OSCILLATION CRITERIA FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH $p(t)$-LAPLACIAN

Yutaka Shoukaku, Kanazawa

Received November 14, 2013
Communicated by Leonid Berezansky

Dedicated to Professor Norio Yoshida on the occasion of his 65th birthday

Abstract. Recently there has been an increasing interest in studying $p(t)$-Laplacian equations, an example of which is given in the following form

$$(|u'(t)|^{p(t)-2}u'(t))' + c(t)|u(t)|^{q(t)-2}u(t) = 0, \quad t > 0.$$

In particular, the first study of sufficient conditions for oscillatory solution of $p(t)$-Laplacian equations was made by Zhang (2007), but to our knowledge, there has not been a paper which gives the oscillatory conditions by utilizing Riccati inequality. Therefore, we establish sufficient conditions for oscillatory solution of nonlinear differential equations with $p(t)$-Laplacian via Riccati method. The results obtained are new and rare, except for a work of Zhang (2007). We present more detailed results than Zhang (2007).

Keywords: $p(t)$-Laplacian; oscillation theory; Riccati inequality

MSC 2010: 34K11, 35B05, 35R10

1. Introduction

The purpose of this paper is to generalize p-Laplacian equations to the case of $p(t)$-Laplacian equations of the form

$$\text{(E)} \qquad (|u'(t)|^{p(t)-2}u'(t))' + c(t)|u(t)|^{q(t)-2}u(t) = 0, \quad t > 0,$$

where $c(t) \in C((0,\infty);(0,\infty))$.

We assume throughout this paper that: $p(t), q(t) \in C^1(\mathbb{R};(1,\infty))$ and satisfy

$$1 < \inf_{t\in\mathbb{R}} p(t), \quad \sup_{t\in\mathbb{R}} p(t) < \infty, \quad 1 < \inf_{t\in\mathbb{R}} q(t), \quad \sup_{t\in\mathbb{R}} q(t) < \infty.$$

DOI: 10.21136/MB.2016.5

Definition 1.1. A function $u \in C^1(0,\infty)$ with the property that $|u'|^{p(t)-2}u' \in C^1(0,\infty)$ is said to be a *solution* of (E) if u satisfies (E) at every point in $(0,\infty)$.

Definition 1.2. A nontrivial solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros ((E) is an *oscillatory equation* if its solution is oscillatory), otherwise it is nonoscillatory.

The first interest in $p(\cdot)$-type Laplacian was in function spaces called variable exponent spaces. Variable exponent space, which appeared in W. Orlicz's article of 1931, was studied afterwards by many authors (see, e.g., [1]).

In 1999, Růžička studied equations with non-standard $p(x)$-growth in the modeling of the so-called electrorheological fluids (see [2]). After this article, the importance of $p(\cdot)$-type Laplacian was recognized [3].

In recent years, Zhang in [9] investigated the oscillation problem for the $p(t)$-Laplacian equation, and obtained the following:

Theorem A (Zhang [9])**.** *Assume that* $c(t) = t^{-\theta(t)}$ *and*

(A1) $\lim_{t\to\infty} f(t) = f_\infty$, $t^{|f(t)-f_\infty|} < M$ ($f(t)$ *satifies the log-Hölder decay condition*);

(A2) $g(t,\cdot) \in C((0,\infty)\times\mathbb{R})$ *is increasing for any fixed* $t > 0$ *and* $0 < \liminf_{t\to\infty} g(t,u)u \leqslant \limsup_{t\to\infty} g(t,u)u < \infty$, $u \in \mathbb{R}\setminus\{0\}$.

If $p(t)$ *possesses* (A1) *and* $\limsup_{t\to\infty}\theta(t) < \liminf_{t\to\infty} q(t)$, *where*

$$1 < \limsup_{t\to\infty} q(t) < \liminf_{t\to\infty} p(t)$$

or

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} p(t), \quad q(t) \text{ is possesses (A1)},$$

then every solution of (E) *is oscillatory.*

Motivated by this article [9], Yoshida established oscillation theorems, Picone identities and Sturmian comparison theorems for half-linear elliptic inequalities with $p(x)$-Laplacians (see, for example, [7] and [8]). Recently, Şahiner and Zafer [4], [5] also studied forced oscillation of half-linear elliptic inequlities with $p(x)$-Laplacians under the condition $q(t) > p(t) > 1$. However, there is a few part having to study the results of Zhang [9] in detail. Therefore, we provide new oscillation criteria for the solution of (E).

