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A-BROWDER-TYPE THEOREMS FOR DIRECT SUMS OF OPERATORS

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Abstract. We study the stability of a-Browder-type theorems for orthogonal direct sums of operators. We give counterexamples which show that in general the properties (SBaw), (SBab), (SBw) and (SBb) are not preserved under direct sums of operators.

However, we prove that if S and T are bounded linear operators acting on Banach spaces and having the property (SBab), then $S \oplus T$ has the property (SBab) if and only if $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$, where $\sigma_{\mathrm{SBF}_+^-}(T)$ is the upper semi-B-Weyl spectrum of T.

We obtain analogous preservation results for the properties (SBaw), (SBb) and (SBw) with extra assumptions.

Keywords: property (SBaw); property (SBab); upper semi-B-Weyl spectrum; direct sum

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1. Introduction

Several authors have been concerned with the study of Weyl-type properties and theorems (generalized or not) for operator matrices, see for example [6], [10], [12], [13], [16]. In the present work we focus on the problem of giving conditions on the direct summands to ensure that the variants of a-Browder-type theorems (defined and studied very recently in [7]) hold for the direct sum, and the paper is organized as follows. In the second part, we give counterexamples which show that generally the properties (SBaw) and (SBab) are not preserved under direct sum. Moreover, in the case of a-isoloid operators, we characterize the stability of property (SBaw) under direct sum via the union of upper semi-B-Weyl spectra of its components, and we obtain an analogous preservation result for property (SBab). In the third part,

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we characterize the stability of property (SBw) under direct sum via the union of upper semi-B-Weyl spectra of its summands, and under the assumption of equality of their point spectrum. Moreover, and under an extra assumption, we obtain a similar preservation result for property (SBb).

Preliminarily we give some definitions that will be needed later. Let X and Y be Banach spaces, let $L(X,Y)$ denote the set of bounded linear operators from X to Y, and abbreviate the Banach algebra $L(X,X)$ to $L(X)$. For $T \in L(X)$ we will denote by $\mathcal{N}(T)$ the null space of T, by $\mathcal{R}(T)$ the range of T, by $n(T)$ the nullity of T and by $d(T)$ its defect. We will also denote by $\sigma(T)$ the spectrum of T, by $\sigma_{\rm a}(T)$ the approximate point spectrum of T, by $\sigma_{\rm p}^0(T)$ the set of all eigenvalues of T of finite multiplicity. An operator $T \in L(X)$ is called an *upper semi-Fredholm* if $\mathcal{R}(T)$ is closed and $n(T) < \infty$, and is called *lower semi-Fredholm* if $\mathcal{R}(T)$ is closed and $d(T) < \infty$. If $T \in L(X)$ is either upper or lower semi Fredholm, then T is called a *semi-Fredholm* operator, and the *index* of T is defined by $\operatorname{ind}(T) = n(T) - d(T)$. If both $n(T)$ and $d(T)$ are finite, then T is called a *Fredholm* operator. For $T \in L(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_{[0]} = T$).

If for some integer n the range space $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is an upper or a lower semi-Fredholm operator, then T is called *upper* or a *lower semi-B-Fredholm* operator, respectively, see [9]. In this case, $\mathcal{R}(T^m)$ is closed, $T_{[m]}$ is a semi-Fredholm operator and $\operatorname{ind}(T_{[m]}) = \operatorname{ind}(T_{[n]})$ for each $m \geqslant n$. This enables us to define the index of the semi-B-Fredholm T as the index of the semi-Fredholm operator $T_{[n]}$, see [3], [9]. Let $\mathrm{SF}_+(X)$ denotes the class of all upper semi-Fredholm operators and let $\mathrm{SF}_+^-(X) = \{T \in \mathrm{SF}_+(X)\colon \operatorname{ind}(T) \leqslant 0\}$. The *upper semi-Weyl spectrum* $\sigma_{\mathrm{SF}_+^-}(T)$ of T is defined by $\sigma_{\mathrm{SF}_+^-}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \notin \mathrm{SF}_+^-(X)\}$. Similarly we define the *upper semi-B-Weyl spectrum* $\sigma_{\mathrm{SBF}_+^-}(T)$ of T.

The *ascent* $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N}\colon \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, and the *descent* $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N}\colon \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. According to [14], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has a finite ascent and finite descent, and in this case they are equal. According to [8], a complex number $\lambda \in \sigma_{\rm a}(T)$ is a *left pole* of T if $a(T - \lambda I) < \infty$ and $\mathcal{R}(T^{a(T-\lambda I)+1})$ is closed.

