ON THE STRONGLY AMBIGUOUS CLASSES OF SOME BIQUADRATIC NUMBER FIELDS

Abdelmalek Azizi, Oujda, Abdelkader Zekhnini, Nador,
Mohammed Taous, Errachidia

Received February 23, 2014. First published June 20, 2016.
Communicated by Radomír Halaš

Abstract. We study the capitulation of 2-ideal classes of an infinite family of imaginary bicyclic biquadratic number fields consisting of fields $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$, where $\mathrm{i} = \sqrt{-1}$ and $p \equiv -q \equiv 1 \pmod 4$ are different primes. For each of the three quadratic extensions $\mathbb{K}/\Bbbk$ inside the absolute genus field $\Bbbk^{(*)}$ of $\Bbbk$, we determine a fundamental system of units and then compute the capitulation kernel of $\mathbb{K}/\Bbbk$. The generators of the groups $\mathrm{Am}_s(\Bbbk/F)$ and $\mathrm{Am}(\Bbbk/F)$ are also determined from which we deduce that $\Bbbk^{(*)}$ is smaller than the relative genus field $(\Bbbk/\mathbb{Q}(\mathrm{i}))^*$. Then we prove that each strongly ambiguous class of $\Bbbk/\mathbb{Q}(\mathrm{i})$ capitulates already in $\Bbbk^{(*)}$, which gives an example generalizing a theorem of Furuya (1977).

Keywords: absolute genus field; relative genus field; fundamental system of units; 2-class group; capitulation; quadratic field; biquadratic field; multiquadratic CM-field

MSC 2010: 11R11, 11R16, 11R20, 11R27, 11R29, 11R37

1. Introduction

Let k be an algebraic number field and let $\mathrm{Cl}_2(k)$ denote its 2-class group, that is the 2-Sylow subgroup of the ideal class group, $\mathrm{Cl}(k)$, of k. We denote by $k^{(*)}$ the absolute genus field of k. Suppose F is a finite extension of k, then we say that an ideal class of k capitulates in F if it is in the kernel of the homomorphism

$$J_F\colon \mathrm{Cl}(k) \longrightarrow \mathrm{Cl}(F)$$

induced by the extension of ideals from k to F. An important problem in Number Theory is to determine explicitly the kernel of J_F, which is usually called the capitulation kernel. If F is the relative genus field of a cyclic extension K/k, which

DOI: 10.21136/MB.2016.0022-14

we denote by $(K/k)^*$ and that is the maximal unramified extension of K which is obtained by composing K and an abelian extension over k, Terada states in [19] that all ambiguous ideal classes of K/k, which are classes of K fixed under any element of $\mathrm{Gal}(K/k)$, capitulate in $(K/k)^*$. If F is the absolute genus field of an abelian extension $K/\mathbb{Q}$, then Furuya confirms in [9] that every strongly ambiguous class of $K/\mathbb{Q}$ which is an ambiguous ideal class containing at least one ideal invariant under any element of $\mathrm{Gal}(K/\mathbb{Q})$, capitulates in F. In this paper, we construct a family of number fields k for which $\mathrm{Cl}_2(k) \simeq (2,2,2)$ and all the strongly ambiguous classes of $k/\mathbb{Q}(\mathrm{i})$ capitulate in $k^{(*)} \subsetneq (k/\mathbb{Q}(\mathrm{i}))^*$.

Let p and q be different primes, $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$ and let $\mathbb{K}$ be an unramified quadratic extension of $\Bbbk$ that is abelian over $\mathbb{Q}$. Denote by $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i}))$ the group of the strongly ambiguous classes of $\Bbbk/\mathbb{Q}(\mathrm{i})$. In [1], the first author studied the capitulation problem in $\mathbb{K}/\Bbbk$ assuming $p \equiv -q \equiv 1 \pmod 4$ and $\mathrm{Cl}_2(\Bbbk) \simeq (2,2)$. On the other hand, in [4], we have dealt with the same problem assuming $p \equiv q \equiv 1 \pmod 4$, and in [5], we have studied the capitulation problem of the 2-ideal classes of $\Bbbk$ in its fourteen unramified extensions, within the first Hilbert 2-class field of $\Bbbk$, assuming $p \equiv q \equiv 5 \pmod 8$. It is the purpose of the present article to pursue this research project further for all types of $\mathrm{Cl}_2(\Bbbk)$, assuming $p \equiv -q \equiv 1 \pmod 4$, we compute the capitulation kernel of $\mathbb{K}/\Bbbk$ and deduce that $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) \subseteq \ker J_{\Bbbk^{(*)}}$. As an application we will determine these kernels when $\mathrm{Cl}_2(\Bbbk)$ is of type $(2,2,2)$.

Let k be a number field. During this paper, we adopt the following notation:

- $p \equiv -q \equiv 1 \pmod 4$ are different primes.
- $\Bbbk$: denotes the field $\mathbb{Q}(\sqrt{2pq}, \sqrt{-1})$.
- κ_K: the capitulation kernel of an unramified extension $K/\Bbbk$.
- $\mathcal{O}_k$: the ring of integers of k.
- E_k: the unit group of $\mathcal{O}_k$.
- W_k: the group of roots of unity contained in k.
- F.S.U.: the fundamental system of units.
- k^+: the maximal real subfield of k, if it is a CM-field.
- $Q_k = [E_k\colon W_k E_{k^+}]$ is Hasse's unit index, if k is a CM-field.
- $q(k/\mathbb{Q}) = \left[E_k\colon \prod_{i=1}^{s} E_{k_i}\right]$ is the unit index of k, if k is multiquadratic, where $k_1, \dots, k_s$ are the quadratic subfields of k.
- $k^{(*)}$: the absolute genus field of k.
- $\mathrm{Cl}_2(k)$: the 2-class group of k.
- $\mathrm{i} = \sqrt{-1}$.
- ε_m: the fundamental unit of $\mathbb{Q}(\sqrt{m})$, if $m > 1$ is a square-free integer.
- $N(a)$: denotes the absolute norm of a number a, i.e. $N_{k/\mathbb{Q}}(a)$, where $k = \mathbb{Q}(\sqrt{a})$.
- $x \pm y$ means $x + y$ or $x - y$ for numbers x and y.

2. Preliminary results

Let us first collect some results that will be useful in what follows.

Let k_j, $1 \leqslant j \leqslant 3$ be the three real quadratic subfields of a biquadratic bicyclic real number field K_0 and let $\varepsilon_j > 1$ be the fundamental unit of k_j. Since $\alpha^2 N_{K_0/\mathbb{Q}}(\alpha) = \prod_{j=1}^{3} N_{K_0/k_j}(\alpha)$ for any $\alpha \in K_0$, the square of any unit of K_0 is in the group generated by the ε_j's, $1 \leqslant j \leqslant 3$. Hence, to determine a fundamental system of units of K_0 it suffices to determine which of the units in $B := \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3, \varepsilon_1\varepsilon_2\varepsilon_3\}$ are squares in K_0 (see [20] or [16]). Put $K = K_0(\mathrm{i})$, then to determine a F.S.U. of K, we will use the following result (see [2], page 18) that the first author has deduced from a theorem of Hasse [11], Section 21, Satz 15.

