NONLINEAR DIFFERENTIAL MONOMIALS SHARING TWO VALUES

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Abstract. Using the notion of weighted sharing of values which was introduced by Lahiri (2001), we deal with the uniqueness problem for meromorphic functions when two certain types of nonlinear differential monomials namely $h^nh^{(k)}$ $(h=f,g)$ sharing a nonzero polynomial of degree less than or equal to 3 with finite weight have common poles and obtain two results. The results in this paper significantly rectify, improve and generalize the results due to Cao and Zhang (2012).

Keywords: uniqueness; meromorphic function; weighted sharing; nonlinear differential polynomials

MSC 2010: 30D35

1. Introduction: definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM (counting multiplicities), and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM (ignoring multiplicities).

We adopt the standard notation of the value distribution theory (see [11]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The symbol $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of

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finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to f provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM if we do not consider the multiplicities.

We say that a finite value z_0 is a fixed point of f if $f(z_0) = z_0$ or z_0 is a zero of $f(z)-z$.

The following well known theorem in the value distribution theory was posed by Hayman and settled by several authors almost at the same time [4], [6].

Theorem A. *Let $f(z)$ be a transcendental meromorphic function, $n \geqslant 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua [8], Yang and Hua [22], obtained the following result.

Theorem B. *Let f and g be two non-constant entire (meromorphic) functions, $n \geqslant 6$ $(n \geqslant 11)$ a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 \mathrm{e}^{cz}$, $g(z) = c_2 \mathrm{e}^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Considering the uniqueness problem of entire or meromorphic functions having fixed points, Fang and Qiu [9] obtained the following theorem.

Theorem C. *Let f and g be two non-constant meromorphic (entire) functions, $n \geqslant 11$ $(n \geqslant 6)$ a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Gradually the research work in the above directions gained pace and today it has become one of the most prominent branches of the uniqueness theory. During the last couple of years a large amount of research papers have been published by different authors (see [2], [3], [5], [7]–[9], [16]–[18], [20], [22], [25], [28], [27]).

We recall the following result by Xu, Yi and Zhang [20].

Theorem D. *Let f be a transcendental meromorphic function, let n $(n \geqslant 2)$, k be two positive integers. Then $f^n f^{(k)}$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Recently, Cao and Zhang [5] proved the following theorems.

Theorem E. *Let f and g be two transcendental meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer. Let $n > \max\{2k-1, k+4/k+4\}$ be a positive integer. If $f^n f^{(k)} - z$ and $g^n g^{(k)} - z$ share* 0 CM, *f and g share ∞* IM, *then one of the following two conclusions holds*

(i) $f^n f^{(k)} \equiv g^n g^{(k)}$;

(ii) $f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, *where c_1, c_2 and c are constants such that* $4(c_1c_2)^{n+1}c^2 = -1$.

Theorem F. *Let f and g be two non-constant meromorphic functions, whose zeros are of multiplicities at least $k+1$, where k is a positive integer with $1 \leqslant k \leqslant 5$. Let $n \geqslant 10$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share* 1 CM, *$f^{(k)}$ and $g^{(k)}$ share* 1 CM, *f and g share ∞* IM, *then one of the following two conclusions holds:*

(i) $f \equiv tg$, *where t is a constant such that $t^{n+1} = 1$;*

(ii) $f(z) = c_3 \mathrm{e}^{dz}$, $g(z) = c_4 \mathrm{e}^{-dz}$, *where c_3, c_4 and d are constants such that* $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$.

Remark 1.1. Theorems E ([5], Theorem 1.2) and F ([5], Theorem 1.3) are new and seemingly fine. However, in the statements of both of them there is a contradiction. It is assumed that f and g have zeros of multiplicities at least k in Theorem E and $k+1$ in Theorem F. But further authors concluded that "$f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$" in Theorem E and "$f(z) = c_3 \mathrm{e}^{dz}$, $g(z) = c_4 \mathrm{e}^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$" in Theorem F. Here we see that f and g have no zeros, so their multiplicities are equal to $k = 0$. Furthermore, it is assumed that k is a positive integer, but in both Theorems E and F the case $k = 0$ is also considered, which is very strange.

Also, on the other hand, in the proof of Theorem E there is again a mistake.

For example, on page 8 under Step 2 we prove (b):

In this section, we would like to point out a gap. On the 3rd line below equation (5.5), the authors said: "From (5.1) and the assumptions that $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, $f^{(k)}$ and $g^{(k)}$ share 1 CM, f and g share ∞ IM, we can deduce that f and g share 0 IM." But this is not true. Actually f and g share 0 IM only when $f^{(k)}$ and $g^{(k)}$ share 0 CM.

The above discussion is sufficient enough to make one inquisitive to investigate the accurate forms of Theorems E and F.

Also it is quite natural to ask the following questions.

Question 1: Can the lower bound of n be further reduced in Theorems E and F?

Question 2: Does Theorem F hold for $k = 6$ keeping all conclusions intact?

We now explain the notation of weighted sharing as introduced in [13] and [14].

Definition 1.1 ([13], [14]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m ($\leqslant k$) if and only if it is an a-point of g with multiplicity m ($\leqslant k$) and z_0 is an a-point of f with multiplicity m ($> k$) if and only if it is an a-point of g with multiplicity n ($> k$), where m is not necessarily equal to n.

We write f, g share (a, k) meaning that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

If $a(z)$ is a small function with respect to $f(z)$ and $g(z)$, we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight l according to whether $f(z) - a(z)$ and $g(z) - a(z)$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$, respectively.

2. Main results

The following theorems are the main results of the present paper which improve Theorems E and F.

Theorem 2.1. *Let n, k, $1 \leqslant k \leqslant 6$, be two positive integers such that $n \geqslant (k^2 + 3k + 7)/(k + 1)$ and let $p(z)$ be a nonzero polynomial such that* $\deg(p) \leqslant 3$. *Let f and g be two transcendental meromorphic functions such that either f and g have no zeros or the zeros of f and g are of multiplicities at least $k+1$. If $f^n f^{(k)} - p(z)$ and $g^n g^{(k)} - p(z)$ share $(0, m)$, where $m = [(k + 2)/(n - k)] + 3$, f, g share ∞* IM, *and when $k \geqslant 2$, $f^{(k)}$ and $g^{(k)}$ share* 0 CM, *then one of the following two conclusions holds:*

(i) *$f \equiv tg$, where t is a constant such that $t^{n+1} = 1$;*

(ii) *if $p(z)$ is not a constant, then $f = c_1 \mathrm{e}^{cQ(z)}$, $g = c_2 \mathrm{e}^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)\,\mathrm{d}t$, c_1, c_2 and c are constants such that $c^2 (c_1 c_2)^{n+1} = -1$;*
if $p(z)$ is a nonzero constant b, then $f = c_3 \mathrm{e}^{dz}$, $g = c_4 \mathrm{e}^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = b^2$.