An operator $T \in L(X)$ is called *upper semi-Browder* if it is an upper semi-Fredholm operator of finite ascent, and is called *Browder* if it is Fredholm of finite ascent and descent. The *upper semi-Browder spectrum* $\sigma_{\rm ub}(T)$ of T is defined by $\sigma_{\rm ub}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I$ is not upper semi-Browder$\}$, and the *Browder spectrum* $\sigma_{\rm b}(T)$ of T is defined by $\sigma_{\rm b}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I$ is not Browder$\}$.

An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f\colon \mathcal{U} \to X$ which satisfies the equation $(T-\lambda I)f(\lambda)=0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$ (see [15] for more details about this property).

Hereafter, the symbol $\sqcup$ stands for the disjoint union, while $\operatorname{iso} A$ means the set of isolated points of a given subset A of $\mathbb{C}$.

Definition 1.1 ([10]). Let $S \in L(X)$ and $T \in L(Y)$. We will say that S and T are of *jointly stable sign index* if for each $\lambda \in \varrho_{\rm SBF}(T)$ and $\mu \in \varrho_{\rm SBF}(S)$, $\operatorname{ind}(T-\lambda I)$ and $\operatorname{ind}(S-\mu I)$ have the same sign, where $\varrho_{\rm SBF}(T)=\mathbb{C}\setminus\sigma_{\rm SBF}(T)$ and $\sigma_{\rm SBF}(T)=\{\lambda\in\mathbb{C}\colon\ T-\lambda I$ is not semi-B-Fredholm$\}$.

For example, from [4], Proposition 2.3, two hyponormal operators T and S acting on a Hilbert space are of jointly stable sign index, since $\operatorname{ind}(S-\lambda I)\leqslant 0$ and $\operatorname{ind}(T-\mu I)\leqslant 0$ for every $\lambda\in\varrho_{\rm SBF}(S)$ and $\mu\in\varrho_{\rm SBF}(T)$. Recall that $T\in L(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is said to be *hyponormal* if $T^*T-TT^*\geqslant 0$ (or equivalently $\|T^*x\|\leqslant\|Tx\|$) for all $x\in\mathcal{H}$. The class of hyponormal operators includes also *subnormal* operators and *quasinormal* operators, see [11].

The inclusion of the following list which contains all symbols and notation we will use and the meaning of the properties we will study in this paper, is motivated by giving the reader an overview of the subject.

- ▷ $\sigma_{{\rm SF}_+^-}(T)$: upper semi-Weyl spectrum of T,
- ▷ $\sigma_{{\rm SBF}_+^-}(T)$: upper semi-B-Weyl spectrum of T,
- ▷ $\sigma_{\rm b}(T)$: Browder spectrum of T,
- ▷ $\sigma_{\rm ub}(T)$: upper semi-Browder spectrum of T,
- ▷ $\Pi^0(T)$: poles of T of finite rank,
- ▷ $\Pi^0_{\rm a}(T)$: left poles of T of finite rank,
- ▷ $\Pi_{\rm a}(T)$: left poles of T,
- ▷ $E^0(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma(T)$,
- ▷ $E^0_{\rm a}(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma_{\rm a}(T)$,
- ▷ $E_{\rm a}(T)$: eigenvalues of T that are isolated in $\sigma_{\rm a}(T)$,
- ▷ $\sigma_{\rm a}(T)=\sigma_{{\rm SBF}_+^-}(T)\sqcup E^0(T) \Leftrightarrow$ property (SBw) holds for T,
- ▷ $\sigma_{\rm a}(T)=\sigma_{{\rm SBF}_+^-}(T)\sqcup \Pi^0(T) \Leftrightarrow$ property (SBb) holds for T,
- ▷ $\sigma_{\rm a}(T)=\sigma_{{\rm SBF}_+^-}(T)\sqcup E^0_{\rm a}(T) \Leftrightarrow$ property (SBaw) holds for T,
- ▷ $\sigma_{\rm a}(T)=\sigma_{{\rm SBF}_+^-}(T)\sqcup \Pi^0_{\rm a}(T) \Leftrightarrow$ property (SBab) holds for T,
- ▷ $\sigma_{\rm a}(T)=\sigma_{{\rm SBF}_+^-}(T)\sqcup \Pi_{\rm a}(T) \Leftrightarrow$ generalized a-Browder's theorem holds for T.