Lemma 2.1. *Let $n \geqslant 2$ be an integer and ξ_n a 2^n-th primitive root of unity, then*

$$\xi_n = \frac{1}{2}(\mu_n + \lambda_n \mathrm{i}), \quad \text{where } \mu_n = \sqrt{2+\mu_{n-1}},\ \lambda_n = \sqrt{2-\mu_{n-1}},$$
$$\mu_2 = 0,\ \lambda_2 = 2 \text{ and } \mu_3 = \lambda_3 = \sqrt{2}.$$

Let n_0 be the greatest integer such that ξ_{n_0} is contained in K, $\{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}$ a F.S.U. of K_0 and ε a unit of K_0 such that $(2+\mu_{n_0})\varepsilon$ is a square in K_0 (if it exists). Then a F.S.U. of K is one of the following systems:

1. *$\{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}$ if ε does not exist,*
2. *$\{\varepsilon'_1, \varepsilon'_2, \sqrt{\xi_{n_0}\varepsilon}\}$ if ε exists; in this case $\varepsilon = {\varepsilon'_1}^{i_1}{\varepsilon'_2}^{i_2}\varepsilon'_3$, where $i_1, i_2 \in \{0,1\}$ (up to a permutation).*

Lemma 2.2 ([1], Lemma 5). *Let $d > 1$ be a square-free integer and $\varepsilon_d = x + y\sqrt{d}$, where x, y are integers or semi-integers. If $N(\varepsilon_d) = 1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$ and $2d(x-1)$ are not squares in $\mathbb{Q}$.*

Lemma 2.3 ([1], Lemma 6). *Let $q \equiv -1 \pmod 4$ be a prime and $\varepsilon_q = x + y\sqrt{q}$ the fundamental unit of $\mathbb{Q}(\sqrt{q})$. Then x is an even integer, $x \pm 1$ is a square in $\mathbb{N}$ and $2\varepsilon_q$ is a square in $\mathbb{Q}(\sqrt{q})$.*

Lemma 2.4 ([1], Lemma 7). *Let p be an odd prime and $\varepsilon_{2p} = x + y\sqrt{2p}$. If $N(\varepsilon_{2p}) = 1$, then $x \pm 1$ is a square in $\mathbb{N}$ and $2\varepsilon_{2p}$ is a square in $\mathbb{Q}(\sqrt{2p})$.*

Lemma 2.5 ([2], page 19, Section 3. (1)). *Let $d > 2$ be a square-free integer and $k = \mathbb{Q}(\sqrt{d}, \mathrm{i})$, put $\varepsilon_d = x + y\sqrt{d}$.*

(1) *If* $N(\varepsilon_d) = -1$, *then* $\{\varepsilon_d\}$ *is a F.S.U. of* k.
(2) *If* $N(\varepsilon_d) = 1$, *then* $\{\sqrt{\mathrm{i}\varepsilon_d}\}$ *is a F.S.U. of* k *if and only if* $x \pm 1$ *is a square in* $\mathbb{N}$, *i.e.* $2\varepsilon_d$ *is a square in* $\mathbb{Q}(\sqrt{d})$. *Else* $\{\varepsilon_d\}$ *is a F.S.U. of* k (*this result is also in* [14]).

3. F.S.U. of some CM-fields

As $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$, so $\Bbbk$ admits three unramified quadratic extensions that are abelian over $\mathbb{Q}$, which are $\mathbb{K}_1 = \Bbbk(\sqrt{p}) = \mathbb{Q}(\sqrt{p}, \sqrt{2q}, \mathrm{i})$, $\mathbb{K}_2 = \Bbbk(\sqrt{q}) = \mathbb{Q}(\sqrt{q}, \sqrt{2p}, \mathrm{i})$ and $\mathbb{K}_3 = \Bbbk(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{pq}, \mathrm{i})$. Put $\varepsilon_{2pq} = x + y\sqrt{2pq}$. The first author gave in [1] the F.S.U.'s of these three fields, if $2\varepsilon_{2pq}$ is not a square in $\mathbb{Q}(\sqrt{2pq})$, i.e. $x+1$ and $x-1$ are not squares in $\mathbb{N}$. In what follows, we determine the F.S.U.'s of $\mathbb{K}_j$, $1 \leqslant j \leqslant 3$, in all cases.

3.1. F.S.U. of the field $\mathbb{K}_1$. Let $\mathbb{K}_1 = \Bbbk(\sqrt{p}) = \mathbb{Q}(\sqrt{p}, \sqrt{2q}, \mathrm{i})$.

Proposition 3.1. *Keep the previous notations. Then* $Q_{\mathbb{K}_1} = 2$ *and just one of the following two cases holds:*
(1) *If* $x \pm 1$ *or* $p(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2q}\varepsilon_{2pq}}\}$ *is a F.S.U. of* $\mathbb{K}_1^+$ *and that of* $\mathbb{K}_1$ *is* $\{\varepsilon_p, \sqrt{\mathrm{i}\varepsilon_{2q}}, \sqrt{\varepsilon_{2q}\varepsilon_{2pq}}\}$.
(2) *If* $2p(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2pq}}\}$ *is a F.S.U. of* $\mathbb{K}_1^+$ *and that of* $\mathbb{K}_1$ *is* $\{\varepsilon_p, \sqrt{\mathrm{i}\varepsilon_{2q}}, \sqrt{\varepsilon_{2pq}}\}$.

Proof. As $p \equiv 1 \pmod 4$, then ε_p is not a square in $\mathbb{K}_1^+$; but ε_{2pq} and $\varepsilon_{2q}\varepsilon_{2pq}$ can be. Moreover, according to Lemma 2.4, $2\varepsilon_{2q}$ is a square in $\mathbb{Q}(\sqrt{2q})$. On the other hand, we know that $N(\varepsilon_{2pq}) = 1$, hence $(x \pm 1)(x \mp 1) = 2pqy^2$. Hence, by Lemma 2.2 and according to the decomposition uniqueness in $\mathbb{Z}$, there are three possibilities: $x \pm 1$ or $p(x \pm 1)$ or $2p(x \pm 1)$ is a square in $\mathbb{N}$, the only remaining case is the first one. If $x \pm 1$ is a square in $\mathbb{N}$ (for the other cases see [1]), then, by Lemma 2.5, $2\varepsilon_{2pq}$ is a square in $\mathbb{K}_1$. Consequently, $\sqrt{\varepsilon_{2q}\varepsilon_{2pq}} \in \mathbb{K}_1^+$; hence $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2q}\varepsilon_{2pq}}\}$ is a F.S.U. of $\mathbb{K}_1^+$, and since $2\varepsilon_{2q}$ is a square in $\mathbb{K}_1^+$, so Lemma 2.1 yields that $\{\varepsilon_p, \sqrt{\mathrm{i}\varepsilon_{2q}}, \sqrt{\varepsilon_{2q}\varepsilon_{2pq}}\}$ is a F.S.U. of $\mathbb{K}_1$. Thus $Q_{\mathbb{K}_1} = 2$. □

3.2. F.S.U. of the field $\mathbb{K}_2$. Let $\mathbb{K}_2 = \Bbbk(\sqrt{q}) = \mathbb{Q}(\sqrt{q}, \sqrt{2p}, \mathrm{i})$.