Theorem 2.2. *Let n, k be two positive integers such that $n \geqslant (k^2 + 2k + 6)/k$ and let $p(z)$ be a nonzero polynomial such that* $\deg(p) \leqslant 3$. *Let f and g be two transcendental meromorphic functions such that either f and g have no zeros or the zeros of f and g are of multiplicities at least k. If $f^n f^{(k)} - p(z)$ and $g^n g^{(k)} - p(z)$*

share $(0,m)$, *where* $m = [(k+2)/(n-k)] + 3$, f, g *share* ∞ IM, *then one of the following two conclusions holds:*

(i) $f^n f^{(k)} \equiv g^n g^{(k)}$;

(ii) *if* $p(z)$ *is not a constant, then* $f = c_1 \mathrm{e}^{cQ(z)}$, $g = c_2 \mathrm{e}^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(t)\,\mathrm{d}t$, c_1, c_2 *and* c *are constants such that* $c^2(c_1c_2)^{n+1} = -1$;
if $p(z)$ *is a nonzero constant* b, *then* $f = c_3\mathrm{e}^{dz}$, $g = c_4\mathrm{e}^{-dz}$, *where* c_3, c_4 *and* d *are constants such that* $(-1)^k(c_3c_4)^{n+1}d^{2k} = b^2$.

In particular, when $k=1$, *then* $f^n f^{(k)} \equiv g^n g^{(k)}$ *implies* $f \equiv tg$, *where* t *is a constant such that* $t^{n+1} = 1$.

We now explain some definitions and notations, which are used in the paper.

Definition 2.1 ([17]). Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f|\geqslant p)$ $(\overline{N}(r,a;f|\geqslant p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r,a;f|\leqslant p)$ $(\overline{N}(r,a;f|\leqslant p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 2.2. We denote by $\overline{N}(r,a;f|{=}k)$ the reduced counting function of those a-points of f whose multiplicities are (or equal) exactly k, where $k \geqslant 2$ is an integer.

Definition 2.3 ([24]). For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f|\geqslant 2) + \ldots + \overline{N}(r,a;f|\geqslant p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 2.4 ([1]). Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p = q \geqslant 2$; each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Definition 2.5 ([14]). Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

3. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H and V the functions.

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big), \tag{3.1}$$

and

$$V = \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big) - \Big(\frac{G'}{G-1} - \frac{G'}{G}\Big) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{3.2}$$

Lemma 3.1 ([26]). *Let f be a non-constant meromorphic function and p, k positive integers. Then*

$$N_p(r,0;f^{(k)}) \leqslant N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 3.2 ([15]). *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}|f\neq 0) \leqslant k\overline{N}(r,\infty;f) + N(r,0;f|{<}k) + k\overline{N}(r,0;f|{\geqslant}k) + S(r,f).$$

Lemma 3.3 ([11]). *Suppose that f is a non-constant meromorphic function, k $(\geqslant 2)$ is an integer. If*

$$N(r,\infty;f) + N(r,0;f) + N(r,0;f^{(k)}) = S\Big(r,\frac{f'}{f}\Big),$$

then $f = \mathrm{e}^{az+b}$, where $a \neq 0$, b are constants.

Lemma 3.4 ([10]). *Let $f(z)$ be a non-constant entire function and let k $(\geqslant 2)$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = \mathrm{e}^{az+b}$, where $a \neq 0$, b are constants.*

Lemma 3.5 ([19]). *Let f and g be two non-constant meromorphic functions, k, n $(> 2k+1)$ two positive integers. If $(f^n)^{(k)} \equiv (g^n)^{(k)}$, then $f \equiv tg$ for a constant t such that $t^n = 1$.*

Lemma 3.6 ([11], [23]). *Let f be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r,a_i)=S(r,f)$, $i=1,2$. Then*

$$T(r,f)\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f).$$

Lemma 3.7. *Let f, g be two non-constant meromorphic functions and $F=f^nf^{(k)}$, $G=g^ng^{(k)}$, where n and k are positive integers. Let $H\not\equiv 0$. If F, G share $(1,m)$, f, g share (∞,p), where $0\leqslant m\leqslant\infty$, $0\leqslant p\leqslant\infty$. Then*

$$\begin{aligned}((n+1)(p+1)+k-1)&\overline{N}(r,\infty;f|\geqslant p+1)\\ &\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

Proof. Since $H\not\equiv 0$, it follows that $F\not\equiv G$. We assert that $V\not\equiv 0$. If not, let $V\equiv 0$. Then by integration we obtain

$$1-\frac{1}{F}\equiv A\Big(1-\frac{1}{G}\Big).$$

If z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $1/F(z_0)=0$ and $1/G(z_0)=0$. So $A=1$ and hence $F\equiv G$, which is a contradiction. Hence $V\not\equiv 0$.

Let z_1 be a pole of f with multiplicity q and a pole of g with multiplicity r. Clearly z_1 is a pole of F with multiplicity $(n+1)q+k$ and a pole of G with multiplicity $(n+1)r+k$. Noting that f, g share (∞,p), from the definition of V it is clear that z_1 is a zero of V with multiplicity at least $(n+1)(p+1)+k-1$, provided $q\geqslant p+1$.

So from the definition of V we have

$$\begin{aligned}((n+1)(p+1)+k-1)&\overline{N}(r,\infty;f|\geqslant p+1)\\ &\leqslant N(r,0;V)\leqslant N(r,\infty;V)+S(r,f)+S(r,g)\\ &\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

□

Lemma 3.8. *Let f, g be two non-constant meromorphic functions whose zeros are of multiplicities at least s, and $F=f^nf^{(k)}$, $G=g^ng^{(k)}$, where n and k are positive integers such that $n>k$. Let $H\not\equiv 0$. If F, G share $(1,m)$, f, g share $(\infty,0)$, where $0\leqslant m\leqslant\infty$, then*

$$\begin{aligned}\overline{N}(r,\infty;f)\leqslant{}&\frac{\min\{s,k\}+1}{s(n-k)}(N(r,0;f)+N(r,0;g))\\ &+\frac{1}{n-k}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