2. Main results

In order to discuss our main results, we need the following lemma, which is due to Usami [6].

Lemma 2.1. *If there exists a function* $\varphi(t) \in C^1([T_0,\infty);(0,\infty))$ *such that*

$$\int_{T_1}^{\infty} \Big(\frac{\bar p(t)|\varphi'(t)|^{\beta}}{\varphi(t)}\Big)^{1/(\beta-1)} \,\mathrm{d}t < \infty, \quad \int_{T_1}^{\infty} \varphi(t)\bar q(t)\,\mathrm{d}t = \infty, \tag{2.1}$$

and

$$\int_{T_1}^{\infty} \frac{1}{\bar p(t)(\varphi(t))^{\beta-1}}\,\mathrm{d}t = \infty$$

for some $T_1 \geqslant T_0 > 0$, *then the Riccati inequality*

$$x'(t) + \frac{1}{\beta}\frac{1}{\bar p(t)}|x(t)|^{\beta} \leqslant -\bar q(t), \tag{2.2}$$

where $\beta > 1$, $\bar p(t) \in C([T_0,\infty);(0,\infty))$ *and* $\bar q(t) \in C([T_0,\infty);\mathbb{R})$, *has no solution on* $[T,\infty)$ *for all large* T.

Theorem 2.1. *Let*

$$p^- \equiv \inf_{t\geqslant T} p(t), \quad p^+ \equiv \sup_{t\geqslant T} p(t), \quad q^- \equiv \inf_{t\geqslant T} q(t), \quad q^+ \equiv \sup_{t\geqslant T} q(t)$$

for some $T > 0$. *If one of the following cases holds:*

(i) $p(t)$ *is increasing,* $1 < p(t) = q(t)$ *or* $1 < p(t) < q(t)$, *there exists* $\varphi(t) \in C^1((0,\infty);(0,\infty))$ *such that*

$$\int^{\infty} \Big\{\frac{|\varphi'(t)|^{p^+/(p^+-1)}}{(p(t)-1+p'(t)t)\varphi(t)}\Big\}^{p^+-1} \,\mathrm{d}t < \infty, \tag{2.3}$$

$$\int^{\infty} \Big\{\frac{|\varphi'(t)|^{p^-/(p^--1)}}{(p(t)-1+p'(t)t)\varphi(t)}\Big\}^{p^--1} \,\mathrm{d}t < \infty, \tag{2.4}$$

$$\int^{\infty} \frac{p(t)-1+p'(t)t}{\varphi(t)^{1/(p^+-1)}}\,\mathrm{d}t = \infty, \quad \int^{\infty} \frac{p(t)-1+p'(t)t}{\varphi(t)^{1/(p^--1)}}\,\mathrm{d}t = \infty,$$

$$\int^{\infty} \varphi(t)c(t)\,\mathrm{d}t = \infty; \tag{2.5}$$

(ii) $q(t)$ *is increasing,* $1 < q(t) < p(t)$, *there exists* $\varphi(t) \in C^1((0,\infty);(0,\infty))$ *such that*

$$\int^{\infty} \Big\{\frac{|\varphi'(t)|^{q^+/(q^+-1)}}{(q(t)-1+q'(t)t)\varphi(t)}\Big\}^{q^+-1} \,\mathrm{d}t < \infty, \tag{2.6}$$

$$\int^{\infty}\left\{\frac{|\varphi'(t)|^{q^-/(q^--1)}}{(q(t)-1+q'(t)t)\varphi(t)}\right\}^{q^--1}\mathrm{d}t<\infty, \tag{2.7}$$

$$\int^{\infty}\frac{q(t)-1+q'(t)t}{\varphi(t)^{1/(q^+-1)}}\,\mathrm{d}t=\infty,\quad \int^{\infty}\frac{q(t)-1+q'(t)t}{\varphi(t)^{1/(q^--1)}}\,\mathrm{d}t=\infty,$$