Proposition 3.2. *Keep the previous notation. Then* $Q_{\mathbb{K}_2} = 2$.
(1) *Assume that* $N(\varepsilon_{2p}) = 1$. *Then just one of the following two cases holds.*
    (i) *If* $x \pm 1$ *or* $2p(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\{\sqrt{\varepsilon_q\varepsilon_{2p}}, \sqrt{\varepsilon_q\varepsilon_{2pq}}, \sqrt{\varepsilon_{2p}\varepsilon_{2pq}}\}$ *is a F.S.U. of* $\mathbb{K}_2^+$ *and that of* $\mathbb{K}_2$ *is* $\{\sqrt{\mathrm{i}\varepsilon_q}, \sqrt{\mathrm{i}\varepsilon_{2p}}, \sqrt{\mathrm{i}\varepsilon_{2pq}}\}$.

(ii) *If* $p(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\{\varepsilon_q, \sqrt{\varepsilon_q\varepsilon_{2p}}, \sqrt{\varepsilon_{2pq}}\}$ *is a F.S.U. of* $\mathbb{K}_2^+$ *and that of* $\mathbb{K}_2$ *is* $\{\sqrt{\mathrm{i}\varepsilon_q}, \sqrt{\mathrm{i}\varepsilon_{2p}}, \sqrt{\varepsilon_{2pq}}\}$.

(2) *Assume that* $N(\varepsilon_{2p}) = -1$. *Then just one of the following two cases holds.*

(i) *If* $x \pm 1$ *or* $2p(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\{\varepsilon_q, \varepsilon_{2p}, \sqrt{\varepsilon_q\varepsilon_{2pq}}\}$ *is a F.S.U. of* $\mathbb{K}_2^+$ *and that of* $\mathbb{K}_2$ *is* $\{\sqrt{\mathrm{i}\varepsilon_q}, \varepsilon_{2p}, \sqrt{\varepsilon_q\varepsilon_{2pq}}\}$.

(ii) *If* $p(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\{\varepsilon_q, \varepsilon_{2p}, \sqrt{\varepsilon_{2pq}}\}$ *is a F.S.U. of* $\mathbb{K}_2^+$ *and that of* $\mathbb{K}_2$ *is* $\{\sqrt{\mathrm{i}\varepsilon_q}, \varepsilon_{2p}, \sqrt{\varepsilon_{2pq}}\}$.

Proof. According to Lemma 2.5, if $x \pm 1$ is a square in $\mathbb{N}$, then $2\varepsilon_{2pq}$ is a square in $\mathbb{Q}(\sqrt{2pq})$. Moreover, Lemma 2.3 implies that $2\varepsilon_q$ is also a square in $\mathbb{Q}(\sqrt{q})$.

(1) If $N(\varepsilon_{2p}) = 1$, then Lemma 2.4 yields that $2\varepsilon_{2p}$ is a square in $\mathbb{Q}(\sqrt{2p})$, thus $\varepsilon_{2p}\varepsilon_{2pq}$, $\varepsilon_q\varepsilon_{2pq}$ and $\varepsilon_q\varepsilon_{2p}$ are squares in $\mathbb{K}_2^+$, which gives the F.S.U. of $\mathbb{K}_2^+$, and that of $\mathbb{K}_2$ is deduced by Lemma 2.1.

(2) If $N(\varepsilon_{2p}) = -1$, then $\varepsilon_q\varepsilon_{2pq}$ is a square in $\mathbb{K}_2^+$, which gives the F.S.U. of $\mathbb{K}_2^+$, and that of $\mathbb{K}_2$ is deduced by Lemma 2.1.

For the other cases see [1]. □

3.3. F.S.U. of the field $\mathbb{K}_3$.

Let $\mathbb{K}_3 = \Bbbk(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{pq}, \mathrm{i})$.

Proposition 3.3. *Put* $\varepsilon_{pq} = a + b\sqrt{pq}$, *where* a *and* b *are in* $\mathbb{Z}$.

(1) *If both of* $x \pm 1$ *and* $a \pm 1$ *are squares in* $\mathbb{N}$, *then*

(i) *if* $Q_{\mathbb{K}_3} = 1$, *then* $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\}$ *is a F.S.U. of both* $\mathbb{K}_3^+$ *and* $\mathbb{K}_3$.

(ii) *if* $Q_{\mathbb{K}_3} = 2$, *then* $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\}$ *is a F.S.U. of* $\mathbb{K}_3^+$ *and that of* $\mathbb{K}_3$ *is* $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\xi\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}}\}$, *where* ξ *is an 8-th root of unity.*

(2) *If* $x \pm 1$ *is a square in* $\mathbb{N}$ *and* $a+1$, $a-1$ *are not, then* $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{2pq}}\}$ *is a F.S.U. of both* $\mathbb{K}_3^+$ *and* $\mathbb{K}_3$; *hence* $Q_{\mathbb{K}_3} = 1$.

(3) *If* $a \pm 1$ *is a square in* $\mathbb{N}$ *and* $x+1$, $x-1$ *are not, then* $\{\varepsilon_2, \varepsilon_{2pq}, \sqrt{\varepsilon_{pq}}\}$ *is a F.S.U. of both* $\mathbb{K}_3^+$ *and* $\mathbb{K}_3$; *hence* $Q_{\mathbb{K}_3} = 1$.

(4) *If* $x+1$, $x-1$, $a+1$ *and* $a-1$ *are not squares in* $\mathbb{N}$, *then* $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ *is a F.S.U. of both* $\mathbb{K}_3^+$ *and* $\mathbb{K}_3$; *hence* $Q_{\mathbb{K}_3} = 1$.

Before proving this proposition, we quote the following result.

Remark 3.4. Keep the notation and hypotheses of Proposition 3.3.

(1) If at most one of the numbers $x+1$, $x-1$, $a+1$ and $a-1$ is a square in $\mathbb{N}$, then according to [1], page 391, Remark 13, $\mathbb{K}_3^+$ and $\mathbb{K}_3$ have the same F.S.U.

(2) From [13], page 348, Theorem 2, if both of $x \pm 1$ and $a \pm 1$ are squares in $\mathbb{N}$, then the unit index of $\mathbb{K}_3$ is 1 or 2.

Proof. We know that $N(\varepsilon_2) = -1$ and $N(\varepsilon_{pq}) = N(\varepsilon_{2pq}) = 1$. Moreover, $(2+\sqrt{2})\varepsilon_2^i\varepsilon_{pq}^j\varepsilon_{2pq}^k$ cannot be a square in $\mathbb{K}_3^+$ for all i, j and k of $\{0,1\}$; as otherwise with some $\alpha \in \mathbb{K}_3^+$ we would have $\alpha^2 = (2+\sqrt{2})\varepsilon_2^i\varepsilon_{pq}^j\varepsilon_{2pq}^k$, so $(N_{\mathbb{K}_3^+/\mathbb{Q}(\sqrt{pq})}(\alpha))^2 = 2(-1)^i\varepsilon_{pq}^{2j}$, yielding that $\sqrt{\pm 2} \in \mathbb{Q}(\sqrt{pq})$, which is absurd.

As $a^2 - 1 = pqb^2$, so by Lemma 2.2 and according to the decomposition uniqueness in $\mathbb{Z}$, there are three possible cases: $a \pm 1$ or $p(a \pm 1)$ or $2p(a \pm 1)$ is a square in $\mathbb{N}$.