2. Properties (SBaw) and (SBab) for direct sums of operators

We start this part by establishing the following lemma to be used in the proof of the main results in this paper.

Lemma 2.1 ([6], [10]). *Let $S \in L(X)$ and $T \in L(Y)$. Then*

(i) $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) \subseteq \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$. *Moreover, if S and T are of jointly stable sign index, then $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$.*

(ii) *If $S \oplus T$ satisfies the generalized a-Browder's theorem then $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$.*

Example 2.2. Let R be the unilateral right shift operator defined on $l^2(\mathbb{N})$ and L its adjoint, then property (SBaw) holds for both R and L since $\sigma_{\mathrm{a}}(R) = \sigma_{\mathrm{SBF}_+^-}(R) \sqcup E_{\mathrm{a}}^0(R) = C(0,1)$, where $C(0,1)$ is the unit circle of $\mathbb{C}$, $\sigma_{\mathrm{a}}(L) = \sigma_{\mathrm{SBF}_+^-}(L) \sqcup E_{\mathrm{a}}^0(L) = D(0,1)$, where $D(0,1)$ is the closed unit disc in $\mathbb{C}$. However, the property (SBaw) does not hold for $R \oplus L$, in fact $\sigma_{\mathrm{a}}(R \oplus L) = D(0,1)$, $\sigma_{\mathrm{SBF}_+^-}(R \oplus L) = C(0,1)$ and $E_{\mathrm{a}}^0(R \oplus L) = \emptyset$. Note that the inclusion $\sigma_{\mathrm{SBF}_+^-}(R \oplus L) \subset \sigma_{\mathrm{SBF}_+^-}(R) \cup \sigma_{\mathrm{SBF}_+^-}(L)$ is proper, since $\sigma_{\mathrm{SBF}_+^-}(R \oplus L) = C(0,1)$ and $\sigma_{\mathrm{SBF}_+^-}(R) \cup \sigma_{\mathrm{SBF}_+^-}(L) = D(0,1)$. Observe also that R and L are a-isoloid.

Nonetheless, we give in the following result a characterization of the stability of property (SBaw) under the direct sum. Before that we recall that $T \in L(X)$ is said to be a-isoloid if all isolated point in the approximate point spectrum is an eigenvalue of T.

Theorem 2.3. *Let $S \in L(X)$ and $T \in L(Y)$. If S and T have property* (SBaw) *and are a-isoloid, then the following assertions are equivalent:*

(i) *$S \oplus T$ has property* (SBaw);

(ii) $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$.

Proof. (i) $\Longrightarrow$ (ii) The property (SBaw) for $S \oplus T$ implies the statement (ii) with no other restriction on either S or T. To show this, from the diagram presented in [7], $S \oplus T$ satisfies the generalized a-Browder's theorem, and hence by Lemma 2.1, $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$.

(ii) $\Longrightarrow$ (i) Suppose that $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$. Since S and T are a-isoloid and since $\sigma_{\mathrm{p}}^0(S \oplus T) = \{\lambda \in \sigma_{\mathrm{p}}^0(S) \cup \sigma_{\mathrm{p}}^0(T) \colon n(S - \lambda I) + n(T - \lambda I) < \infty\}$, we have

$$\begin{aligned} E_{\mathrm{a}}^0(S \oplus T) &= \operatorname{iso} \sigma_{\mathrm{a}}(S \oplus T) \cap \sigma_{\mathrm{p}}^0(S \oplus T) = \operatorname{iso}[\sigma_{\mathrm{a}}(S) \cup \sigma_{\mathrm{a}}(T)] \cap [\sigma_{\mathrm{p}}^0(S) \cup \sigma_{\mathrm{p}}^0(T)] \\ &= [E_{\mathrm{a}}^0(S) \cap \varrho_{\mathrm{a}}(T)] \cup [E_{\mathrm{a}}^0(T) \cap \varrho_{\mathrm{a}}(S)] \cup [E_{\mathrm{a}}^0(S) \cap E_{\mathrm{a}}^0(T)], \end{aligned}$$

where $\varrho_{\rm a}(\cdot) = \mathbb{C} \setminus \sigma_{\rm a}(\cdot)$. As both S and T have property (SBaw), we have

$$\begin{aligned}\sigma_{\rm a}(S \oplus T) \setminus \sigma_{\rm SBF_+^-}(S \oplus T) &= [\sigma_{\rm a}(S) \cup \sigma_{\rm a}(T)] \setminus [\sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T)] \\ &= [E_{\rm a}^0(S) \setminus \sigma_{\rm a}(T)] \cup [E_{\rm a}^0(T) \setminus \sigma_{\rm a}(S)] \cup [E_{\rm a}^0(S) \cap E_{\rm a}^0(T)].\end{aligned}$$