P r o o f. We note, provided f has zeros of multiplicities $\geqslant s$, that when $s > k$, $N_k(r,0;f) \leqslant k\overline{N}(r,0;f|\geqslant s) \leqslant ks^{-1}N(r,0;f|\geqslant s) = ks^{-1}N(r,0;f)$ and when $s \leqslant k$, $N_k(r,0;f) \leqslant N(r,0;f)$ is obvious. Using Lemmas 3.2 and 3.7 for $p=0$ we get

$$\begin{aligned}(n+k)\overline{N}(r,\infty;f) &\leqslant \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\ &\leqslant \overline{N}(r,0;f) + \overline{N}(r,0;f^{(k)}|f\neq 0) + \overline{N}(r,0;g)\\ &\quad + \overline{N}(r,0;g^{(k)}|g\neq 0) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\ &\leqslant \overline{N}(r,0;f) + k\overline{N}(r,\infty;f) + N_k(r,0;f) + \overline{N}(r,0;g)\\ &\quad + k\overline{N}(r,\infty;g) + N_k(r,0;g) + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\ &\leqslant \frac{1}{s}N(r,0;f) + \frac{\min\{s,k\}}{s}N(r,0;f) + \frac{1}{s}N(r,0;g)\\ &\quad + \frac{\min\{s,k\}}{s}N(r,0;g) + 2k\overline{N}(r,\infty;f)\\ &\quad + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\ &\leqslant \frac{\min\{s,k\}+1}{s}(N(r,0;f) + N(r,0;g)) + 2k\overline{N}(r,\infty;f)\\ &\quad + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).\end{aligned}$$

Hence the lemma follows. □

Lemma 3.9 ([21]). *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1f + a_2f^2 + \ldots + a_nf^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Lemma 3.10. *Let f be a non-constant meromorphic function and $F = f^nf^{(k)}$, where n and k are positive integer. Then*

$$(n-1)T(r,f) \leqslant T(r,F) - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f).$$

P r o o f. Note that

$$\begin{aligned}N(r,\infty;F) &= N(r,\infty;f^n) + N(r,\infty;f^{(k)})\\ &= N(r,\infty;f^n) + N(r,\infty;f) + k\overline{N}(r,\infty;f) + S(r,f).\end{aligned}$$

That is,

$$N(r,\infty;f^n) = N(r,\infty;F) - N(r,\infty;f) - k\overline{N}(r,\infty;f) + S(r,f).$$

Also

$$\begin{aligned}m(r,f^n) = m\Big(r,\frac{F}{f^{(k)}}\Big) &\leqslant m(r,F) + m\Big(r,\frac{1}{f^{(k)}}\Big) + S(r,f)\\ &= m(r,F) + T(r,f^{(k)}) - N(r,0;f^{(k)}) + S(r,f)\end{aligned}$$

$$
\begin{aligned}
&= m(r,F) + N(r,\infty;f^{(k)}) + m(r,f^{(k)}) - N(r,0;f^{(k)}) + S(r,f)\\
&\leqslant m(r,F) + N(r,\infty;f) + k\overline{N}(r,\infty;f) + m\Big(r,\frac{f^{(k)}}{f}\Big)\\
&\quad + m(r,f) - N(r,0;f^{(k)}) + S(r,f)\\
&= m(r,F) + T(r,f) + k\overline{N}(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f).
\end{aligned}
$$

Now

$$
\begin{aligned}
nT(r,f) &= N(r,\infty;f^n) + m(r,f^n)\\
&\leqslant T(r,F) + T(r,f) - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f),
\end{aligned}
$$

i.e.,

$$(n-1)T(r,f) \leqslant T(r,F) - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f).$$

□

Lemma 3.11. *Let $p(z)$ be a nonzero polynomial and n, k two positive integers such that $n > (k^2+3k+3)/(k+1)$. Let f, g be two transcendental meromorphic functions such that either f and g have no zeros or the zeros of f and g are of multiplicities at least $k+1$ and let $F = f^n f^{(k)}/p$, $G = g^n g^{(k)}/p$. If f, g share $(\infty,0)$ and $H \equiv 0$, then either $FG \equiv 1$ or $F \equiv G$.*

P r o o f. Since $H \equiv 0$, by integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \tag{3.3}$$

where a, b are constants and $a \neq 0$. From (3.3) it is clear that F and G share $(1,\infty)$.

We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (3.3) we have

$$F \equiv \frac{-a}{G-a-1}$$

together with

$$\overline{N}(r,a+1;G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + \overline{N}(r,0;p).$$

So, in view of Lemma 3.10 and the second fundamental theorem we get

$$
\begin{aligned}
(n-1)T(r,g) &\leqslant T(r,G) - N(r,\infty;g) - N(r,0;g^{(k)}) + T(r,p) + S(r,g)\\
&\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G)\\
&\quad - N(r,\infty;g) - N(r,0;g^{(k)}) + O(\log r) + S(r,g)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \overline{N}(r,0;g)+\overline{N}(r,0;g^{(k)})+\overline{N}(r,\infty;f)\\
&\quad+\overline{N}(r,0;p)-N(r,0;g^{(k)})+S(r,g)\\
&\leqslant \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+S(r,g)\\
&\leqslant \frac{1}{k+1}N(r,0;g)+N(r,\infty;g)+S(r,g)\leqslant \frac{k+2}{k+1}T(r,g)+S(r,g),
\end{aligned}
$$

which is a contradiction since $n>(k^2+3k+3)/(k+1)$.

If $b\neq -1$, from (3.3) we obtain that

$$
F-\Big(1+\frac{1}{b}\Big)\equiv \frac{-a}{b^2\Big(G+\dfrac{a-b}{b}\Big)}.
$$

So,

$$
\overline{N}\Big(r,\frac{b-a}{b};G\Big)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f)+\overline{N}(r,0;p).
$$

Using Lemma 3.10 and the same argument as the one used in the case when $b=-1$ we get a contradiction.

Case 2. Let $b\neq 0$ and $a=b$.

If $b=-1$, then from (3.3) we have

$$
FG\equiv 1,
$$

i.e.,

$$
f^nf^{(k)}g^ng^{(k)}\equiv p^2.
$$

If $b\neq -1$, from (3.3) we have

$$
\frac{1}{F}\equiv \frac{bG}{(1+b)G-1}.
$$

Therefore

$$
\overline{N}\Big(r,\frac{1}{1+b};G\Big)=\overline{N}(r,0;F).
$$

So, in view of Lemmas 3.2, 3.10 and the second fundamental theorem we get

$$
\begin{aligned}
(n-1)T(r,g)&\leqslant T(r,G)-N(r,\infty;g)-N(r,0;g^{(k)})+S(r,g)\\
&\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\Big(r,\frac{1}{1+b};G\Big)\\
&\quad-N(r,\infty;g)-N(r,0;g^{(k)})+S(r,g)\\
&\leqslant \overline{N}(r,0;g)+\overline{N}(r,0;g^{(k)})+\overline{N}(r,0;F)-N(r,0;g^{(k)})+S(r,g)\\
&\leqslant \overline{N}(r,0;g)+\overline{N}(r,0;f)+\overline{N}(r,0;f^{(k)}|f\neq 0)+S(r,g)
\end{aligned}
$$

$$\leqslant \overline{N}(r,0;g) + \overline{N}(r,0;f) + k\overline{N}(r,0;f|\geqslant k) + k\overline{N}(r,\infty;f) + S(r,g)$$
$$\leqslant \frac{1}{k+1}T(r,g) + \frac{1}{k+1}T(r,f) + \frac{k}{k+1}T(r,f) + kT(r,f) + S(r,f).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leqslant T(r,g)$ for $r \in I$.