$$\int^{\infty}\varphi(t)c(t)\,\mathrm{d}t=\infty, \tag{2.8}$$

then every solution $u(t)$ of (E) *is oscillatory.*

Proof. Suppose that u is a nonoscillatory solution. We prove only the case $u>0$, $t\geqslant t_0$ for some $t_0>0$, as the proof of the case $u<0$ is similar. It follows that

$$(|u'|^{p(t)-2}u')'=-c(t)|u|^{q(t)-2}u<0,\quad t\geqslant t_0. \tag{2.9}$$

Now we claim that $u'(t)>0$, $t>t_1$ for some $t_1>t_0$. In fact, if $u'(t)>0$ does not hold, then $u'(t)\leqslant 0$, $t>t_1$. Hence we show that

$$|u'(t)|^{p(t)-2}u'(t)<|u'(t_1)|^{p(t_1)-2}u'(t_1)\leqslant 0.$$

Thus we can find a $t_2>t_1$ such that $u'(t_2)<0$. Integrating (2.9) over $[t_2,t]$ yields

$$|u'(t)|^{p(t)-2}u'(t)\leqslant|u'(t_2)|^{p(t_2)-2}u'(t_2)<0$$

for $t>t_2$, and therefore,

$$-(-u'(t))^{p(t)-1}\leqslant-|u'(t_2)|^{p(t_2)-1},\quad t>t_2.$$

This shows that

$$\begin{aligned}u'(t)&\leqslant-|u'(t_2)|^{(p(t_2)-1)/(p(t)-1)}\\&\leqslant-\min_{t\geqslant t_2}|u'(t_2)|^{(p(t_2)-1)/(p(t)-1)}:=-a<0.\end{aligned}$$

Integrate the above inequality to obtain

$$u(t)\leqslant-a(t-t_2)+u(t_2)\to-\infty$$

as $t\to\infty$. This contradicts the assumption. Hence, we have $u'(t)>0$, $t>t_3$ for some $t_3>t_0$.

(i) If $p(t) = q(t)$ or $1 < p(t) < q(t)$, then we define the function $w_1(t)$ such that

$$w_1(t) = \Big(\frac{u'(t)}{u(t)}\Big)^{p(t)-1} > 0, \tag{2.10}$$

which is led by

$$u(t) = \exp\bigg(\int_0^t w_1^{1/(p(s)-1)}(s)\,\mathrm{d}s\bigg). \tag{2.11}$$

This means that $u(t) > 0$, $t > 0$. Making use of the above argument we easily see that $u'(t) > 0$. Differentiating both sides of (2.10), we see that

$$\begin{aligned} w_1'(t) = &-c(t)u(t)^{q(t)-p(t)} \\ &-w_1(t)\big\{(p(t)-1)w_1^{1/(p(t)-1)}(t) + p'(t)\log u(t)\big\} \end{aligned} \tag{2.12}$$

for $t > t_3$. In view of (2.10) and (2.11), we have $\log u > 0$. Accordingly, we see that $u(t) > 1$ and

$$w_1'(t) \leqslant 0, \quad t \geqslant t_4$$

for some $t_4 > t_3$. Therefore, we see that $\lim\limits_{t\to\infty} w_1(t) := w_1(\infty)$ exists, and we can separate the two case of $0 < w_1(\infty) < 1$ and $1 \leqslant w_1(\infty) < \infty$. First, we take the case when $0 < w_1(\infty) < 1$. Then it follows from (2.11) that

$$\log u(t) = \int_0^t w_1^{1/(p(s)-1)}(s)\,\mathrm{d}s,$$

and consequently

$$\log u(t) \geqslant t w_1^{1/(p^- -1)}(t), \quad t \geqslant t_5 \tag{2.13}$$

for $t_5 > t_4$. Combining (2.12) with (2.13), we have

$$w_1'(t) \leqslant -c(t) - ((p(t)-1) + p'(t)t)w_1^{p^-/(p^- -1)}(t), \quad t \geqslant t_5.$$

Next, for the case when $1 \leqslant w_1(\infty) < \infty$, it can be shown by using a similar method that

$$w_1'(t) \leqslant -c(t) - (p(t) - 1 + p'(t)t)w_1^{p^+/(p^+ -1)}(t), \quad t \geqslant t_5.$$

By applying Lemma 2.1, we see that (2.3)–(2.5) imply that the above Riccati inequalities cannot have a solution. This is a contradiction.

