C^1 SELF-MAPS ON CLOSED MANIFOLDS WITH FINITELY MANY PERIODIC POINTS ALL OF THEM HYPERBOLIC

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Abstract. Let X be a connected closed manifold and f a self-map on X. We say that f is almost quasi-unipotent if every eigenvalue λ of the map f_{*k} (the induced map on the k-th homology group of X) which is neither a root of unity, nor a zero, satisfies that the sum of the multiplicities of λ as eigenvalue of all the maps f_{*k} with k odd is equal to the sum of the multiplicities of λ as eigenvalue of all the maps f_{*k} with k even.

We prove that if f is C^1 having finitely many periodic points all of them hyperbolic, then f is almost quasi-unipotent.

Keywords: hyperbolic periodic point; differentiable map; Lefschetz number; Lefschetz zeta function; quasi-unipotent map; almost quasi-unipotent map

MSC 2010: 37C05, 37C25, 37C30

1. Introduction and statement of the main result

Let X be a topological space and $f\colon X \to X$ a continuous map on X. We say that $x \in X$ is a *periodic point of period* p if $f^p(x) = x$ and $f^j(x) \neq x$ for $1 \leqslant j \leqslant p-1$.

Let X be a differentiable manifold and f a differentiable map. We say that a periodic point of period p is *hyperbolic* if the derivative of f^p at x, i.e. $Df^p_x\colon TX_x \to TX_x$, has no eigenvalues of modulus equal to 1.

If the dimension of X is n, the map f induces a homomorphism on the k-th rational homology group of X for $0 \leqslant k \leqslant n$, i.e. $f_{*k}\colon H_k(X,\mathbb{Q}) \to H_k(X,\mathbb{Q})$. Here

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$H_k(X, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$ and f_{*k} is a linear map whose matrix has integer entries.

A linear transformation is called *quasi-unipotent* if its eigenvalues are roots of unity. We say that a continuous map $f\colon X \to X$ is *quasi-unipotent* if the maps f_{*k} are quasi-unipotent for $0 \leqslant k \leqslant n$, where n is the dimension of the manifold X. We say that a continuous map $f\colon X \to X$ is *almost quasi-unipotent* if every eigenvalue λ of a map f_{*k} which is neither a root of unity nor a zero, satisfies that the sum of the multiplicities of λ as an eigenvalue of all the maps f_{*k} with k odd is equal to the sum of the multiplicities of λ as an eigenvalue of all the maps f_{*k} with k even. Clearly the quasi-unipotent maps are almost quasi-unipotent. In Section 3, we show an example of an almost quasi-unipotent map which is not quasi-unipotent.

We say that a manifold is *closed* if it is compact and without boundary.

Theorem 1.1. *Let X be a connected closed manifold and f a C^1 self-map on X with finitely many periodic points all of them hyperbolic. Then f is almost quasi-unipotent.*

The reciprocal of Theorem 1.1 is false. Consider the classical construction of Smale's horseshoe (cf. [12]); there is a diffeomorphism $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that it has infinitely many periodic points all of them hyperbolic and with all possible periods. However, the map f is quasi-unipotent. There are maps on the n-dimensional torus which are minimal (all orbits are dense) and quasi-unipotent (cf. [4]).

In [8] sufficient conditions are given for the existence of almost quasi-unipotent maps on various closed manifolds having infinitely many periodic points all of them hyperbolic. We list some of these results in Section 3.

We note that Theorem 1.1 can be extended to manifolds with boundary which have no periodic points on the boundary.

We remark that Theorem 1.1 allows to weaken the hypothesis of the results of [2], [7], [9], [10]. In these articles the periodic orbits of Morse-Smale diffeomorphisms on n-dimensional torus, orientable and non-orientable surfaces are studied. So Theorem 1.1 allows to extend those results to C^1 maps having finitely many periodic points all of them hyperbolic. Clearly the Morse-Smale diffeomorphisms satisfy this last condition (cf. [11]).

In Theorem 2.1 we give a characterization of almost quasi-unipotent maps in terms of the Lefschetz zeta function. We show that a C^1 map on a closed manifold is almost quasi-unipotent if and only if the zeros and poles of its Lefschetz zeta function are roots of unity, or the Lefschetz zeta function is equal to 1.

2. Definitions and proof of Theorem 1.1

The *Lefschetz number* of f is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \operatorname{trace}(f_{*k}).$$

The Lefschetz Fixed Point Theorem states that if $L(f) \neq 0$ then f has a fixed point (cf. [3]).