So, for $r \in I$ we have

$$(n-1)T(r,g) \leqslant \frac{k^2+2k+2}{k+1}T(r,g) + S(r,g),$$

which is a contradiction since $n > (k^2+3k+3)/(k+1)$.

Case 3. Let $b = 0$. From (3.3) we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{3.4}$$

If $a \neq 1$ then from (3.4) we obtain

$$\overline{N}(r,1-a;G) = \overline{N}(r,0;F).$$

We can deduce a contradiction similarly to Case 2. Therefore $a = 1$ and from (3.4) we obtain $F \equiv G$. □

Lemma 3.12. *Let $p(z)$ be a nonzero polynomial and n, k two positive integers such that $n > (k^2+2k+2)/k$. Let f, g be two transcendental meromorphic functions such that either f and g have no zeros or the zeros of f and g are of multiplicities at least k and let $F = f^n f^{(k)}/p$, $G = g^n g^{(k)}/p$. If f, g share $(\infty, 0)$ and $H \equiv 0$, then either $FG \equiv 1$ or $F \equiv G$.*

Proof. We omit the proof since it can be carried out along the lines of the proof of Lemma 3.11. □

Lemma 3.13 ([12]). *Let f and g be two non-constant meromorphic functions. Suppose that f and g share 0 and ∞ CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM for $k = 1, 2, \ldots, 6$. Then f and g satisfy one of the following cases:*

(i) *$f \equiv tg$, where t ($\neq 0$) is a constant,*
(ii) *$f(z) = \mathrm{e}^{az+b}$, $g(z) = \mathrm{e}^{cz+d}$, where a, b, c and d are constants such that $ac \neq 0$,*
(iii) *$f(z) = a/(1-b\mathrm{e}^{\alpha(z)})$, $g(z) = a/(\mathrm{e}^{-\alpha(z)} - b)$, where a, b are nonzero constants and $\alpha(z)$ is a non-constant entire function,*
(iv) *$f(z) = a(1-b\mathrm{e}^{cz})$, $g(z) = d(\mathrm{e}^{-cz} - b)$, where a, b, c and d are nonzero constants.*

Lemma 3.14. *Let n and k $(1 \leqslant k \leqslant 6)$ be two positive integers such that $n \geqslant k+2$. Let f and g be two transcendental meromorphic functions such that either f and g have no zeros or the zeros of f and g are of multiplicities at least $k+1$. Suppose f, g share ∞ IM and when $k \geqslant 2$, $f^{(k)}$ and $g^{(k)}$ share 0 CM. If $f^n f^{(k)} \equiv g^n g^{(k)}$, then $f \equiv tg$, where t is a constant such that $t^{n+1} = 1$.*

Proof. Suppose

$$f^n f^{(k)} \equiv g^n g^{(k)}. \tag{3.5}$$

When $k=1$, from (3.5) we get $(f^{n+1})' \equiv (g^{n+1})'$ and so the result follows from Lemma 3.5.

Next we suppose $2 \leqslant k \leqslant 6$.

Since f and g share ∞ IM, it follows from (3.5) that f and g share ∞ CM. Also, since $f^{(k)}$ and $g^{(k)}$ share 0 CM, it follows from (3.5) that f and g share 0 CM. From (3.5) it is clear that $S(r,f) = S(r,g)$.

Let $h = g/f$. We now consider the following cases:

Case 1: Suppose h is not constant. Since f and g share 0 and ∞ CM, it follows that $h = \mathrm{e}^{\alpha}$, where α is a non-constant entire function. Now in view of Lemma 3.13 we have to consider the following subcases.

Subcase 1.1: Suppose $f(z) = \mathrm{e}^{az+b}$ and $g(z) = \mathrm{e}^{cz+d}$, where a, b, c and d are constants such that $ac \neq 0$. Now from (3.5) we get

$$\Bigl(\frac{a}{c}\Bigr)^k \mathrm{e}^{(n+1)(b-d)}\mathrm{e}^{(n+1)(a-c)z} \equiv 1, \tag{3.6}$$

which implies that $a = c$. Thus $h = \mathrm{e}^{d-b}$, which is a contradiction since h is not a constant.

Subcase 1.2: $f(z) = a/(1 - b\mathrm{e}^{\alpha})$ and $g(z) = a/(\mathrm{e}^{-\alpha} - b)$, where a, b are constants and $\alpha(z)$ is a non-constant entire function. Clearly $h = g/f = \mathrm{e}^{\alpha(z)}$ and f, g have no zeros. Also we have

$$T(r,f) = T(r,h) + S(r,h), \quad T(r,g) = T(r,h) + S(r,h),$$

and so

$$S(r,f) = S(r,g) = S(r,h).$$

From (3.5) we see that

$$\begin{aligned} \overline{N}(r,1;h^n) &= \overline{N}\Bigl(r,1;\frac{f^{(k)}}{g^{(k)}}\Bigr) \leqslant T\Bigl(r,\frac{g^{(k)}}{f^{(k)}}\Bigr) + O(1) \\ &\leqslant N\Bigl(r,\infty;\frac{g^{(k)}}{f^{(k)}}\Bigr) + m\Bigl(r,\infty;\frac{g^{(k)}}{f^{(k)}}\Bigr) \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&\leqslant m\Big(r,\infty;\frac{g^{(k)}}{g}\Big)+m\Big(r,\infty;\frac{f}{f^{(k)}}\Big)+m\Big(r,\infty;\frac{g}{f}\Big)\\
&\leqslant T\Big(r,\frac{f^{(k)}}{f}\Big)+S(r,g)+T(r,h)\\
&\leqslant N\Big(r,\infty;\frac{f^{(k)}}{f}\Big)+T(r,h)+S(r,h)\\
&\leqslant k\overline{N}(r,\infty;f)+T(r,h)+S(r,h)\leqslant (k+1)T(r,h)+S(r,h).
\end{aligned}
$$

Now by the second fundamental theorem and using Lemma 3.9, (3.7) we get

$$
\begin{aligned}
nT(r,h)=T(r,h^n)&\leqslant \overline{N}(r,0;h^n)+\overline{N}(r,\infty;h^n)+\overline{N}(r,1;h^n)+S(r,h)\\
&\leqslant (k+1)T(r,h)+S(r,h),
\end{aligned}
$$

which is a contradiction since $n>k+1$.