(a) If $a \pm 1$ is a square in $\mathbb{N}$, then there exist b_1 and b_2 in $\mathbb{N}$ with $b = b_1b_2$ such that

$$\begin{cases} a \pm 1 = b_1^2, \\ a \mp 1 = pqb_2^2, \end{cases} \quad \text{hence} \quad \sqrt{\varepsilon_{pq}} = \tfrac{1}{2}(b_1\sqrt{2} + b_2\sqrt{2pq}) \in \mathbb{K}_3^+.$$

(b) If $p(a \pm 1)$ is a square in $\mathbb{N}$, then there exist b_1 and b_2 in $\mathbb{N}$ with $b = b_1b_2$ such that

$$\begin{cases} a \pm 1 = pb_1^2, \\ a \mp 1 = qb_2^2, \end{cases} \quad \text{hence} \quad \begin{cases} \sqrt{\varepsilon_{pq}} = \tfrac{1}{2}(b_1\sqrt{2p} + b_2\sqrt{2q}) \notin \mathbb{K}_3^+, \\ \sqrt{p\varepsilon_{pq}} \in \mathbb{K}_3^+ \quad \text{and} \quad \sqrt{q\varepsilon_{pq}} \in \mathbb{K}_3^+. \end{cases}$$

(c) If $2p(a \pm 1)$ is a square in $\mathbb{N}$, then there exist b_1 and b_2 in $\mathbb{N}$ with $b = 2b_1b_2$ such that

$$\begin{cases} a \pm 1 = 2pb_1^2, \\ a \mp 1 = 2qb_2^2, \end{cases} \quad \text{hence} \quad \begin{cases} \sqrt{\varepsilon_{pq}} = b_1\sqrt{p} + b_2\sqrt{q} \notin \mathbb{K}_3^+; \\ \sqrt{p\varepsilon_{pq}} \in \mathbb{K}_3^+ \quad \text{and} \quad \sqrt{q\varepsilon_{pq}} \in \mathbb{K}_3^+. \end{cases}$$

Similarly, we get:

(a′) If $x \pm 1$ is a square in $\mathbb{N}$, then $\sqrt{\varepsilon_{2pq}} \in \mathbb{K}_3^+$.

(b′) If $p(x \pm 1)$ is a square in $\mathbb{N}$, then $\sqrt{\varepsilon_{2pq}} \notin \mathbb{K}_3^+$, $\sqrt{p\varepsilon_{2pq}} \in \mathbb{K}_3^+$ and $\sqrt{q\varepsilon_{2pq}} \in \mathbb{K}_2^+$.

(c′) If $2p(x \pm 1)$ is a square in $\mathbb{N}$, then $\sqrt{\varepsilon_{2pq}} \notin \mathbb{K}_3^+$, $\sqrt{p\varepsilon_{2pq}} \in \mathbb{K}_2^+$ and $\sqrt{q\varepsilon_{2pq}} \in \mathbb{K}_2^+$.

Consequently, we find:

(1) If $a \pm 1$ and $x \pm 1$ are squares in $\mathbb{N}$, then $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\}$ is a F.S.U. of $\mathbb{K}_3^+$.

(i) If $Q_{\mathbb{K}_3} = 1$, then $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\}$ is also a F.S.U. of $\mathbb{K}_3$.

(ii) If $Q_{\mathbb{K}_3} = 2$, then, according to [13], $\mathbb{K}_3^+(\sqrt{2+\sqrt{2}}) = \mathbb{K}_3^+\big(\sqrt{\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}}\big)$, so there exists $\alpha \in \mathbb{K}_3^+$ such that $2 + \sqrt{2} = \alpha^2\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$. This implies that $(2+\sqrt{2})\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}$ is a square in $\mathbb{K}_3^+$. Hence Lemma 2.1 yields that $\{\varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\xi\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}}\}$ is a F.S.U. of $\mathbb{K}_3$, where ξ is an 8-th root of unity.

(2) If $x \pm 1$ is a square in $\mathbb{N}$ and $a+1$, $a-1$ are not, then $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{2pq}}\}$ is a F.S.U. of $\mathbb{K}_3^+$ and, by Remark 3.4, of $\mathbb{K}_3$.

(3) If $a \pm 1$ is a square in $\mathbb{N}$ and $x+1$, $x-1$ are not, then $\{\varepsilon_2, \varepsilon_{2pq}, \sqrt{\varepsilon_{pq}}\}$ is a F.S.U. of $\mathbb{K}_3^+$ and, by Remark 3.4, of $\mathbb{K}_3$.

(4) If $x+1$, $x-1$, $a+1$ and $a-1$ are not squares in $\mathbb{N}$, then $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ is a F.S.U. of $\mathbb{K}_3^+$ and, by Remark 3.4, of $\mathbb{K}_3$. $\square$

4. The ambiguous classes of $\Bbbk/\mathbb{Q}(\mathrm{i})$

Let $F = \mathbb{Q}(\mathrm{i})$ and $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$. We denote by $\mathrm{Am}(\Bbbk/F)$ the group of the ambiguous classes of $\Bbbk/F$ and by $\mathrm{Am}_s(\Bbbk/F)$ the subgroup of $\mathrm{Am}(\Bbbk/F)$ generated by the strongly ambiguous classes. As $p \equiv 1 \pmod 4$, so there exist e and f in $\mathbb{N}$ such that $p = e^2 + 4f^2 = \pi_1\pi_2$. Put $\pi_1 = e + 2\mathrm{i}f$ and $\pi_2 = e - 2\mathrm{i}f$. Let $\mathcal{H}_j$ and $\mathcal{H}_0$, respectively, be the prime ideal of $\Bbbk$ above π_j and $1+\mathrm{i}$, $j \in \{1,2\}$. It is easy to see that $\mathcal{H}_j^2 = (\pi_j)$ and $\mathcal{H}_0^2 = (1+\mathrm{i})$. Therefore $[\mathcal{H}_j] \in \mathrm{Am}_s(\Bbbk/F)$ for all $j \in \{0,1,2\}$. Keep the notation $\varepsilon_{2pq} = x + y\sqrt{2pq}$. In this section, we will determine generators of $\mathrm{Am}_s(\Bbbk/F)$ and $\mathrm{Am}(\Bbbk/F)$. Let us first prove the following result.

Lemma 4.1. *Consider the prime ideals $\mathcal{H}_j$ of $\Bbbk$, $0 \leqslant j \leqslant 2$.*

(1) *If $x \pm 1$ is a square in $\mathbb{N}$, then $|\langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2]\rangle| = 8$.*
(2) *Else, $[\mathcal{H}_1] = [\mathcal{H}_2]$ and $|\langle[\mathcal{H}_0], [\mathcal{H}_1]\rangle| = 4$.*

Proof. Since $\mathcal{H}_0^2 = (1+\mathrm{i})$, $\mathcal{H}_l^2 = (\pi_l)$ and $(\mathcal{H}_0\mathcal{H}_l)^2 = ((1+\mathrm{i})\pi_l) = ((e \mp 2f) + \mathrm{i}(e \pm 2f))$, where $1 \leqslant l \leqslant 2$, and since also $\sqrt{2} \notin \mathbb{Q}(\sqrt{2pq})$, $\sqrt{e^2 + (2f)^2} = \sqrt{p} \notin \mathbb{Q}(\sqrt{2pq})$ and $\sqrt{(e \mp 2f)^2 + (e \pm 2f)^2} = \sqrt{2p} \notin \mathbb{Q}(\sqrt{2pq})$, so according to [6], Proposition 1, $\mathcal{H}_0$, $\mathcal{H}_l$ and $\mathcal{H}_0\mathcal{H}_l$ are not principal in $\Bbbk$.