(ii) If $1 < q(t) < p(t)$, then we define the function $w_2(t)$ such that

$$w_2(t) = \frac{u'(t)^{p(t)-1}}{u(t)^{q(t)-1}} = \Big(\frac{u'(t)}{u(t)}\Big)^{q(t)-1} u'(t)^{p(t)-q(t)} > 0, \tag{2.14}$$

which is led by

$$u(t) = \exp\Bigg(\int_0^t \Big(\frac{w_2(s)}{u'(t)^{p(s)-q(s)}}\Big)^{1/(q(s)-1)} \mathrm{d}s\Bigg). \tag{2.15}$$

Differentiating both sides of (2.14), we see that

$$w_2'(t) = \frac{(|u'(t)|^{p(t)-2}u'(t))'}{u(t)^{q(t)-1}} - w_2(t)\frac{(u(t)^{q(t)-1})'}{u(t)^{q(t)-1}}. \tag{2.16}$$

This implies that

$$w_2'(t) = -c(t) - w_2(t)\Big\{(q(t)-1)\frac{u'(t)}{u(t)} + q'(t)\log u(t)\Big\}, \quad t \geqslant t_3, \tag{2.17}$$

which together with (2.14) and (2.15) ensures that $\log u > 0$ and

$$w_2'(t) \leqslant 0, \quad t \geqslant t_6$$

for some $t_6 > t_3$. At this point, it is clear that $\lim\limits_{t\to\infty} w_2(t) := w_2(\infty)$ exists, and we can separate the two cases of $0 < w_2(\infty) < 1$ and $1 \leqslant w_2(\infty) < \infty$. First, we take the case when $0 < w_2(\infty) < 1$. From (2.9) it follows that

$$u'(t)^{p(t)-1} \leqslant u'(t_2)^{p(t_2)-1} \equiv k_0$$

for some constant $k_0 > 0$. From (2.14) we see that

$$w_2(t) \leqslant \Big(\frac{u'(t)}{u(t)}\Big)^{q(t)-1} {k_0}^{(p(t)-q(t))/(p(t)-1)} \leqslant \Big(\frac{u'(t)}{u(t)}\Big)^{q(t)-1} k_1,$$

and so

$$\frac{u'(t)}{u(t)} \geqslant \Big(\frac{w_2(t)}{k_1}\Big)^{1/(q(t)-1)} \geqslant k_2 w_2^{1/(q^- -1)}(t), \quad t \geqslant t_7 \tag{2.18}$$

for some $t_7 > t_6$. On the other hand, by (2.15), we also obtain

$$\begin{aligned}\log u(t) &\geqslant \int_0^t \Big(\frac{w_2(s)}{{k_0}^{(p(s)-q(s))/(p(s)-1)}}\Big)^{1/(q(s)-1)} \mathrm{d}s \\ &\geqslant k_3 t w_2^{1/(q^- -1)}(t), \quad t \geqslant t_8\end{aligned} \tag{2.19}$$

for some constant $k_i > 0$, $i = 1,2,3$, and some $t_8 > t_7$. Combining (2.17), (2.18) with (2.19), we have

$$w_2'(t) \leqslant -c(t) - K(q(t) - 1 + q'(t)t)w_2^{q^-/(q^--1)}(t), \quad t \geqslant t_8.$$

Finally, for the case when $1 \leqslant w_2(\infty) < \infty$, it is easy to verify that

$$w_2'(t) \leqslant -c(t) - K(q(t) - 1 + q'(t)t)w_2^{q^+/(q^+-1)}(t), \quad t \geqslant t_8$$

for some constant $K > 0$. By applying Lemma 2.1, we see that (2.6)–(2.8) imply that the above Riccati inequalities cannot have a solution. This contradiction completes the proof of the theorem. $\square$

Corollary 2.1. *Assume that* $c(t) = t^{-\theta(t)}$, *where* $\theta(t) \in C((0,\infty);\mathbb{R})$, *and that*

$$1 + \limsup_{t\to\infty} \theta(t) < \liminf_{t\to\infty} q(t).$$

If one of the following cases holds:

(i) $p(t)$ *is increasing,* $1 < p(t) < q(t)$ *or* $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} q(t)$;