Subcase 1.3: $f(z)=a(1-b\mathrm{e}^{cz})$ and $g(z)=d(\mathrm{e}^{-cz}-b)$, where a, b, c and d are nonzero constants. Then all zeros of f and g are simple, which contradicts our assumption.

Case 2: Suppose h is a constant. Then from (3.5) we get $h^{n+1}=1$. Thus we have $f\equiv tg$, where t is a constant such that $t^{n+1}=1$. $\square$

Lemma 3.15 ([1]). *Let f and g be non-constant meromorphic functions sharing $(1,k_1)$, where $2\leqslant k_1\leqslant\infty$. Then*

$$
\begin{aligned}
&\overline{N}(r,1;f|{=}2)+2\overline{N}(r,1;f|{=}3)+\ldots+(k_1-1)\overline{N}(r,1;f|{=}k_1)+k_1\overline{N}_L(r,1;f)\\
&\qquad+(k_1+1)\overline{N}_L(r,1;g)+k_1\overline{N}_E^{(k_1+1}(r,1;g)\leqslant N(r,1;g)-\overline{N}(r,1;g).
\end{aligned}
$$

Lemma 3.16. *Let $p(z)$ be a nonzero polynomial with $\deg(p(z))=l$ $(\leqslant 3)$ and n, k two positive integers such that $n>2l-1$. Let f and g be two transcendental meromorphic functions such that either f and g have no zeros or the zeros of f and g are of multiplicities at least k. If $f^nf^{(k)}g^ng^{(k)}\equiv p^2$ and f, g share ∞ IM, then*

(i) *if $p(z)$ is not a constant, then $f=c_1\mathrm{e}^{cQ(z)}$, $g=c_2\mathrm{e}^{-cQ(z)}$, where $Q(z)=\int_0^z p(t)\,\mathrm{d}t$, c_1, c_2 and c are constants such that $c^2(c_1c_2)^{n+1}=-1$,*

(ii) *if $p(z)$ is a nonzero constant b, then $f=c_3\mathrm{e}^{dz}$, $g=c_4\mathrm{e}^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^{n+1}d^{2k}=b^2$.*

P r o o f. Suppose

$$
f^nf^{(k)}g^ng^{(k)}\equiv p^2. \tag{3.8}
$$

Since f and g share ∞ IM we have from (3.8) that f and g are transcendental entire functions. We consider the following cases.

Case 1: Let $\deg(p(z)) = l$ $(\geqslant 1)$.

First we suppose $k \geqslant 2$.

Let us assume that the zeros of f and g are of multiplicities at least k. Let z_0 be a zero of $f(g)$ with multiplicity q $(\geqslant k)$. Then z_0 is a zero of $f^n f^{(k)}$ $(g^n g^{(k)})$ with multiplicity at least $nq + q - k$. Now from (3.8) it is clear that z_0 must be a zero of $p^2(z)$ with multiplicity at most $2l$.

Note that since $n \geqslant 2l$, we must have

$$nq + q - k \geqslant nk \geqslant 2lk. \tag{3.9}$$

Since $k \geqslant 2$ we cannot have $2lk = 2l$ and so one can conclude that f (g) has no zero, which is a contradiction. Thus the case "zeros of f and g are of multiplicities at least k" is discarded automatically. Hence when $k \geqslant 2$, one can easily conclude that f and g have no zeros.

Next we suppose $k = 1$.

In this case we also want to prove that neither f nor g have zeros. Suppose f (g) has a zero. Let z_1 be a zero of f (g) with multiplicity q_1. Then z_1 is a zero of $f^n f'$ $(g^n g')$ with multiplicity at least $nq_1 + q_1 - 1$. Now from (3.8) it is clear that z_1 must be a zero of $p^2(z)$ with multiplicity at most $2l$.

Note that since $n \geqslant 2l$, we must have

$$nq_1 + q_1 - 1 \geqslant n \geqslant 2l. \tag{3.10}$$

Hence we must have $q_1 = 1$ and $n = 2l$. So from (3.8) it is clear that z_1 is a simple zero of f (g) and at the same time z_1 is the only zero of $p(z)$. Also it is evident that g (f) has no zero.

Therefore we can take

$$f(z) = (z - z_1)\mathrm{e}^{\alpha_1(z)}, \quad g(z) = \mathrm{e}^{\beta_1(z)}, \tag{3.11}$$

where $\alpha_1(z)$ and $\beta_1(z)$ are two non-constant entire functions. Then from (3.8) we get

$$(1 + (z - z_1)\alpha_1'(z))\beta_1'(z)\mathrm{e}^{(2l+1)(\alpha_1(z)+\beta_1(z))} \equiv b_1. \tag{3.12}$$

First we suppose that α_1 and β_1 are transcendental.

Let $\alpha_1 + \beta_1 = \gamma$. Clearly γ is not a constant. Then from (3.12) we get

$$(1 + (z - z_1)\alpha_1')(\gamma' - \alpha_1')\mathrm{e}^{(2l+1)\gamma} \equiv b_1. \tag{3.13}$$

We have $T(r,\gamma')=m(r,s\gamma')+O(1)=m(r,(\mathrm{e}^{s\gamma})'/\mathrm{e}^{s\gamma})=S(r,\mathrm{e}^{s\gamma})$, where $s=2l+1$. Thus from (3.13) we get

$$\begin{aligned}T(r,\mathrm{e}^{s\gamma}) &\leqslant T\Big(r,\frac{1}{(1+(z-z_1)\alpha_1')(\gamma'-\alpha_1')}\Big)+O(1)\\ &\leqslant 2T(r,\alpha_1')+S(r,\alpha_1')+S(r,\mathrm{e}^{s\gamma}),\end{aligned}$$

which implies that $T(r,\mathrm{e}^{s\gamma})=O(T(r,\alpha_1'))$ and so $S(r,\mathrm{e}^{s\gamma})$ can be replaced by $S(r,\alpha_1')$. Thus we get $T(r,\gamma')=S(r,\alpha_1')$ and so γ' is a small function with respect to α_1'. In view of (3.13) and by Lemma 3.6, we get

$$\begin{aligned}T(r,\alpha_1') \leqslant{}& \overline{N}(r,\infty;\alpha_1')+\overline{N}(r,0;1+(z-z_1)\alpha_1')\\ &+\overline{N}(r,0;\alpha_1'-\gamma')+S(r,\alpha_1')=S(r,\alpha_1'),\end{aligned}$$

which shows that α_1 is a polynomial. Similarly we can prove that β_1 is also a polynomial. This contradicts the fact that α_1 and β_1 are transcendental.