The *Lefschetz zeta function* of f is defined as

$$\zeta_f(t) = \exp\left(\sum_{m\geqslant 1} \frac{L(f^m)}{m} t^m\right).$$

Since $\zeta_f(t)$ is the generating function of the Lefschetz numbers, $L(f^m)$, it keeps the information of the Lefschetz number for all iterates of f. There is an alternative way to compute it:

$$\zeta_f(t) = \prod_{k=0}^{n} \det(\mathrm{Id}_k - t f_{*k})^{(-1)^{k+1}}, \tag{2.1}$$

where $n = \dim X$, $m_k = \dim H_k(X, \mathbb{Q})$, Id_k is the identity map on $H_k(X, \mathbb{Q})$, and by convention $\det(\mathrm{Id}_k - t f_{*k}) = 1$ if $m_k = 0$ (cf. [5]).

Let X be a connected closed manifold and $f\colon X \to X$ a C^1 map. We say that f is of *Franks type* if its Lefschetz zeta function $\zeta_f(t)$ is of the form $\prod_{i=1}^{m} (1 - \Delta_i t^{r_i})^{(-1)^{s_i}}$ for some positive integer m, where $\Delta_i = \pm 1$, r_i and s_i are positive integers. Note that $\zeta_f(t) = 1$ is a possible Lefschetz zeta function for a Franks type map.

We remark that f is Franks type if and only if the zeros and poles of $\zeta_f(t)$ are roots of unity or $\zeta_f(t) = 1$.

Theorem 2.1. *Let X be a connected closed manifold and $f\colon X \to X$ a C^1 map. The map f is of Franks type if and only if f is almost quasi-unipotent.*

Proof. Let $\zeta_f(t)$ be the Lefschetz zeta function of f; according to (2.1) it is of the form

$$(1-t)^{-1} p_1(t) p_2(t)^{-1} \dots p_n(t)^{(-1)^{n+1}}, \tag{2.2}$$

where $p_k(t) = \det(\mathrm{Id}_k - t f_{*k})$. Let m_k be the dimension of $H_k(X, \mathbb{Q})$. Then $p_k(t) = t^{m_k} q_k(1/t)$, or equivalently $p_k(1/t) = t^{-m_k} q_k(t)$, where $q_k(t) = \det(t \, \mathrm{Id}_k - f_{*k})$, i.e. the characteristic polynomial of f_{*k}. So, if all eigenvalues of f_{*k} are zero, then $p_k(t) = 1$.

First we shall prove that if f is almost quasi-unipotent, then f is of Franks type. We separate the proof into three cases.

Case 1: Assume that all eigenvalues of all the maps f_{*k} are roots of unity, i.e. f is quasi-unipotent. Then all roots of the polynomials $q_i(t)$ are roots of unity for all i, so the roots of the polynomials $p_i(t)$ are also roots of unity. Hence the zeros and poles of $\zeta_f(t)$ are roots of unity, or $\zeta_f(t) = 1$. Then f is of Franks type.

Case 2: Assume that all eigenvalues of all the maps f_{*k} are roots of unity or zero. In general the characteristic polynomial of f_{*k} can be written as $q_k(t) = t^{l_k} r_k(t)$, where $r_k(t) = \prod_{j=1}^{s_k} (t - \lambda_j)$, with λ_j the nonzero eigenvalues of f_{*k}, l_k is the dimension of the kernel of f_{*k} and $s_k = m_k - l_k$. Due to this $p_k(1/t) = t^{-s_k} r_k(t)$. The coefficients of the polynomial $r_k(t)$ are integers, since $q_k(t)$ has integers coefficients.

If all eigenvalues of f_{*k} are zero, then $p_k(t) = 1$. If some eigenvalue f_{*k} is not zero, then the roots of $r_k(t)$ are roots of unity. Since $p_k(1/t) = t^{-s_k} r_k(t)$ and the degree of $r_k(t)$ is s_k, we have $p_k(1/t) = r_k(1/t)$. Therefore, zero is not a root of $p_k(t)$, and all roots of $p_k(t)$ are roots of unity. Hence the zeros and poles of $\zeta_f(t)$ are roots of unity, or $\zeta_f(t) = 1$. Again f is of Franks type.

Case 3: Assume that for some f_{*k} there is an eigenvalue λ different from a root of unity and from zero. Then λ^{-1} is a root of the polynomial $p_k(t)$. By the definition of being almost quasi-unipotent, the multiplicity of λ^{-1} as a root of the polynomial $p_1(t)p_3(t)\dots p_l(t)$, if l is the largest odd positive integer less than or equal to $\dim X = n$, is equal to the multiplicity of λ^{-1} as a root of the polynomial $p_2(t)p_4(t)\dots p_m(t)$, if m is the largest even positive integer less than or equal to $\dim X$. This implies that the factor $t - \lambda^{-1}$ cancels in the expression (2.2). So, by the arguments of the proof of Case 2 the unique poles or zeros of $\zeta_f(t)$ are the roots of unity, or $\zeta_f(t) = 1$. Hence f is of Franks type.