Hence $\sigma_{\rm a}(S \oplus T) = \sigma_{\rm SBF_+^-}(S \oplus T) \sqcup E_{\rm a}^0(S \oplus T)$ and so property (SBaw) holds by $S \oplus T$. □

R e m a r k 2.4. The assumption "S and T are a-isoloid" is essential in Theorem 2.3. For example, let $T \in L(l^2(\mathbb{N}))$ and let $S \in L(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$ be defined as

$$T(x_1, x_2, x_3, \dots) = \Big(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\Big) \quad \text{and} \quad S = R \oplus U,$$

where $U \in L(l^2(\mathbb{N}))$ is defined by $U(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$, and R is the unilateral right shift. Then property (SBaw) holds for T because $\sigma_{\rm a}(T) = \sigma_{\rm SBF_+^-}(T) \sqcup E_{\rm a}^0(T) = \{0\}$. The property (SBaw) holds also for S because $\sigma_{\rm a}(S) = \sigma_{\rm SBF_+^-}(S) \sqcup E_{\rm a}^0(S) = C(0,1) \cup \{0\}$. But it does not hold for $T \oplus S$, since $\sigma_{\rm a}(T \oplus S) = \sigma_{\rm SBF_+^-}(T \oplus S) = C(0,1) \cup \{0\}$ and $E_{\rm a}^0(T \oplus S) = \{0\}$. Here $\sigma_{\rm SBF_+^-}(T \oplus S) = \sigma_{\rm SBF_+^-}(T) \cup \sigma_{\rm SBF_+^-}(S)$, S is a-isoloid and T is not a-isoloid.

Corollary 2.5. *Suppose that $S \in L(X)$ and $T \in L(Y)$ are a-isoloid operators of jointly stable sign index. If S and T have property* (SBaw), *then $S \oplus T$ has property* (SBaw).

P r o o f. Assume that S and T are a-isoloid and have property (SBaw). Since S and T are of jointly stable sign index, from Lemma 2.1 we have $\sigma_{\rm SBF_+^-}(S \oplus T) = \sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T)$. But by Theorem 2.3 this is equivalent to say that property (SBaw) holds for $S \oplus T$. □

E x a m p l e 2.6. On the Banach space $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, let $S = R \oplus U$ be defined as above and let T be defined by $T(x_1, x_2, x_3, \dots) = 0 \oplus (x_2/2, x_3/3, x_4/4, \dots)$. Clearly, T and S are a-isoloid and $\sigma_{\rm SBF_+^-}(S \oplus T) = \sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T) = C(0,1)$. As $\sigma_{\rm a}(T) = \sigma_{\rm SBF_+^-}(T) = \{0\}$ and $E_{\rm a}^0(T) = \emptyset$ we have $\sigma_{\rm a}(T) \setminus \sigma_{\rm SBF_+^-}(T) = E_{\rm a}^0(T)$ and T has property (SBaw). As was already mentioned, S has property (SBaw). Hence by Theorem 2.3, $S \oplus T$ has property (SBaw).

Generally, the property (SBab) is not transmitted from the direct summands to the direct sum. For instance, the unilateral shift operators R and L defined in Example 2.2 have property (SBab), but their direct sum $R \oplus L$ does not have this property because $\Pi_{\rm a}^0(R \oplus L) = \emptyset$ and $\sigma_{\rm a}(R \oplus L) \setminus \sigma_{\rm SBF_+^-}(R \oplus L) \neq \emptyset$. Note that as was already mentioned, $\sigma_{\rm SBF_+^-}(R \oplus L) = C(0,1)$ and $\sigma_{\rm SBF_+^-}(R) \cup \sigma_{\rm SBF_+^-}(L) = D(0,1)$.

However, we characterize in the next result the stability of property (SBab) under direct sum via the union of upper semi-B-Weyl spectra of its components.

Theorem 2.7. *Let $S \in L(X)$ and $T \in L(Y)$. If S and T have property* (SBab), *then the following assertions are equivalent:*

(i) *$S \oplus T$ has property* (SBab);

(ii) $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$.