(ii) $q(t)$ *is increasing,* $1 < q(t) < p(t)$,

then every solution $u(t)$ *of* (E) *is oscillatory.*

Proof. If $1 < p(t) < q(t)$ or $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} q(t)$, then we can derive by applying Theorem 2.1 with $\varphi(t) = t^{q^- - k}$ for some $1 + q^- - p^+ < k < q^- - \theta^+$ that

$$\int^\infty \left\{\frac{((q^- - k)t^{q^- - k - 1})^{p^+/(p^+-1)}}{(p(t) - 1 + p'(t)t)t^{q^- - k}}\right\}^{p^+-1} \mathrm{d}t < c_0 \int^\infty t^{-p^+ + (q^- - p^+ - k + 1)} \,\mathrm{d}t < \infty,$$

$$\int^\infty \left\{\frac{((q^- - k)t^{q^- - k - 1})^{p^-/(p^--1)}}{(p(t) - 1 + p'(t)t)t^{q^- - k}}\right\}^{p^--1} \mathrm{d}t < \infty,$$

$$\int^\infty \frac{p(t) - 1 + p'(t)t}{(t^{p^- - k})^{1/(p^+-1)}} \,\mathrm{d}t > c_1 \int^\infty t^{1-(q^- - k)/(p^+-1)} \,\mathrm{d}t = \infty,$$

$$\int^\infty \frac{p(t) - 1 + p'(t)t}{(t^{p^- - k})^{1/(p^--1)}} \,\mathrm{d}t > \infty,$$

$$\int^\infty t^{-\theta(t) + q^- - k} \,\mathrm{d}t > \int^\infty t^{q^- - \theta^+ - k} \,\mathrm{d}t. \tag{2.20}$$

On the other hand, if $1 < q(t) < p(t)$, then we choose $1 + q^- - q^+ < k < q^- - \theta^+$ such that

$$\int^{\infty}\Big\{\frac{((q^- - k)t^{q^- -k-1})^{q^+/(q^+-1)}}{(q(t)-1+q'(t)t)t^{q^- -k}}\Big\}^{q^+-1}\,\mathrm{d}t < c_2\int^{\infty} t^{-q^+ + (q^- - q^+ - k+1)}\,\mathrm{d}t < \infty,$$

$$\int^{\infty}\Big\{\frac{((q^- - k)t^{q^- -k-1})^{q^-/(q^- -1)}}{(q(t)-1+q'(t)t)t^{q^- -k}}\Big\}^{q^- -1}\,\mathrm{d}t < \infty,$$

$$\int^{\infty}\frac{q(t)-1+q'(t)t}{(t^{q^- -k})^{1/(q^+-1)}}\,\mathrm{d}t > c_3\int^{\infty} t^{1-(q^- -k)/(q^+-1)}\,\mathrm{d}t = \infty,$$

$$\int^{\infty}\frac{q(t)-1+p'(t)t}{(t^{q^- -k})^{1/(q^- -1)}}\,\mathrm{d}t > \infty,$$

$$\int^{\infty} t^{-\theta(t)+q^- -k}\,\mathrm{d}t > \int^{\infty} t^{q^- -\theta^+ -k}\,\mathrm{d}t \tag{2.21}$$

for some positive constants c_i, $i = 0, 1, 2, 3$. Now we assume that $1 + \theta^+ < q^-$ holds, then integral calculus conditions (2.20) and (2.21) become infinite. Clearly, we see that the conditions of Theorem 2.1 hold. Therefore the conclusion follows from Theorem 2.1. $\square$

Evidently, Theorem 2.1 does not apply to Theorem A. Hence we will improve the Lemma 2.1 as follows.

Lemma 2.2. *If there exists a function $\varphi(t) \in C^1([T_0,\infty);(0,\infty))$ such that* (2.1) *holds for some $T_1 \geqslant T_0 > 0$, then the Riccati inequality* (2.2) *has no positive solution on $[T,\infty)$ for all large T.*

Proof. Let $x(t)$ be a positive solution of (2.2). We assume that $\varphi(t)$ is defined for $t \geqslant T_0$. Multiplying (2.2) by $\varphi(t)$ and integrating over $[T_0, t]$, we obtain

$$x(T_0)\varphi(T_0) \geqslant -\int_{T_0}^{t} x(s)\varphi'(s)\,\mathrm{d}s + \int_{T_0}^{t}\frac{\varphi(s)x(s)^{\beta}}{\bar p(s)}\,\mathrm{d}s + \int_{T_0}^{t}\varphi(s)\bar q(s)\,\mathrm{d}s. \tag{2.22}$$