Next we suppose that α_1 is a transcendental entire function and β_1 is a polynomial.

Then $\alpha_1+\beta_1$ is transcendental, i.e., γ is transcendental. Now proceeding as above one can easily prove that α_1 is also a polynomial. Clearly in this case we also arrive at a contradiction.

Next we suppose that α_1 is a polynomial and β_1 is a transcendental entire function.

Then $\alpha_1+\beta_1$ is transcendental, i.e., γ is transcendental. Now from (3.12) we get

$$(1+(z-z_1)\gamma'-(z-z_1)\beta_1')\beta_1'\mathrm{e}^{(2l+1)\gamma}\equiv b_1. \tag{3.14}$$

Also we have $T(r,\gamma')=S(r,\mathrm{e}^{s\gamma})$, where $s=2l+1$. Thus from (3.14) we get

$$\begin{aligned}T(r,\mathrm{e}^{s\gamma}) &\leqslant T\Big(r,\frac{1}{(1+(z-z_1)\gamma'-(z-z_1)\beta_1')\beta_1'}\Big)+O(1)\\ &\leqslant 2T(r,\beta_1')+S(r,\beta_1')+S(r,\mathrm{e}^{s\gamma}),\end{aligned}$$

which implies that $T(r,\mathrm{e}^{s\gamma})=O(T(r,\beta_1'))$ and so $S(r,\mathrm{e}^{s\gamma})$ can be replaced by $S(r,\beta_1')$. Thus we get $T(r,\gamma')=S(r,\beta_1')$ and so γ' is a small function with respect to β_1'. In view of (3.14) and by Lemma 3.6, we get

$$\begin{aligned}T(r,\beta_1') \leqslant{}& \overline{N}(r,\infty;\beta_1')+\overline{N}(r,0;1+(z-z_1)\gamma'-(z-z_1)\beta_1')\\ &+\overline{N}(r,0;\beta_1')+S(r,\beta_1')=S(r,\beta_1'),\end{aligned}$$

which shows that β_1 is a polynomial. This contradicts the fact that β_1 is transcendental.

Finally we suppose that both α_1 and β_1 are polynomials. From (3.13) we can conclude that $\alpha_1(z) + \beta_1(z) \equiv C$ for a constant C. If α_1' is a polynomial then $1+(z-z_1)\alpha_1'$ must be a polynomial, which is impossible as b_1 is a nonzero constant. Clearly in this case we also arrive at a contradiction. Thus when $k=1$, we conclude that f and g have no zeros.

Finally let neither f nor g have zeros whatever the values of k. Therefore we can take

$$f(z) = \mathrm{e}^{\alpha(z)}, \quad g(z) = \mathrm{e}^{\beta(z)}, \tag{3.15}$$

where α and β are non-constant entire functions.

We consider the following subcases.

Subcase 1.1: Let $k \geqslant 2$. From (3.8) we see that

$$N(r,0;(f)^{(k)}) \leqslant N(r,0;p^2) = O(\log r).$$

First we suppose that both α and β are transcendental. Using (3.8) we have

$$N(r,\infty;f) + N(r,0;f) + N(r,0;(f)^{(k)}) = S(r,\alpha') = S\Bigl(r,\frac{f'}{f}\Bigr). \tag{3.16}$$

Clearly (3.16) also holds for g instead of f. Then from (3.16) and Lemma 3.3 we must have

$$f(z) = \mathrm{e}^{az+b}, \quad g(z) = \mathrm{e}^{cz+d}, \tag{3.17}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But these types of f and g do not agree with the relation (3.8).

Next we suppose that both α and β are polynomials such that $\deg(\alpha) \geqslant 2$ and $\deg(\beta) \geqslant 2$.

From (3.8) we deduce that $\alpha + \beta \equiv C$, where C is a constant and

$$((\alpha')^k + P_{k-1}(\alpha'))((\beta')^k + Q_{k-1}(\beta'))\mathrm{e}^{(n+1)C} \equiv p^2, \tag{3.18}$$

where $P_{k-1}(\alpha')$ and $Q_{k-1}(\beta')$ are differential polynomials in α' and β' of degree at most $k-1$, i.e.,

$$C_1(\alpha')^{2k} \equiv p^2 + \overline{P}_{2k-1}(\alpha'), \tag{3.19}$$

where C_1 is a nonzero constant and $\overline{P}_{2k-1}(\alpha')$ is a differential polynomial in α' of degree at most $2k-1$.

If $l = 1$, then we arrive at a contradiction with (3.19).

Suppose $l = 2$. Then from (3.19) we get $\deg(\alpha') = 1$ and $k = 2$. Thus α'' is a nonzero constant. Now from (3.18) taking $e^{(n+1)C}$ by 1 we get

$$((s\alpha')^2)^2 - (s\alpha'')^2 \equiv p^2. \tag{3.20}$$

From (3.20) we see that $2\deg(\alpha'') = \deg((\alpha'')^2) \geqslant \deg((\alpha')^2) > 2\deg(\alpha'')$ and so $\alpha'' = 0$, which is a contradiction.

Suppose $l = 3$. Then from (3.19) we get $\deg(\alpha') = 1$ and $k = 3$. Thus α'' is a nonzero constant. Now from (3.18) taking $e^{(n+1)C}$ by 1 we get

$$(3s^2\alpha'\alpha'')^2 - ((s\alpha')^3)^2 \equiv p^2. \tag{3.21}$$

From (3.21) we see that $2\deg(\alpha'\alpha'') = \deg((\alpha'\alpha'')^2) \geqslant \deg((\alpha')^3) > 2\deg(\alpha'\alpha'')$ and so $\alpha'' = 0$, which is a contradiction.