(1) If $x \pm 1$ is a square in $\mathbb{N}$, then $p(x+1)$, $p(x-1)$, $2p(x+1)$ and $2p(x-1)$ are not squares in $\mathbb{N}$. Moreover, $(\mathcal{H}_1\mathcal{H}_2)^2 = (p)$, hence according to [6], Proposition 2, $\mathcal{H}_1\mathcal{H}_2$ is not principal in $\Bbbk$, and the result follows.

(2) If $x+1$ and $x-1$ are not squares in $\mathbb{N}$, then $p(x \pm 1)$ or $2p(x \pm 1)$ is a square in $\mathbb{N}$; as $(\mathcal{H}_1\mathcal{H}_2)^2 = (p)$, hence according to [6], Proposition 2, $\mathcal{H}_1\mathcal{H}_2$ is principal in $\Bbbk$. This completes the proof. $\square$

Determine now the generators of $\mathrm{Am}_s(\Bbbk/F)$ and $\mathrm{Am}(\Bbbk/F)$. According to the ambiguous class number formula (see [8]), the genus number, $[(\Bbbk/F)^*\colon \Bbbk]$, is given by

$$|\mathrm{Am}(\Bbbk/F)| = [(\Bbbk/F)^*\colon \Bbbk] = \frac{h(F)2^{t-1}}{[E_F\colon E_F \cap N_{\Bbbk/F}(\Bbbk^\times)]}, \tag{4.1}$$

where $h(F)$ is the class number of F and t is the number of finite and infinite primes of F ramified in $\Bbbk/F$. Moreover, as the class number of F is equal to 1, the formula (4.1) yields that

$$|\mathrm{Am}(\Bbbk/F)| = [(\Bbbk/F)^*\colon \Bbbk] = 2^r, \tag{4.2}$$

where $r = \operatorname{rank}\mathrm{Cl}_2(\Bbbk) = t - e - 1$ and $2^e = [E_F\colon E_F \cap N_{\Bbbk/F}(\Bbbk^\times)]$ (see for example [17]). The relation between $|\mathrm{Am}(\Bbbk/F)|$ and $|\mathrm{Am}_s(\Bbbk/F)|$ is given by the following formula (see for example [15]):

$$\frac{|\mathrm{Am}(\Bbbk/F)|}{|\mathrm{Am}_s(\Bbbk/F)|} = [E_F \cap N_{\Bbbk/F}(\Bbbk^{\times})\colon\ N_{\Bbbk/F}(E_{\Bbbk})]. \tag{4.3}$$

To continue, we need the following lemma.

Lemma 4.2. *Let* $p \equiv -q \equiv 1 \pmod 4$ *be different primes,* $F = \mathbb{Q}(\mathrm{i})$ *and* $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$.

(1) *If* $p \equiv 1 \pmod 8$, *then* i *is a norm in* $\Bbbk/F$.

(2) *If* $p \equiv 5 \pmod 8$, *then* i *is not a norm in* $\Bbbk/F$.

Proof. Let $\mathfrak{p}$ be a prime ideal of $F = \mathbb{Q}(\mathrm{i})$ such that $\mathfrak{p} \neq 2_F$, where 2_F is the prime ideal of F above 2. Then the Hilbert symbol yields that $((2pq, \mathrm{i})/\mathfrak{p}) = ((pq, \mathrm{i})/\mathfrak{p})$, since $2\mathrm{i} = (1+\mathrm{i})^2$. Hence, by Hilbert symbol properties and according to [10], page 205, we get:

- ▷ If $\mathfrak{p}$ is not above p and q, then $v_{\mathfrak{p}}(pq) = 0$, thus $((pq, \mathrm{i})/\mathfrak{p}) = 1$.
- ▷ If $\mathfrak{p}$ lies above p, then $v_{\mathfrak{p}}(pq) = 1$, so $((pq, \mathrm{i})/\mathfrak{p}) = (\mathrm{i}/\mathfrak{p}) = (2/p)$, indeed $(2/p)(\mathrm{i}/\mathfrak{p}) = (2/\mathfrak{p})(\mathrm{i}/\mathfrak{p}) = (2\mathrm{i}/\mathfrak{p}) = 1$.
- ▷ If $\mathfrak{p}$ lies above q, then $v_{\mathfrak{p}}(pq) = 1$, so $((pq, \mathrm{i})/\mathfrak{p}) = (\mathrm{i}/\mathfrak{p}) = (N_{F/\mathbb{Q}}(\mathrm{i})/q) = (1/q) = 1$, since q remained inert in $F/\mathbb{Q}$.

So for every prime ideal $\mathfrak{p} \in F$ and by the product formula for the Hilbert symbol, we deduce that $((pq, \mathrm{i})/\mathfrak{p}) = 1$, hence:

(1) If $p \equiv 1 \pmod 8$, then i is a norm in $\Bbbk/F$.

(2) If $p \equiv 5 \pmod 8$, then i is not a norm in $\Bbbk/F$. □

Proposition 4.3. *Let* $(\Bbbk/F)^*$ *denote the relative genus field of* $\Bbbk/F$.

(1) $\Bbbk^{(*)} \subseteq (\Bbbk/F)^*$ *and* $[(\Bbbk/F)^*\colon\ \Bbbk^{(*)}] \leqslant 2$.

(2) *Assume* $p \equiv 1 \pmod 8$.

  (i) *If* $x \pm 1$ *is a square in* $\mathbb{N}$, *then* $\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2]\rangle$.

  (ii) *Else, there exists an unambiguous ideal* $\mathcal{I}$ *in* $\Bbbk/\mathbb{Q}(\mathrm{i})$ *of order* 2 *such that* $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle[\mathcal{H}_0], [\mathcal{H}_1]\rangle$ *and* $\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{I}]\rangle$.

(3) *Assume* $p \equiv 5 \pmod 8$, *then neither* $x+1$ *nor* $x-1$ *is a square in* $\mathbb{N}$ *and* $\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle[\mathcal{H}_0], [\mathcal{H}_1]\rangle$.

Proof. (1) As $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$, so $[\Bbbk^{(*)}\colon\ \Bbbk] = 4$. Moreover, according to [17], page 90, Proposition 2, $r = \operatorname{rank}\mathrm{Cl}_2(\Bbbk) = 3$ if $p \equiv 1 \pmod 8$ and $r = \operatorname{rank}\mathrm{Cl}_2(\Bbbk) = 2$ if $p \equiv 5 \pmod 8$, so $[(\Bbbk/F)^*\colon\ \Bbbk] = 4$ or 8. Hence $[(\Bbbk/F)^*\colon\ \Bbbk^{(*)}] \leqslant 2$, and the result follows.