P r o o f. (i) $\Longrightarrow$ (ii) Property (SBab) for $S \oplus T$ implies from [7], Theorem 2.14, that the generalized a-Browder's theorem holds for $S \oplus T$. From Lemma 2.1, $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$.

(ii) $\Longrightarrow$ (i) Since we know that the upper semi-Browder spectrum of a direct sum is the union of the upper semi-Browder spectra of its components, that is, $\sigma_{\mathrm{ub}}(S \oplus T) = \sigma_{\mathrm{ub}}(S) \cup \sigma_{\mathrm{ub}}(T)$, hence

$$\begin{aligned}\Pi_{\mathrm{a}}^0(S \oplus T) &= \sigma_{\mathrm{a}}(S \oplus T) \setminus \sigma_{\mathrm{ub}}(S \oplus T) = [\sigma_{\mathrm{a}}(S) \cup \sigma_{\mathrm{a}}(T)] \setminus [\sigma_{\mathrm{ub}}(S) \cup \sigma_{\mathrm{ub}}(T)] \\ &= [\Pi_{\mathrm{a}}^0(S) \setminus \sigma_{\mathrm{a}}(T)] \cup [\Pi_{\mathrm{a}}^0(T) \setminus \sigma_{\mathrm{a}}(S)] \cup [\Pi_{\mathrm{a}}^0(S) \cap \Pi_{\mathrm{a}}^0(T)].\end{aligned}$$

As S and T have property (SBab) and $\sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)$, we have

$$\begin{aligned}\sigma_{\mathrm{a}}(S \oplus T) \setminus \sigma_{\mathrm{SBF}_+^-}(S \oplus T) &= [\sigma_{\mathrm{a}}(S) \cup \sigma_{\mathrm{a}}(T)] \setminus [\sigma_{\mathrm{SBF}_+^-}(S) \cup \sigma_{\mathrm{SBF}_+^-}(T)] \\ &= [\Pi_{\mathrm{a}}^0(S) \cap \varrho_{\mathrm{a}}(T)] \cup [\Pi_{\mathrm{a}}^0(T) \cap \varrho_{\mathrm{a}}(S)] \cup [\Pi_{\mathrm{a}}^0(S) \cap \Pi_{\mathrm{a}}^0(T)].\end{aligned}$$

Hence $\sigma_{\mathrm{a}}(S \oplus T) \setminus \sigma_{\mathrm{SBF}_+^-}(S \oplus T) = \Pi_{\mathrm{a}}^0(S \oplus T)$ and this means that $S \oplus T$ has property (SBab). $\square$

From Theorem 2.7 and Lemma 2.1, we have immediately the following corollary:

Corollary 2.8. *If $S \in L(X)$ and $T \in L(Y)$ are of jointly stable sign index and have property* (SBab), *then $S \oplus T$ has property* (SBab).

3. Properties (SBb) and (SBw) for direct sums of operators

In this section we study the preservation of properties (SBb) and (SBw) under orthogonal direct sums. Among other, we show that generally, if $T \in L(X)$ and $S \in L(Y)$ have property (SBb), then it is not guaranteed that their orthogonal direct sum $S \oplus T$ has property (SBb), as we can see in the following example. Moreover, we explore certain sufficient conditions which ensure their preservation under direct sums.

E x a m p l e 3.1. Let $T \in L(\mathbb{C}^n)$ be a quasinilpotent operator and let $R \in L(l^2(\mathbb{N}))$ be the unilateral right shift operator. Then $\sigma_{\rm a}(T) = \{0\}$, $\sigma_{\rm SBF_+^-}(T) = \emptyset$, $\Pi^0(T) = \{0\}$. Thus $\sigma_{\rm a}(T) = \sigma_{\rm SBF_+^-}(T) \sqcup \Pi^0(T)$ and so the property (SBb) holds by T. Moreover, $\sigma_{\rm a}(R) = C(0,1)$, $\sigma_{\rm SBF_+^-}(R) = C(0,1)$, $\Pi^0(R) = \emptyset$. So $\sigma_{\rm a}(R) = \sigma_{\rm SBF_+^-}(R) \sqcup \Pi^0(R)$ and R has property (SBb). But their orthogonal direct sum $T \oplus R$ defined on the Banach space $\mathbb{C}^n \oplus l^2(\mathbb{N})$ does not have property (SBb), because $\sigma_{\rm a}(T \oplus R) = C(0,1) \cup \{0\}$, $\sigma_{\rm SBF_+^-}(T \oplus R) = C(0,1)$ and $\Pi^0(T \oplus R) = \emptyset$, since $\sigma(T \oplus R) = D(0,1)$, the closed unit disc in $\mathbb{C}$ which has no isolated points. We notice here that $\Pi^0(T) \cap \varrho_{\rm a}(R) = \{0\}$ and $\sigma_{\rm SBF_+^-}(T \oplus R) = \sigma_{\rm SBF_+^-}(T) \cup \sigma_{\rm SBF_+^-}(R)$.