Finally we prove that if f is of Franks type, then f is almost quasi-unipotent. Assume that f is not almost quasi-unipotent, then there exists an eigenvalue λ of some map f_{*k}, different from zero, from a root of unity and such that the multiplicity of λ^{-1} as a root of the polynomial $p_1(t)p_3(t)\dots p_l(t)$ if l is the largest odd positive integer less than or equal to $\dim X$ is different from the multiplicity of λ^{-1} as a root of the polynomial $p_2(t)p_4(t)\dots p_m(t)$ if m is the largest even positive integer less than or equal to $\dim X$. Therefore the factor $t - \lambda^{-1}$ does not cancel in the expression (2.2). So λ^{-1} is a zero or a pole of $\zeta_f(t)$. Hence $\zeta_f(t)$ has a zero or a pole, which is not a root of unity. Consequently f is not of Franks type. This completes the proof of the theorem. □

Let M be a C^1 compact manifold and let $f\colon M \to M$ be a C^1 map. Let x be a hyperbolic periodic point of period p of f and E_x^u its unstable linear space, i.e. the subspace of the tangent space T_xM generated by the eigenvalues of $Df^p(x)$ of norm larger than 1. Let γ be the orbit of x, the *index* u of γ is the dimension of E_x^u. We define the orientation type Δ of γ as $+1$ if $Df^p(x)\colon E_x^u \to E_x^u$ preserves orientation and -1 if reverses the orientation. The collection of the triples (p, u, Δ) belonging to all the periodic orbits of f is called the *periodic data* of f. The same triple can appear more than once if it corresponds to different periodic orbits.

Theorem 2.2 (Franks [6]). *Let f be a C^1 map on a closed manifold having finitely many periodic points all of them hyperbolic, and let Σ be the periodic data of f. Then the Lefschetz zeta function $\zeta_f(t)$ of f satisfies*

$$\zeta_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}. \tag{2.3}$$

P r o o f of Theorem 1.1. Let X be a connected closed manifold and $f\colon X \to X$ a C^1 map having finitely many periodic points all of them hyperbolic. Due to Theorem 2.2 it is of Franks type. By Theorem 2.1 the map f is almost quasi-unipotent. □

3. Remarks

(i) An example of an almost quasi-unipotent map which is not quasi-unipotent, can be obtained as follows:

The linear map $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$A = \begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where m is an integer different from -1, 0 and 1, covers a unique algebraic endomorphism $f = f_A\colon \mathbb{T}^3 \to \mathbb{T}^3$ such that the action which it induces on its first homology map is $f_{*1} = A$, for more details see for instance [1]. So f_{*1} has as eigenvalues m, 1 and -1. Since the homology groups of the torus form an exterior algebra (cf. [13]), we have that the eigenvalues of f_{*2} are m, $-m$ and -1, and the eigenvalue of f_{*3} is $-m$. Therefore the map $f_A\colon \mathbb{T}^3 \to \mathbb{T}^3$ is not quasi-unipotent since m is an eigenvalue, but it is almost quasi-unipotent because the eigenvalues m and $-m$ satisfy the definition.

Since the corresponding characteristic polynomials of f_{*1}, f_{*2} and f_{*3} are $q_1(t) = (t-m)(t-1)(t+1)$, $q_2(t) = (t-m)(t+m)(t+1)$ and $q_3(t) = t+m$, according to (2.2) the Lefschetz zeta function of f_A is

$$\zeta_{f_A}(t) = \frac{t^3 q_1(1/t) t q_3(1/t)}{(1-t)t^3 q_2(1/t)} = 1.$$

(ii) The authors established in [8] sufficient conditions for quasi-unipotent maps to have infinitely many periodic points. In the following lines we list some of these conditions.

Theorem 3.1 ([8]). *Let $f\colon X \to X$ be a C^1 map with all its periodic points hyperbolic. Let f have at most one periodic orbit of even index with period a power of 2 different from 1 and*

$$\zeta_f(t) = \frac{p(t)}{(1-t)^m},$$

where $m \geqslant 2$ and $p(t)$ is a polynomial that can have one of the following forms:

(a) $p(t) = 1$,

(b) $p(t) = \prod_{i=1}^{l_1} (1 \pm t^{n_i})$, *where the n_i's are odd integers greater than 2,*

(c) $p(t) = \prod_{j=1}^{l_2} (1 + t^{2^{k_j}})$, *where the k_j's are positive integers,*

(d)

$$p(t) = \left(\prod_{i=1}^{l_1}(1 \pm t^{n_i})\right)\left(\prod_{j=1}^{l_2}(1 + t^{2^{k_j}})\right),$$

where the k_j's are positive integers and the n_i's are odd integers greater than 2.