(2) Assume that $p \equiv 1 \pmod 8$, hence i is a norm in $\Bbbk/\mathbb{Q}(\mathrm{i})$, thus formula (4.3) yields that

$$\frac{|\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i}))|}{|\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i}))|} = [E_{\mathbb{Q}(\mathrm{i})} \cap N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(\Bbbk^\times) \colon N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(E_\Bbbk)] = \begin{cases} 1 & \text{if } x \pm 1 \text{ is a square in } \mathbb{N}, \\ 2 & \text{if not,} \end{cases}$$

since in the case when $x \pm 1$ is a square in $\mathbb{N}$, we have $E_\Bbbk = \langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_{2pq}} \rangle$, hence $[E_{\mathbb{Q}(\mathrm{i})} \cap N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(\Bbbk^\times) \colon N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(E_\Bbbk)] = [\langle \mathrm{i} \rangle \colon \langle \mathrm{i} \rangle] = 1$, and if not we have $E_\Bbbk = \langle \mathrm{i}, \varepsilon_{2pq} \rangle$, hence $[E_{\mathbb{Q}(\mathrm{i})} \cap N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(\Bbbk^\times) \colon N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(E_\Bbbk)] = [\langle \mathrm{i} \rangle \colon \langle -1 \rangle] = 2$.

On the other hand, as $p \equiv 1 \pmod 8$, so according to [17], page 90, Proposition 2, $r = 3$. Therefore $|\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i}))| = 2^3$.

(i) If $x \pm 1$ is a square in $\mathbb{N}$, then $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i}))$, hence by Lemma 4.1 we get $\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.

(ii) If $x + 1$ and $x - 1$ are not squares in $\mathbb{N}$, then

$$|\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i}))| = 2|\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i}))| = 8,$$

hence Lemma 4.1 yields that $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$.

Consequently, there exists an unambiguous ideal $\mathcal{I}$ in $\Bbbk/F$ of order 2 such that

$$\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{I}] \rangle.$$

By Chebotarev theorem, $\mathcal{I}$ can always be chosen as a prime ideal of $\Bbbk$ above a prime l in $\mathbb{Q}$, which splits completely in $\Bbbk$. So we can determine $\mathcal{I}$ by using the following lemma.

Lemma 4.4 ([18]). *Let $p_1, p_2, \ldots, p_n$ be distinct primes and for each j, let $e_j = \pm 1$. Then there exist infinitely many primes l such that $(p_j/l) = e_j$ for all j.*

Let $l \equiv 1 \pmod 4$ be a prime satisfying $(2pq/l) = -(q/l) = 1$, then l splits completely in $\Bbbk$. Let $\mathcal{I}$ be a prime ideal of $\Bbbk$ above l; hence $\mathcal{I}$ remained inert in $\mathbb{K}_2$ and $(2p/l) = -1$. We need to prove that $\mathcal{I}$, $\mathcal{H}_0\mathcal{I}$, $\mathcal{H}_1\mathcal{I}$ and $\mathcal{H}_0\mathcal{H}_1\mathcal{I}$ are not principal in $\Bbbk$.

$\triangleright$ As $\mathcal{I}$ remained inert in $\mathbb{K}_2$, so $\varphi_{\mathbb{K}_2/\Bbbk}(\mathcal{I}) \neq 1$, where $\varphi_{\mathbb{K}_2/\Bbbk}$ denotes the Artin map of $\mathbb{K}_2$ over $\Bbbk$; similarly, we have $\varphi_{\mathbb{K}_2/\Bbbk}(\mathcal{H}_1\mathcal{I}) \neq 1$ (note that $(p/q) = 1$, since $p(x \pm 1)$ or $2p(x \pm 1)$ is a square in $\mathbb{N}$). Therefore $\mathcal{I}$ and $\mathcal{H}_1\mathcal{I}$ are not principal in $\Bbbk$.

$\triangleright$ Let us prove that $\mathcal{H}_0\mathcal{I}$ is not principal in $\Bbbk$. For this, we consider the following cases:

(a) Assume $(2/l) = 1$, then $(p/l) = -1$; thus if $(2/q) = -1$, then $\varphi_{\mathbb{K}_3/\Bbbk}(\mathcal{H}_0\mathcal{I}) \neq 1$, and if $(2/q) = 1$, then $\varphi_{\mathbb{K}_1/\Bbbk}(\mathcal{H}_0\mathcal{I}) \neq 1$. Hence $\mathcal{H}_0\mathcal{I}$ is not principal in $\Bbbk$.

(b) Assume now $(2/l) = -1$, hence $(p/l) = 1$. Thus if $(2/q) = 1$, then $\varphi_{\mathbb{K}_2/\Bbbk}(\mathcal{H}_0\mathcal{I}) \neq 1$. If $(2/q) = -1$, so we need the following two quadratic extensions of $\Bbbk$: $\mathbb{K}_4 = \Bbbk(\sqrt{\pi_1})$ and $\mathbb{K}_5 = \Bbbk(\sqrt{2\pi_1}) = \Bbbk(\sqrt{\pi_2 q})$, where $p = e^2 + 16f^2 = \pi_1\pi_2 = (e + 4\mathrm{i}f)(e - 4\mathrm{i}f)$, since $p \equiv 1 \pmod 8$. Note that $\mathbb{K}_4/\Bbbk$ and $\mathbb{K}_5/\Bbbk$ are unramified (see [7]). As $(2/p) = 1$, we have $((1+\mathrm{i})/\pi_1) = ((1+\mathrm{i})/\pi_2)$, hence the quadratic residue symbol implies that

$$\Big(\frac{\pi_1}{\mathcal{H}_0\mathcal{I}}\Big) = \Big(\frac{1+\mathrm{i}}{\pi_1}\Big) = -\Big(\frac{\pi_2 q}{\mathcal{H}_0\mathcal{I}}\Big).$$

Therefore, if $((1+\mathrm{i})/\pi_1) = -1$, then $\varphi_{\mathbb{K}_4/\Bbbk}(\mathcal{H}_0\mathcal{I}) \neq 1$, else we have $\varphi_{\mathbb{K}_5/\Bbbk}(\mathcal{H}_0\mathcal{I}) \neq 1$. Thus $\mathcal{H}_0\mathcal{I}$ is not principal in $\Bbbk$.

By the same argument, we show that $\mathcal{H}_0\mathcal{H}_1\mathcal{I}$ is not principal in $\Bbbk$.

(3) Assume that $p \equiv 5 \pmod 8$, hence i is not a norm in $\Bbbk/\mathbb{Q}(\mathrm{i})$ and $x+1$, $x-1$ are not squares in $\mathbb{N}$, for if $x \pm 1$ is a square in $\mathbb{N}$, then the Legendre symbol implies that

$$1 = \Big(\frac{x \pm 1}{p}\Big) = \Big(\frac{x \mp 1 \pm 2}{p}\Big) = \Big(\frac{2}{p}\Big),$$

which is absurd. Thus $|\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i}))| = 2^2$ and

$$\frac{|\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i}))|}{|\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i}))|} = [E_{\mathbb{Q}(\mathrm{i})} \cap N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(\Bbbk^\times)\colon\ N_{\Bbbk/\mathbb{Q}(\mathrm{i})}(E_\Bbbk)] = 1.$$

Hence by Lemma 4.1 we get $\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle[\mathcal{H}_0], [\mathcal{H}_1]\rangle$. This completes the proof. $\square$

5. Capitulation

In this section, we will determine the classes of $\mathrm{Cl}_2(\Bbbk)$, the 2-class group of $\Bbbk$, that capitulate in $\mathbb{K}_j$ for all $j \in \{1, 2, 3\}$. For this we need the following theorem.