However, and under an extra assumption, we characterize in the next theorem the stability of property (SBb) under direct sum.

Theorem 3.2. *Suppose that $S \in L(X)$ and $T \in L(Y)$ are such that $\Pi^0(S) \cap \varrho_{\rm a}(T) = \Pi^0(T) \cap \varrho_{\rm a}(S) = \emptyset$. If both S and T have property* (SBb)*, then the following assertions are equivalent:*

(i) *$S \oplus T$ has property* (SBb)*;*

(ii) $\sigma_{\rm SBF_+^-}(S \oplus T) = \sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T)$.

P r o o f. (ii) $\Longrightarrow$ (i) Since S and T both have property (SBb), we have

$$\begin{aligned}&[\sigma_{\rm a}(S) \cup \sigma_{\rm a}(T)] \setminus [\sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T)]\\ &\qquad = [\Pi^0(S) \cap \varrho_{\rm a}(T)] \cup [\Pi^0(T) \cap \varrho_{\rm a}(S)] \cup [\Pi^0(S) \cap \Pi^0(T)] = \Pi^0(S) \cap \Pi^0(T).\end{aligned}$$

On the other hand, as we know that $\sigma_{\rm b}(S \oplus T) = \sigma_{\rm b}(S) \cup \sigma_{\rm b}(T)$ for any pair of operators, we have

$$\begin{aligned}\Pi^0(S \oplus T) &= \sigma(S \oplus T) \setminus \sigma_{\rm b}(S \oplus T) = [\sigma(S) \cup \sigma(T)] \setminus [\sigma_{\rm b}(S) \cup \sigma_{\rm b}(T)]\\ &= [\Pi^0(S) \cap \varrho(T)] \cup [\Pi^0(T) \cap \varrho(S)] \cup [\Pi^0(S) \cap \Pi^0(T)],\end{aligned}$$

where $\varrho(\cdot) = \mathbb{C} \setminus \sigma(\cdot)$. Since we also have that $\Pi^0(T) \cap \varrho(S) = \emptyset$ and $\Pi^0(S) \cap \varrho(T) = \emptyset$, it follows that $\Pi^0(S \oplus T) = \Pi^0(S) \cap \Pi^0(T)$. Hence

$$\Pi^0(S \oplus T) = [\sigma_{\rm a}(S) \cup \sigma_{\rm a}(T)] \setminus [\sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T)].$$

As by hypothesis $\sigma_{\rm SBF_+^-}(S \oplus T) = \sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T)$, we have $\Pi^0(S \oplus T) = \sigma_{\rm a}(S \oplus T) \setminus \sigma_{\rm SBF_+^-}(S \oplus T)$ and $S \oplus T$ has property (SBb).

(i) $\Longrightarrow$ (ii) If $S \oplus T$ has property (SBb) then from [7], Corollary 2.11, $S \oplus T$ has property (SBab). Consequently, we have the equality $\sigma_{\rm SBF_+^-}(S \oplus T) = \sigma_{\rm SBF_+^-}(S) \cup \sigma_{\rm SBF_+^-}(T)$ as seen in the proof of Theorem 2.7. □

R e m a r k 3.3. Remark that generally, we cannot ensure the transmission of property (SBab) from two operators S and T to the direct sum $S \oplus T$ even if $\Pi^0(S)\cap\varrho_{\rm a}(T) = \Pi^0(T)\cap\varrho_{\rm a}(S) = \emptyset$. Indeed, the shift operators R and L defined in Example 2.2 both have property (SBb), because $\sigma_{\rm a}(R) = \sigma_{{\rm SBF}_+^-}(R) \sqcup \Pi^0(R) = C(0,1)$ and $\sigma_{\rm a}(L) = \sigma_{{\rm SBF}_+^-}(L) \sqcup \Pi^0(L) = D(0,1)$. But this property does not hold by their direct sum, because $\sigma_{{\rm SBF}_+^-}(R\oplus L) \sqcup \Pi^0(R\oplus L) = C(0,1)$ and $\sigma_{\rm a}(R\oplus L) = D(0,1)$. Note that $\Pi^0(R) \cap \varrho_{\rm a}(L) = \Pi^0(L) \cap \varrho_{\rm a}(R) = \emptyset$.

