqslant \Delta\Big(F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), \\ &\qquad \psi\Big(F_{x_{n(k)},x_{n(k)+1},a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{z,Tz,a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big)\Big) \\ &\geqslant \Delta\Big(F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), \\ &\qquad \psi\Big(F_{x_{n(k)},x_{n(k)+1},a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{z,Tz,a}(\varphi(t))\Big)\Big). \end{aligned} \tag{3.41}$$

By (3.29), (3.30) and (3.41), there exists a positive integer N_1 such that

$$F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), F_{x_{n(k)},x_{n(k)+1},a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big) > F_{z,Tz,a}(\varphi(t))$$

for all $n(k) > N_1$.

Then, we have from (3.42),

$$F_{z,Tz,a}(\varphi(t)) > F_{z,Tz,a}(\varphi(t)),$$

which is a contradiction.

Case Ib: Let $n(k)$ be odd. Then $x_{n(k)} \in B$ and $z \in A \cap B \Rightarrow z \in A$.

Again, since $0 < \alpha < 1$, we can get $\varphi(t_1/\alpha) > \varphi(t)$.

Then, we have

$$\begin{aligned}
(3.42)\ F_{z,Tz,a}(\varphi(t)) &\geqslant \Delta(F_{z,Tz,x_{n(k)+1}}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), F_{x_{n(k)+1},Tz,a}(\varphi(t_1+t_2))) \\
&= \Delta(F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), F_{Tz,Tx_{n(k)},a}(\varphi(t_1+t_2))) \\
&\geqslant \Delta\Big(F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), \\
&\qquad \psi\Big(F_{z,Tz,a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{x_{n(k)},Tx_{n(k)},a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big)\Big) \\
&\geqslant \Delta\Big(F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), \\
&\qquad \psi\Big(F_{z,Tz,a}(\varphi(t)), F_{x_{n(k)},x_{n(k)+1},a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big)\Big).
\end{aligned}$$

By (3.29), (3.30) and (3.41), there exists a positive integer N_2 such that

$$F_{z,x_{n(k)+1},Tz}(\xi_1), F_{z,x_{n(k)+1},a}(\xi_2), F_{x_{n(k)},x_{n(k)+1},a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big) > F_{z,Tz,a}(\varphi(t))$$

for all $n(k) > N_2$.

Then, we have from (3.43),

$$F_{z,Tz,a}(\varphi(t)) > F_{z,Tz,a}(\varphi(t)),$$

which is a contradiction.

Combining both cases we conclude that $F_{z,Tz,a}(\varphi(t)) = 1$ for all $t > 0$, which implies that $z = Tz$.

To prove the uniqueness of the fixed point, let u be another fixed point of T, that is, $Tu = u$ in $A \cap B$. Let $a \in X$ be any element different from z and u.

Then, for all $t > 0$,

$$\begin{aligned}
F_{z,u,a}(\varphi(t)) &= F_{Tz,Tu,a}(\varphi(t)) \\
&\geqslant \psi\Big(F_{z,Tz,a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{u,Tu,a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big) \text{ (for } t_1, t_2 > 0 \text{ and } t_1 + t_2 = t) \\
&= \psi\Big(F_{z,z,a}\Big(\varphi\Big(\frac{t_1}{\alpha}\Big)\Big), F_{u,u,a}\Big(\varphi\Big(\frac{t_2}{\beta}\Big)\Big)\Big) = \psi(1,1) = 1.
\end{aligned}$$

Therefore, $z = u$. □

E x a m p l e 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$, $A = \{x_1, x_2, x_4\}$, $B = \{x_3, x_4\}$, the t-norm Δ be a third order minimum t-norm and F be defined as

$$F_{x_1,x_2,x_3}(t) = F_{x_1,x_2,x_4}(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ 0.40, & \text{if } 0 < t < 4, \\ 1, & \text{if } t \geqslant 4, \end{cases}$$
$$F_{x_1,x_3,x_4}(t) = F_{x_2,x_3,x_4}(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Then (X, F, Δ) is a complete 2-Menger space. If we define $T\colon X \to X$ as $Tx_1 = x_4$, $Tx_2 = x_3$, $Tx_3 = x_4$, $Tx_4 = x_4$, then the mapping T satisfies all the conditions of the Theorem 3.2, where $\varphi(t) = t$, $\psi(x, y) = (\sqrt{x} + \sqrt{y})/2$, $\alpha, \beta > 0$ with $0 < \alpha + \beta < 1$ and x_4 is the unique fixed point of T in $A \cap B$.

Theorem 3.1 is also satisfied by this example with $\Delta(a, b, c) = \min\{a, b, c\}$.

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