Subcase 1.2: Let $k = 1$. Suppose that α and β are transcendental. Then from (3.8) we get

$$\alpha'\beta' e^{(n+1)(\alpha+\beta)} \equiv p^2(z). \tag{3.22}$$

Let $\alpha + \beta = \gamma$ and $s = n + 1$. From (3.22) we know that γ is not a constant since in that case we get a contradiction. Now from (3.22) we get

$$\alpha'(\gamma' - \alpha')e^{s\gamma} \equiv p^2(z). \tag{3.23}$$

We have $T(r, \gamma') = m(r, s\gamma') + O(1) = m(r, (e^{s\gamma})'/e^{s\gamma}) = S(r, e^{s\gamma})$. Thus from (3.23) we get

$$\begin{aligned} T(r, e^{s\gamma}) &\leqslant T\Big(r, \frac{p^2}{\alpha'(\gamma' - \alpha')}\Big) + O(1) \\ &\leqslant T(r, \alpha') + T(r, \gamma' - \alpha') + O(\log r) + O(1) \\ &\leqslant 2T(r, \alpha') + S(r, \alpha') + S(r, e^{s\gamma}), \end{aligned}$$

which implies that $T(r, e^{s\gamma}) = O(T(r, \alpha'))$ and so $S(r, e^{s\gamma})$ can be replaced by $S(r, \alpha')$. Thus we get $T(r, \gamma') = S(r, \alpha')$ and so γ' is a small function with respect to α'. In view of (3.23) and by Lemma 3.6 we get

$$T(r, \alpha') \leqslant \overline{N}(r, \infty; \alpha') + \overline{N}(r, 0; \alpha') + \overline{N}(r, 0; \alpha' - \gamma') + S(r, \alpha') \leqslant O(\log r) + S(r, \alpha'),$$

which shows that α' is a polynomial and so α is a polynomial. Similarly we can prove that β is also a polynomial. This contradicts the fact that α and β are transcendental. Next suppose without loss of generality that α is a polynomial and β is a transcendental entire function. Then γ is transcendental. So in view of (3.23) we obtain

$$\begin{aligned} sT(r, \mathrm{e}^{\gamma}) &\leqslant T\Big(r, \frac{p^2}{\alpha'(\gamma'-\alpha')}\Big) + O(1) \\ &\leqslant T(r,\alpha') + T(r,\gamma'-\alpha') + S(r,\mathrm{e}^{\gamma}) \leqslant T(r,\gamma') + S(r,\mathrm{e}^{\gamma}) = S(r,\mathrm{e}^{\gamma}), \end{aligned}$$

which leads to a contradiction. Thus both α and β are polynomials. From (3.22) we can conclude that $\alpha(z)+\beta(z) \equiv C$ for a constant C and so $\alpha'(z)+\beta'(z) \equiv 0$. Again from (3.22) we get $\mathrm{e}^{s\gamma}\alpha'\beta' \equiv p^2(z)$. By computation we get

$$\alpha' = cp(z), \quad \beta' = -cp(z). \tag{3.24}$$

Hence

$$\alpha = cQ(z) + l_1, \quad \beta = -cQ(z) + l_2, \tag{3.25}$$

where $Q(z) = \int_0^z p(t)\,\mathrm{d}t$ and l_1, l_2 are constants. Finally we take f and g as

$$f(z) = c_1\mathrm{e}^{cQ(z)}, \quad g(z) = c_2\mathrm{e}^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $c^2(c_1c_2)^{n+1} = -1$.

Case 2: Let $p(z)$ be a nonzero constant b. Then from (3.8) we get

$$f^n f^{(k)} g^n g^{(k)} \equiv b^2, \tag{3.26}$$

where f and g are transcendental entire functions. Clearly f and g have no zeros and so we can take f and g as

$$f = \mathrm{e}^{\alpha}, \quad g = \mathrm{e}^{\beta}, \tag{3.27}$$

where $\alpha(z)$, $\beta(z)$ are two non-constant entire functions.

We now consider the following two subcases.

Subcase 2.1: Let $k \geqslant 2$.

From (3.26) it is clear that $ff^{(k)} \neq 0$ and $gg^{(k)} \neq 0$. Then by Lemma 3.4 we have

$$f(z) = \mathrm{e}^{az+b}, \quad g(z) = \mathrm{e}^{cz+d}, \tag{3.28}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But from (3.26) we see that $a + c = 0$.

Subcase 2.2: Let $k = 1$.

Suppose that α and β are transcendental. Then from (3.26) we get

$$\alpha'\beta'\mathrm{e}^{(n+1)(\alpha+\beta)} \equiv b^2. \tag{3.29}$$

Let $\alpha+\beta=\gamma$ and $s=n+1$. Clearly γ is non-constant. Now from (3.29) we get

$$\alpha'(\gamma'-\alpha')\mathrm{e}^{s\gamma}\equiv b^2. \tag{3.30}$$

We have $T(r,\gamma')=m(r,s\gamma')+O(1)=m(r,(\mathrm{e}^{s\gamma})'/\mathrm{e}^{s\gamma})=S(r,\mathrm{e}^{s\gamma})$. Thus from (3.30) we get

$$T(r,\mathrm{e}^{s\gamma})\leqslant T\Big(r,\frac{b^2}{\alpha'(\gamma'-\alpha')}\Big)+O(1)\leqslant 2T(r,\alpha')+S(r,\alpha')+S(r,\mathrm{e}^{s\gamma}),$$

which shows that $S(r,\mathrm{e}^{s\gamma})$ can be replaced by $S(r,\alpha')$. Thus we get $T(r,\gamma')=S(r,\alpha')$ and so γ' is a small function with respect to α'. In view of (3.30) and by Lemma 3.6 we get

$$T(r,\alpha')\leqslant \overline{N}(r,\infty;\alpha')+\overline{N}(r,0;\alpha')+\overline{N}(r,0;\alpha'-\gamma')+S(r,\alpha')=S(r,\alpha'),$$

which shows that α is a polynomial. Similarly we can prove that β is also a polynomial. This contradicts the fact that α and β are transcendental. Next suppose without loss of generality that α is a polynomial and β is a transcendental entire function. Then γ is transcendental. So in view of (3.30) we obtain

$$sT(r,\mathrm{e}^{\gamma})\leqslant T\Big(r,\frac{b^2}{\alpha'(\gamma'-\alpha')}\Big)+O(1)\leqslant T(r,\gamma')+S(r,\mathrm{e}^{\gamma})=S(r,\mathrm{e}^{\gamma}),$$

which leads to a contradiction. Thus both α and β are polynomials. From (3.29) we conclude that $\alpha(z)+\beta(z)\equiv C$ for a constant C and so $\alpha'(z)+\beta'(z)\equiv 0$. Again from (3.29) we get $\mathrm{e}^{s\gamma}\alpha'\beta'\equiv b^2$. By computation we get

$$\alpha'=a_1,\quad \beta'=-a_1. \tag{3.31}$$

Hence

$$\alpha=a_1z+b_1,\quad \beta=-a_1z+b_2 \tag{3.32}$$

and so

$$f(z)=\mathrm{e}^{a_1z+b_1},\quad g(z)=\mathrm{e}^{-a_1z+b_2}, \tag{3.33}$$

where $a_1\neq 0$, b_1 and b_2 are constants.

Thus in either cases we can take f and g as

$$f(z)=c_3\mathrm{e}^{dz},\quad g(z)=c_4\mathrm{e}^{-dz},$$

where c_3, c_4 and d are nonzero constants such that $(-1)^k(c_3c_4)^{n+1}d^{2k}=b^2$.