A bounded linear operator $A \in L(X,Y)$ is said to be *quasi-invertible* if it is injective and has dense range. Two bounded linear operators $T \in L(X)$ and $S \in L(Y)$ on complex Banach spaces X and Y are *quasisimilar* provided there exist quasi-invertible operators $A \in L(X,Y)$ and $B \in L(Y,X)$ such that $AT = SA$ and $BS = TB$. For example and according to [2], if $T \in L(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is invertible and p-hyponormal then there exists $S \in L(\mathcal{H})$ log-hyponormal quasisimilar to T. Recall that an operator $T \in L(\mathcal{H})$ is said to be *p-hyponormal*, with $0 < p \leqslant 1$, if $(T^*T)^p \geqslant (TT^*)^p$, and is said to be *log-hyponormal* if T is invertible and satisfies $\log(T^*T) \geqslant \log(TT^*)$.

Corollary 3.4. *If $S \in L(\mathcal{H})$ and $T \in L(\mathcal{H})$ are quasisimilar hyponormal operators and both have property* (SBb)*, then $S \oplus T$ has property* (SBb).

P r o o f. Since S and T are hyponormal then they are of jointly stable sign index, and this implies by Lemma 2.1 that $\sigma_{{\rm SBF}_+^-}(S\oplus T) = \sigma_{{\rm SBF}_+^-}(S) \cup \sigma_{{\rm SBF}_+^-}(T)$. The quasisimilarity of S and T implies by [10], Lemma 2.8, that $\Pi(S) = \Pi(T)$. So $\Pi^0(S) \cap \varrho_{\rm a}(T) = \emptyset$ and $\Pi^0(T) \cap \varrho_{\rm a}(S) = \emptyset$. Hence by Theorem 3.2, $S \oplus T$ has property (SBb). $\square$

In the next theorem, we characterize the stability of property (SBw) under direct sum via the union of upper semi-B-Weyl spectra of its summands, which in turn are supposed to have the same eigenvalues of finite multiplicity.

Theorem 3.5. *Suppose that both $S \in L(X)$ and $T \in L(Y)$ have property* (SBw)*. If $\sigma_{\rm p}^0(S) = \sigma_{\rm p}^0(T)$ then the following assertions are equivalent:*

(i) *$S \oplus T$ has property* (SBw)*;*

(ii) $\sigma_{{\rm SBF}_+^-}(S\oplus T) = \sigma_{{\rm SBF}_+^-}(S) \cup \sigma_{{\rm SBF}_+^-}(T)$.

P r o o f. (ii) $\Longrightarrow$ (i) Suppose that $\sigma_{{\rm SBF}_+^-}(S\oplus T) = \sigma_{{\rm SBF}_+^-}(S) \cup \sigma_{{\rm SBF}_+^-}(T)$. As both S and T have property (SBw), we have

$$\begin{aligned}\sigma_{\rm a}(S\oplus T) \setminus \sigma_{{\rm SBF}_+^-}(S\oplus T) &= [\sigma_{\rm a}(S) \cup \sigma_{\rm a}(T)] \setminus [\sigma_{{\rm SBF}_+^-}(S) \cup \sigma_{{\rm SBF}_+^-}(T)] \\ &= [E^0(T) \cap \varrho_{\rm a}(S)] \cup [E^0(S) \cap \varrho_{\rm a}(T)] \cup [E^0(S) \cap E^0(T)].\end{aligned}$$