Theorem 5.1 ([12]). *Let K/k be a cyclic extension of prime degree, then the number of classes that capitulate in K/k is $[K\colon k][E_k\colon N_{K/k}(E_K)]$, where E_k and E_K are the unit groups of k and K, respectively.*

Theorem 5.2. *Let $\mathbb{K}_j$, $1 \leqslant j \leqslant 3$ be the three unramified quadratic extensions of $\Bbbk$ defined above.*

(1) *For $j \in \{1, 2\}$ we have:*
   (i) *If $x \pm 1$ is a square in $\mathbb{N}$, then $|\kappa_{\mathbb{K}_j}| = 4$.*
   (ii) *Else $|\kappa_{\mathbb{K}_j}| = 2$.*

(2) *Put* $\varepsilon_{pq} = a + b\sqrt{pq}$ *and let* $Q_{\mathbb{K}_3}$ *denote the unit index of* $\mathbb{K}_3$*.*

(i) *If both* $x \pm 1$ *and* $a \pm 1$ *are squares in* $\mathbb{N}$*, then*

(a) *if* $Q_{\mathbb{K}_3} = 1$*, then* $|\kappa_{\mathbb{K}_3}| = 4$*,*

(b) *if* $Q_{\mathbb{K}_3} = 2$*, then* $|\kappa_{\mathbb{K}_3}| = 2$*.*

(ii) *If one of the four numbers* $x + 1$*,* $x - 1$*,* $a + 1$ *and* $a - 1$ *is a square in* $\mathbb{N}$ *and the others are not, then* $|\kappa_{\mathbb{K}_3}| = 4$*.*

(iii) *If* $x + 1$*,* $x - 1$*,* $a + 1$ *and* $a - 1$ *are not squares in* $\mathbb{N}$*, then* $|\kappa_{\mathbb{K}_3}| = 2$*.*

Proof. (1) According to Proposition 3.1, $E_{\mathbb{K}_1} = \langle \mathrm{i}, \varepsilon_p, \sqrt{\mathrm{i}\varepsilon_{2q}}, \sqrt{\varepsilon_{2q}\varepsilon_{2pq}}\rangle$ or $\langle \mathrm{i}, \varepsilon_p, \sqrt{\mathrm{i}\varepsilon_{2q}}, \sqrt{\varepsilon_{2pq}}\rangle$, so $N_{\mathbb{K}_1/\Bbbk}(E_{\mathbb{K}_1}) = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$. On the other hand, Proposition 3.2 yields that $E_{\mathbb{K}_2} = \langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_q}, \sqrt{\mathrm{i}\varepsilon_{2p}}, \sqrt{\mathrm{i}\varepsilon_{2pq}}\rangle$ or $\langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_q}, \sqrt{\mathrm{i}\varepsilon_{2p}}, \sqrt{\varepsilon_{2pq}}\rangle$ or $\langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_q}, \varepsilon_{2p}, \sqrt{\varepsilon_q\varepsilon_{2pq}}\rangle$ or $\langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_q}, \varepsilon_{2p}, \sqrt{\varepsilon_{2pq}}\rangle$, hence $N_{\mathbb{K}_2/\Bbbk}(E_{\mathbb{K}_2}) = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$.

(i) If $x \pm 1$ is a square in $\mathbb{N}$, then Lemma 2.5 yields that $E_\Bbbk = \langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_{2pq}}\rangle$. Therefore $[E_\Bbbk\colon N_{\mathbb{K}_j/\Bbbk}(E_{\mathbb{K}_j})] = 2$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_j}| = 4$.

(ii) Else $E_\Bbbk = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$, which gives that $[E_\Bbbk\colon N_{\mathbb{K}_j/\Bbbk}(E_{\mathbb{K}_j})] = 1$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_1}| = 2$.

(2) (i) Assume that $x \pm 1$ and $a \pm 1$ are squares in $\mathbb{N}$, so by Lemma 2.5 we get $E_\Bbbk = \langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_{2pq}}\rangle$.

(a) If $Q_{\mathbb{K}_3} = 1$, then Proposition 3.3 implies that $E_{\mathbb{K}_3} = \langle \sqrt{\mathrm{i}}, \varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}\rangle$, hence $N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3}) = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$, from which we deduce that $[E_\Bbbk\colon N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3})] = 2$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_3}| = 4$.

(b) If $Q_{\mathbb{K}_3} = 2$, then Proposition 3.3 implies that

$$E_{\mathbb{K}_3} = \left\langle \sqrt{\mathrm{i}}, \varepsilon_2, \sqrt{\varepsilon_{pq}}, \sqrt{\xi\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}}\right\rangle,$$

thus $N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3}) = \langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_{2pq}}\rangle$, from which we deduce that $[E_\Bbbk\colon N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3})] = 1$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_3}| = 2$.

(ii) If $x \pm 1$ is a square in $\mathbb{N}$ and $a + 1$, $a - 1$ are not, then by Lemma 2.5 we get $E_\Bbbk = \langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_{2pq}}\rangle$. Moreover, Proposition 3.3 implies that $E_{\mathbb{K}_3} = \langle \sqrt{\mathrm{i}}, \varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{2pq}}\rangle$, hence $N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3}) = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$. Therefore $[E_\Bbbk\colon N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3})] = 2$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_3}| = 4$.

If $a \pm 1$ is a square in $\mathbb{N}$ and $x + 1$, $x - 1$ are not, then by Lemma 2.5 we get $E_\Bbbk = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$. Moreover, Proposition 3.3 implies that $E_{\mathbb{K}_3} = \langle \sqrt{\mathrm{i}}, \varepsilon_2, \sqrt{\varepsilon_{pq}}, \varepsilon_{2pq}\rangle$, hence $N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3}) = \langle \mathrm{i}, \varepsilon_{2pq}^2\rangle$. Therefore $[E_\Bbbk\colon N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3})] = 2$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_3}| = 4$.

(iii) Finally, assume that $x + 1$, $x - 1$, $a + 1$ and $a - 1$ are not squares in $\mathbb{N}$, then by Lemma 2.5 we get $E_\Bbbk = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$. Moreover, Proposition 3.3 implies that $E_{\mathbb{K}_3} = \langle \sqrt{\mathrm{i}}, \varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\rangle$, hence $N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3}) = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$. Therefore $[E_\Bbbk\colon N_{\mathbb{K}_3/\Bbbk}(E_{\mathbb{K}_3})] = 1$, and Theorem 5.1 implies that $|\kappa_{\mathbb{K}_3}| = 2$. □

5.1. Capitulation in $\mathbb{K}_1$.

Theorem 5.3. *Keep the notation and hypotheses previously mentioned.*

(1) *If $x \pm 1$ is a square in $\mathbb{N}$, then $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.*
(2) *Else $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1] \rangle$.*

Proof. Let us first prove that $\mathcal{H}_1$ and $\mathcal{H}_2$ capitulate in $\mathbb{K}_1$. As $N(\varepsilon_p) = -1$, we have $s^2 + 4 = t^2 p$, where $\varepsilon_p = (s + t\sqrt{p})/2$, hence $(s - 2\mathrm{i})(s + 2\mathrm{i}) = t^2 p$. According to the decomposition uniqueness in $\mathbb{Z}[\mathrm{i}]$, there exist t_1 and t_2 in $\mathbb{Z}[\mathrm{i}]$ such that:

$$(1) \begin{cases} s \pm 2\mathrm{i} = t_1^2 \pi_1 \\ s \mp 2\mathrm{i} = t_2^2 \pi_2, \end{cases} \quad \text{or} \quad (2) \begin{cases} s \pm 2\mathrm{i} = \mathrm{i} t_1^2 \pi_1 \\ s \mp 2\mathrm{i} = -\mathrm{i} t_2^2 \pi_2, \end{cases} \quad \text{where } t = t_1 t_2.$$