By using Young's inequality we have

$$\begin{aligned} x(s)|\varphi'(s)| &= x(s)\Big(\frac{\varphi(s)}{\bar p(s)}\Big)^{1/\beta}\Big(\frac{\bar p(s)}{\varphi(s)}\Big)^{1/\beta}|\varphi'(s)| \\ &\leqslant \frac{1}{\beta}\frac{\varphi(s)}{\bar p(s)}x(s)^{\beta} + \frac{\beta-1}{\beta}\Big(\Big(\frac{\bar p(s)}{\varphi(s)}\Big)^{1/\beta}|\varphi'(s)|\Big)^{\beta/(\beta-1)}. \end{aligned} \tag{2.23}$$

Combining (2.22) with (2.23) yields

$$\begin{aligned}x(T_0)\varphi(T_0) \geqslant \Big(1-\frac{1}{\beta}\Big)\int_{T_0}^t \frac{\varphi(s)}{\bar p(s)} x(s)^\beta \,\mathrm{d}s \\ -\frac{\beta-1}{\beta}\int_{T_0}^t \Big(\frac{\bar p(s)}{\varphi(s)}|\varphi'(s)|^\beta\Big)^{1/(\beta-1)}\,\mathrm{d}s + \int_{T_0}^t \varphi(s)\bar q(s)\,\mathrm{d}s,\end{aligned}$$

which contradicts the condition (2.1). The proof is complete. □

Theorem 2.2. *If one of the following cases holds:*

(i) $p(t)$ *is increasing,* $1 < p(t) = q(t)$ *or* $1 < p(t) < q(t)$*, there exists* $\varphi(t) \in C^1((0,\infty);(0,\infty))$ *satisfying* (2.3)–(2.5);

(ii) $q(t)$ *is increasing,* $1 < q(t) < p(t)$*, there exists* $\varphi(t) \in C^1((0,\infty);(0,\infty))$ *satisfying* (2.6)–(2.8),

then every solution $u(t)$ *of* (E) *is oscillatory.*

Example 2.1. We consider the equation

$$(|u'(t)|^{2-4/(3t)}u'(t))' + t^{-1/3+1/t}|u(t)|^{1-1/t}u(t) = 0, \quad t \geqslant 1, \tag{2.24}$$

where

$$p(t) = 4 - \frac{4}{3t}, \quad q(t) = 3 - \frac{1}{t}, \quad c(t) = t^{-1/3+1/t}.$$

Letting $\varphi(t) = t^{2/3}$, we see that

$$\begin{gathered}\int^\infty \Big\{\frac{(\frac{2}{3}t^{-1/3})^{3/2}}{(1-1/t+1/t)t^{2/3}}\Big\}^2\,\mathrm{d}t = \int^\infty \Big(\frac{2}{3}\Big)^3 t^{-7/3}\,\mathrm{d}t < \infty, \\ \int^\infty \frac{(\frac{2}{3}t^{-1/3})^2}{(1-1/t+1/t)t^{2/3}}\,\mathrm{d}t = \int^\infty \frac{2}{3}t^{-4/3}\,\mathrm{d}t < \infty, \\ \int^\infty t^{2/3}t^{-1/3+1/t}\,\mathrm{d}t = \infty.\end{gathered}$$

Hence, it follows from (2.3)–(2.5) that all conditions of Theorem 2.2 (ii) are satisfied. However, Theorem 2.1 (ii) is not applicable, since

$$\begin{gathered}\int^\infty \frac{1-1/t+1/t}{(t^{2/3})^{3/2}}\,\mathrm{d}t = \int^\infty t^{-1}\,\mathrm{d}t = \infty, \\ \int^\infty \frac{1-1/t+1/t}{(t^{2/3})^2}\,\mathrm{d}t = \int^\infty t^{-4/3}\,\mathrm{d}t < \infty.\end{gathered}$$

Therefore, from Theorem 2.2 (ii), every solution of (2.24) oscillates.