Since by hypothesis $\sigma_{\rm p}^0(T)=\sigma_{\rm p}^0(S)$, hence $E^0(T)\cap\varrho_{\rm a}(S)=E^0(S)\cap\varrho_{\rm a}(T)=\emptyset$. Therefore $\sigma_{\rm a}(S\oplus T)\setminus\sigma_{\mathrm{SBF}_+^-}(S\oplus T)=E^0(S)\cap E^0(T)$. On the other hand, we have

$$\begin{aligned}E^0(S\oplus T)=&\ \operatorname{iso}\sigma(S\oplus T)\cap\sigma_{\rm p}^0(S\oplus T)=\operatorname{iso}[\sigma(S)\cup\sigma(T)]\cap[\sigma_{\rm p}^0(S)\cup\sigma_{\rm p}^0(T)]\\ &=[E^0(S)\cap\varrho(T)]\cup[E^0(T)\cap\varrho(S)]\cup[E^0(S)\cap E^0(T)]\\ &=E^0(S)\cap E^0(T),\quad \text{because } E^0(S)\cap\varrho(T)=E^0(T)\cap\varrho(S)=\emptyset.\end{aligned}$$

Hence $\sigma_{\rm a}(S\oplus T)\setminus\sigma_{\mathrm{SBF}_+^-}(S\oplus T)=E^0(S\oplus T)$ and $S\oplus T$ has property (SBw).

(i) $\Longrightarrow$ (ii) If $S\oplus T$ has property (SBw), then by [7], Corollary 2.4, $S\oplus T$ has property (SBb). We conclude that $\sigma_{\mathrm{SBF}_+^-}(S\oplus T)=\sigma_{\mathrm{SBF}_+^-}(S)\cup\sigma_{\mathrm{SBF}_+^-}(T)$ as seen in the proof of Theorem 3.2. □

Example 3.6. In general, we cannot expect that property (SBw) will hold for the direct sum $S\oplus T$ for every two operators S and T having property (SBw). To see this, if we consider the operators T and R defined in Example 3.1, then both T and R have property (SBw) because $\sigma_{\rm a}(T)\setminus\sigma_{\mathrm{SBF}_+^-}(T)=E^0(T)=\{0\}$ and $\sigma_{\rm a}(R)\setminus\sigma_{\mathrm{SBF}_+^-}(R)=E^0(R)=\emptyset$. But $T\oplus R$ does not have property (SBw) because $\sigma_{\rm a}(T\oplus R)\setminus\sigma_{\mathrm{SBF}_+^-}(T\oplus R)=\{0\}\neq E^0(T\oplus R)=\emptyset$. Observe that $\sigma_{\mathrm{SBF}_+^-}(T\oplus R)=\sigma_{\mathrm{SBF}_+^-}(T)\cup\sigma_{\mathrm{SBF}_+^-}(R)=C(0,1)$, but $\sigma_{\rm p}^0(R)=\emptyset\neq\sigma_{\rm p}^0(T)=\{0\}$.

Corollary 3.7. *Suppose that $S\in L(X)$ and $T\in L(Y)$ are quasisimilar and both satisfy property* (SBw). *If S or T has the SVEP, then $S\oplus T$ has property* (SBw).

Proof. The quasisimilarity of S and T implies that $\sigma_{\rm p}^0(S)=\sigma_{\rm p}^0(T)$. It implies also due to [1], Theorem 2.15, that both S and T have SVEP. Thus from [5], Theorem 2.5, we conclude that $\operatorname{ind}(T-\lambda I)\leqslant 0$ and $\operatorname{ind}(S-\mu I)\leqslant 0$ for each $\lambda\in\varrho_{\rm SBF}(T)$ and $\mu\in\varrho_{\rm SBF}(S)$. Hence $\sigma_{\mathrm{SBF}_+^-}(S\oplus T)=\sigma_{\mathrm{SBF}_+^-}(S)\cup\sigma_{\mathrm{SBF}_+^-}(T)$. But due to Theorem 3.5, this is equivalent to say that $S\oplus T$ has property (SBw). □

We end this section by the following examples.

Example 3.8. 1) A bounded linear operator $T\in L(\mathcal{H})$ is said to be *paranormal* if $\|Tx\|^2\leqslant\|T^2x\|\|x\|$ for all $x\in\mathcal{H}$. We know that every paranormal operator has SVEP. So every two paranormal operators are of jointly stable sign index. Hence by Corollary 2.8, if S and T are paranormal operators having property (SBab), then $S\oplus T$ has property (SBab).

2) A bounded linear operator $T\in L(\mathcal{H})$ is said to be *M-hyponormal* if there exists $M>0$ such that $MT^*T\geqslant TT^*$. It is well known that these operators have SVEP. So every two M-hyponormal operators are of jointly stable sign index. Hence if S and T are M-hyponormal operators and have property (SBab), then $S\oplus T$ has property (SBab).

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