This completes the proof. □

4. Proofs of the theorems

Proof of Theorem 2.1. Let $F = f^n f^{(k)}/p$ and $G = g^n g^{(k)}/p$.

Note that f and g are transcendental meromorphic functions, so $p(z)$ is a small function with respect to both $f^n f^{(k)}$ and $g^n g^{(k)}$. Also F, G share $(1, m)$ except for the zeros of $p(z)$ and f, g share $(\infty, 0)$.

Case 1. Let $H \not\equiv 0$.

From (3.1) it can be easily calculated that the possible poles of H occur at

(i) multiple zeros of F and G,
(ii) those 1 points of F and G whose multiplicities are different,
(iii) those poles of F and G whose multiplicities are different,
(iv) the zeros of $F'(G')$ which are not zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned}(4.1)\qquad N(r,\infty;H) \leqslant{}& \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F|{\geqslant}2)\\ &+ \overline{N}(r,0;G|{\geqslant}2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'),\end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $p(z_0) \neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So,

$$(4.2)\qquad N(r,1;F|{=}1) \leqslant N(r,0;H) \leqslant N(r,\infty;H) + S(r,f) + S(r,g).$$

Using (4.1) and (4.2) we get

$$\begin{aligned}(4.3)\quad \overline{N}(r,1;F) &\leqslant N(r,1;F|{=}1) + \overline{N}(r,1;F|{\geqslant}2)\\ &\leqslant \overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;F|{\geqslant}2) + \overline{N}(r,0;G|{\geqslant}2) + \overline{N}_*(r,1;F,G)\\ &\quad + \overline{N}(r,1;F|{\geqslant}2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\\ &\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,0;F|{\geqslant}2) + \overline{N}(r,0;G|{\geqslant}2) + \overline{N}_*(r,1;F,G)\\ &\quad + \overline{N}(r,1;F|{\geqslant}2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g).\end{aligned}$$

Now in view of Lemmas 3.2 and 3.15 for $k_1 = m$ we get

$$\begin{aligned}(4.4)\quad &\overline{N}_0(r,0;G') + \overline{N}(r,1;F|{\geqslant}2) + \overline{N}_*(r,1;F,G)\\ &\leqslant \overline{N}_0(r,0;G') + \overline{N}(r,1;F|{=}2) + \overline{N}(r,1;F|{=}3) + \ldots + \overline{N}(r,1;F|{=}m)\\ &\quad + \overline{N}_E^{(m+1}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_*(r,1;F,G)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \overline{N}_0(r,0;G') - \overline{N}(r,1;F|{=}3) - \ldots - (m-2)\overline{N}(r,1;F|{=}m)\\
&\quad - (m-1)\overline{N}_L(r,1;F) - m\overline{N}_L(r,1;G) - (m-1)\overline{N}_E^{(m+1}(r,1;F)\\
&\quad + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_*(r,1;F,G)\\
&\leqslant \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G)\\
&\quad - (m-2)\overline{N}_L(r,1;F) - (m-1)\overline{N}_L(r,1;G)\\
&\leqslant N(r,0;G'|G{\neq}\,0) - (m-2)\overline{N}_L(r,1;F) - (m-1)\overline{N}_L(r,1;G)\\
&\leqslant \overline{N}(r,0;G) + \overline{N}(r,\infty;g) - (m-2)\overline{N}_L(r,1;F) - (m-1)\overline{N}_L(r,1;G)\\
&= \overline{N}(r,0;G) + \overline{N}(r,\infty;g) - (m-2)\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G).
\end{aligned}
$$

Hence using (4.3), (4.4) and Lemma 3.1 we get from the second fundamental theorem that

$$
\begin{aligned}
(4.5)\ T(r,F) &\leqslant \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - N_0(r,0;F')\\
&\leqslant 2\overline{N}(r,\infty,f) + N_2(r,0;F) + \overline{N}(r,0;G|{\geqslant}2) + \overline{N}(r,1;F|{\geqslant}2)\\
&\quad + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f) + N_2(r,0;F) + N_2(r,0;G)\\
&\quad - (m-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + N_2(r,0;f^{(k)}) + 2\overline{N}(r,0;g)\\
&\quad + N_2(r,0;g^{(k)}) - (m-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + N(r,0;f^{(k)}) + 2\overline{N}(r,0;g) + N_{k+2}(r,0;g)\\
&\quad + k\overline{N}(r,\infty;g) - (m-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant (3+k)\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;g)\\
&\quad + N(r,0;f^{(k)}) - (m-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).
\end{aligned}
$$

Now using Lemmas 3.8 for $s = k+1$, 3.10 we get from (4.5)

$$
\begin{aligned}
(4.6)\quad (n-1)T(r,f) &\leqslant T(r,F) - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f)\\
&\leqslant (2+k)\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;g)\\
&\quad - (m-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant \frac{(k+1)(k+2)}{(k+1)(n-k)}(N(r,0;f) + N(r,0;g)) + \frac{2}{k+1}(N(r,0;f)\\
&\quad + N(r,0;g)) + N(r,0;g) + \frac{k+2}{n-k}\overline{N}_*(r,1;F,G)\\
&\quad - (m-2)\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\leqslant \frac{(k+5)n + k^2 + k + 4}{(k+1)(n-k)}T(r) + S(r).
\end{aligned}
$$

In a similar way we obtain

$$(n-1)T(r,g) \leqslant \frac{(k+5)n+k^2+k+4}{(k+1)(n-k)}T(r)+S(r). \tag{4.7}$$

Combining (4.6) and (4.7) we see that

$$(n-1)T(r) \leqslant \frac{(k+5)n+k^2+k+4}{(k+1)(n-k)}T(r)+S(r),$$

i.e.,

$$(k+1)(n-k_1)(n-k_2)T(r) \leqslant S(r), \tag{4.8}$$

where

$$k_1 = \frac{k^2+3k+6+\sqrt{(k^2+3k+6)^2+16(k+1)}}{2(k+1)}$$

and

$$k_2 = \frac{k^2+3k+6-\sqrt{(k^2+3k+6)^2+16(k+1)}}{2(k+1)}.$$

Since $k_1 < (k^2+3k+7)/(k+1)$ and $n \geqslant (k^2+3k+7)/(k+1)$, (4.8) leads to a contradiction.

Case 2. Let $H \equiv 0$. Note that $n \geqslant (k^2+3k+7)/(k+1) > (k^2+3k+3)/(k+1)$. Then, theorem follows from Lemmas 3.11, 3.14 and 3.16. $\square$

P r o o f of Theorem 2.2. When $H \not\equiv 0$ we follow the proof of Theorem 2.1 while for $H \equiv 0$ we follow Lemmas 3.5, 3.12 and 3.16. So we omit the detailed proof. $\square$

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