Corollary 2.2. *Assume that* $c(t) = t^{-\theta(t)}$, *where* $\theta(t) \in C((0,\infty);\mathbb{R})$, *and that*

$$\limsup_{t\to\infty} \theta(t) < \liminf_{t\to\infty} q(t).$$

If one of the following cases holds:

(i) $p(t)$ *is increasing,* $1 < p(t) < q(t)$ *or* $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} q(t)$;

(ii) $q(t)$ *is increasing,* $1 < q(t) < p(t)$,

then every solution $u(t)$ *of* (E) *is oscillatory.*

Example 2.2. Consider the equation

$$(|u'(t)|^{1-2^{-\log t}} u'(t))' + t^{-1/2+\sin t} |u(t)|^{3+\sin t} u(t) = 0, \quad t \geqslant 1, \tag{2.25}$$

where

$$p(t) = 3 - 2^{-\log t}, \quad q(t) = 5 + \sin t, \quad c(t) = t^{-1/2+\sin t}.$$

We choose $\varphi(t) = t^{1/2}$ to find that

$$\int^{\infty} \left\{ \frac{(\frac{1}{2}t^{-1/2})^{3/2}}{(2 - 2^{-\log t} + 2^{-\log t})t^{1/2}} \right\}^2 \mathrm{d}t = \int^{\infty} \left(\frac{1}{2}\right)^5 t^{-5/2} \,\mathrm{d}t < \infty,$$

$$\int^{\infty} \frac{(\frac{1}{2}t^{-1/2})^2}{(2 - 2^{-\log t} + 2^{-\log t})t^{1/2}} \,\mathrm{d}t = \int^{\infty} \left(\frac{1}{2}\right)^3 t^{-3/2} \,\mathrm{d}t < \infty,$$

$$\int^{\infty} t^{1/2} t^{-1/2+\sin t} \,\mathrm{d}t = \int^{\infty} t^{\sin t} \,\mathrm{d}t = \infty.$$

Furthermore, it is easy to check that

$$\limsup_{t\to\infty} \theta(t) = \frac{3}{2} < 4 = \liminf_{t\to\infty} q(t).$$

Since all conditions of Theorem 2.2 (i) and Corollary 2.2 (i) are satisfied, every solution of (2.25) oscillates.

Acknowledgement. The author sincerely thanks Professor Norio Yoshida for continuing interest and encouragement, without him this study could not have been carried out. Also, the author would like to thank referee for helpful comments and suggestions.

References

[1] *L. Diening, P. Harjulehto, P. Hästö, M. Růžička*: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Berlin, 2011. zbl MR

[2] *P. Harjulehto, P. Hästö, Ú. V. Lê, M. Nuortio*: Overview of differential equations with non-standard growth. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *72* (2010), 4551–4574. zbl MR

[3] *P. A. Hästö*: The $p(x)$-Laplacian and applications. J. Anal. *15* (2007), 53–62. zbl MR

[4] *Y. Şahiner, A. Zafer*: Oscillation of nonlinear elliptic inequalities with $p(x)$-Laplacian. Complex Var. Elliptic Equ. *58* (2013), 537–546. zbl MR

[5] *Y. Şahiner, A. Zafer*: Oscillation of $p(x)$-Laplacian elliptic inequalities with mixed variable exponents. Math. Inequal. Appl. *16* (2013), 947–961. zbl MR

[6] *H. Usami*: Some oscillation theorems for a class of quasilinear elliptic equations. Ann. Mat. Pura Appl. (4) *175* (1998), 277–283. zbl MR

[7] *N. Yoshida*: Picone-type inequality and Sturmian comparison theorems for quasilinear elliptic operators with $p(x)$-Laplacians. Electron. J. Differ. Equ. (electronic only) *2012* (2012), Article No. 01, 9 pages. zbl MR

[8] *N. Yoshida*: Oscillation criteria for half-linear elliptic inequalities with $p(x)$-Laplacians via Riccati method. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *74* (2011), 2563–2575. zbl MR

[9] *Q. Zhang*: Oscillatory property of solutions for $p(t)$-Laplacian equations. J. Inequal. Appl. (electronic only) *2007* (2007), Article No. 58548, 8 pages. zbl MR

Author's address: *Yutaka Shoukaku*, Institute of Science and Engineering, Kanazawa University, Kakuma-machi, Kanazawa 920-1192, Ishikawa, Japan, e-mail: `shoukaku@se.kanazawa-u.ac.jp`.