Then f has infinitely many periodic points.

Theorem 3.2 ([8]). *Let $f\colon X \to X$ be a C^1 map with all its periodic points hyperbolic. Let f have neither periodic points of even index with period 2, nor fixed points and let*

$$\zeta_f(t) = \frac{p(t)}{(1-t)^{m_1}},$$

where $p(t)$ is a polynomial that can have one of the following forms:

(a) $p(t) = 1$,

(b) $p(t) = \prod_{i=1}^{l_1} (1 \pm t^{n_i})$, *where the n_i's are odd integers greater than 2,*

(c) $p(t) = \prod_{j=1}^{l_2} (1 + t^{2^{k_j}})$, *where the k_j's are positive integers,*

(d)

$$p(t) = \left(\prod_{i=1}^{l_1}(1 \pm t^{n_i})\right)\left(\prod_{j=1}^{l_2}(1 + t^{2^{k_j}})\right),$$

where the k_j's are positive integers and the n_i's are odd integers greater than 2.

Then f has infinitely many periodic points.

Theorem 3.3 ([8])**.** *Let $f\colon X \to X$ be a C^1 map with all its periodic points hyperbolic such that $\zeta_f(t) = 1$.*

(a) *If f has a periodic point with an odd period p, with index u and has no periodic points of periods a multiple of p with index $v \not\equiv u \pmod 2$, then f has infinitely many periodic points.*

(b) *Assume that f has periodic points of period a power of 2 whose indexes have the same parity. Let u be this parity. If f has no fixed points of index v with $v \not\equiv u \pmod 2$, then f has infinitely many periodic points.*

References

[1] *L. Alsedà, S. Baldwin, J. Llibre, R. Swanson, W. Szlenk*: Minimal sets of periods for torus maps via Nielsen numbers. Pac. J. Math. *169* (1995), 1–32. zbl MR

[2] *P. Berrizbeitia, V. F. Sirvent*: On the Lefschetz zeta function for quasi-unipotent maps on the n-dimensional torus. J. Difference Equ. Appl. *20* (2014), 961–972. zbl MR

[3] *R. F. Brown*: The Lefschetz Fixed Point Theorem. Scott, Foresman, London, 1971. zbl MR

[4] *N. M. dos Santos, R. Urzúa-Luz*: Minimal homeomorphisms on low-dimensional tori. Ergodic Theory Dyn. Syst. *29* (2009), 1515–1528. zbl MR

[5] *J. M. Franks*: Homology and Dynamical Systems. CBMS Regional Conference Series in Mathematics 49, American Mathematical Society, Providence, 1982. zbl MR

[6] *J. M. Franks*: Some smooth maps with infinitely many hyperbolic periodic points. Trans. Am. Math. Soc. *226* (1977), 175–179. zbl MR

[7] *J. L. García Guirao, J. Llibre*: Minimal Lefschetz sets of periods for Morse-Smale diffeomorphisms on the n-dimensional torus. J. Difference Equ. Appl. *16* (2010), 689–703. zbl MR

[8] *J. Llibre, V. F. Sirvent*: C^1 self-maps on closed manifolds with all their points hyperbolic. Houston J. Math *41* (2015), 1119–1127.

[9] *J. Llibre, V. F. Sirvent*: Minimal sets of periods for Morse-Smale diffeomorphisms on non-orientable compact surfaces without boundary. J. Difference Equ. Appl. *19* (2013), 402–417. zbl MR

[10] *J. Llibre, V. F. Sirvent*: Minimal sets of periods for Morse-Smale diffeomorphisms on orientable compact surfaces. Houston J. Math. *35* (2009), 835–855; erratum ibid. *36* (2010), 335–336. zbl MR

[11] *M. Shub, D. Sullivan*: Homology theory and dynamical systems. Topology *14* (1975), 109–132. zbl MR

[12] *S. Smale*: Differentiable dynamical systems. With an appendix to the first part of the paper: "Anosov diffeomorphisms" by J. Mather, Bull. Am. Math. Soc. *73* (1967), 747–817. zbl MR

[13] *J. W. Vick*: Homology Theory. An Introduction to Algebraic Topology. Graduate Texts in Mathematics 145, Springer, New York, 1994. zbl MR

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