▷ The system (1) implies that $2s = t_1^2\pi_1 + t_2^2\pi_2$. Put $\alpha = (t_1\pi_1 + t_2\sqrt{p})/2$ and $\beta = (t_2\pi_2 + t_1\sqrt{p})/2$. Then α and β are in $\mathbb{K}_1 = \Bbbk(\sqrt{p})$ and we have

$$\begin{aligned} \alpha^2 &= \frac{1}{4}(t_1^2\pi_1^2 + t_2^2 p + 2t_1t_2\pi_1\sqrt{p}) \\ &= \frac{1}{4}\pi_1(t_1^2\pi_1 + t_2^2\pi_2 + 2t\sqrt{p}) && \text{since } p = \pi_1\pi_2 \text{ and } t = t_1t_2 \\ &= \frac{1}{4}\pi_1(2s + 2t\sqrt{p}) && \text{since } 2s = t_1^2\pi_1 + t_2^2\pi_2 \\ &= \pi_1\varepsilon_p && \text{since } \varepsilon_p = \frac{1}{2}(s + t\sqrt{p}). \end{aligned}$$

The same argument yields that $\beta^2 = \pi_2\varepsilon_p$.

Consequently, $(\alpha^2) = (\pi_1) = \mathcal{H}_1^2$ and $(\beta^2) = (\pi_2) = \mathcal{H}_2^2$, hence $(\alpha) = \mathcal{H}_1$ and $(\beta) = \mathcal{H}_2$.

▷ Similarly, system (2) yields that $2s = \mathrm{i}t_1^2\pi_2 - \mathrm{i}t_2^2\pi_1$, hence $\sqrt{2\pi_1\varepsilon_p} = (t_1(1+\mathrm{i})\pi_1 + t_2(1-\mathrm{i})\sqrt{p})/2$ and $\sqrt{2\pi_2\varepsilon_p} = (t_1(1+\mathrm{i})\sqrt{p} + t_2(1-\mathrm{i})\pi_2)/2$ are in $\mathbb{K}_1$. Therefore there exist α and β in $\mathbb{K}_1$ such that $2\pi_1\varepsilon_p = \alpha^2$ and $2\pi_2\varepsilon_p = \beta^2$, thus $(\alpha/(1+\mathrm{i})) = \mathcal{H}_1$ and $(\beta/(1+\mathrm{i})) = \mathcal{H}_2$. This yields that $\mathcal{H}_1$ and $\mathcal{H}_2$ capitulate in $\mathbb{K}_1$.

On the other hand, by Lemma 4.1, $\mathcal{H}_j$, $1 \leqslant j \leqslant 2$, are not principal in $\Bbbk$.

(1) If $x \pm 1$ is a square in $\mathbb{N}$, then Lemma 4.1 yields that $[\mathcal{H}_1\mathcal{H}_2] \neq 1$. Hence the result.

(2) If $x+1$ and $x-1$ are not squares in $\mathbb{N}$, then Lemma 4.1 yields that $[\mathcal{H}_1] = [\mathcal{H}_2]$. This completes the proof. □

5.2. Capitulation in $\mathbb{K}_2$. We need the following two lemmas.

Lemma 5.4. *If $N(\varepsilon_{2p}) = 1$, then*

(1) $p \equiv 1 \pmod 8$,

(2) $2p(x-1)$ *is not a square in* $\mathbb{N}$.

P r o o f. (1) Put $\varepsilon_{2p} = \alpha + \beta\sqrt{2p}$, then, if $N(\varepsilon_{2p}) = 1$, Lemma 2.4 yields that

$$\begin{cases} \alpha \pm 1 = \beta_1^2, \\ \alpha \mp 1 = 2p\beta_2^2, \end{cases}$$

hence $1 = ((\alpha \pm 1)/p) = ((\alpha \mp 1 \pm 2)/p) = (2/p)$, so the result.

(2) If $2p(x-1)$ is a square in $\mathbb{N}$, then

$$\begin{cases} x - 1 = 2py_1^2, \\ x + 1 = qy_2^2; \end{cases}$$

thus

$$\begin{cases} \left(\dfrac{2p}{q}\right) = \left(\dfrac{x-1}{q}\right) = -\left(\dfrac{2}{q}\right), \\ \left(\dfrac{q}{p}\right) = \left(\dfrac{x+1}{p}\right) = \left(\dfrac{2}{p}\right); \end{cases}$$

this implies that $(2/p) = -1$, which contradicts the first assertion (1). $\square$

Lemma 5.5. *Put $\varepsilon_{pq} = a + b\sqrt{pq}$. If $a \pm 1$ is a square in $\mathbb{N}$, then $p \equiv 1 \pmod 8$.*

P r o o f. The same argument as in Lemma 5.4 (1) leads to the result. $\square$

Theorem 5.6. *Keep the notation and hypotheses previously mentioned.*

(1) *If $N(\varepsilon_{2p}) = 1$ and $x \pm 1$ is a square in $\mathbb{N}$, then $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2]\rangle$ or $\langle[\mathcal{H}_1], [\mathcal{H}_2]\rangle$.*

(2) *If $N(\varepsilon_{2p}) = 1$ and $x+1$, $x-1$ are not squares in $\mathbb{N}$, then there exists an unambiguous ideal $\mathcal{I}$ in $\Bbbk/F$ of order 2 such that $\kappa_{\mathbb{K}_2} = \langle[\mathcal{I}]\rangle$ or $\langle[\mathcal{H}_0\mathcal{I}]\rangle$ or $\langle[\mathcal{H}_1\mathcal{I}]\rangle$ or $\langle[\mathcal{H}_0\mathcal{H}_1\mathcal{I}]\rangle$.*

(3) *If $N(\varepsilon_{2p}) = -1$, then*

  (i) *if $x \pm 1$ is a square in $\mathbb{N}$, then $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2]\rangle$;*

  (ii) *else, $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_0\mathcal{H}_1]\rangle$.*

P r o o f. Since $(\pi_j) = \mathcal{H}_j^2$, $j \in \{1,2\}$, and $\mathcal{H}_0^2 = (1+\mathrm{i})$, so $(2p) = ((1+\mathrm{i})\mathcal{H}_1\mathcal{H}_2)^2$. Moreover, $2p$ is a square in $\mathbb{K}_2$, so there exists $\alpha \in \mathbb{K}_2$ such that $(2p) = (\alpha^2)$, hence $((1+\mathrm{i})\mathcal{H}_1\mathcal{H}_2)^2 = (\alpha^2)$, therefore $\mathcal{H}_1\mathcal{H}_2 = (\alpha/(1+\mathrm{i}))$ and $\mathcal{H}_1\mathcal{H}_2$ capitulates in $\mathbb{K}_2$.

(1) If $N(\varepsilon_{2p}) = 1$, then by Lemma 5.4 we get $p \equiv 1 \pmod 8$. Moreover, according to Lemma 4.1, if $x \pm 1$ is a square in $\mathbb{N}$, then $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_1\mathcal{H}_2$ are not principal

in $\mathbb{k}$, and according to Theorem 5.2, there are four classes that capitulate in $\mathbb{K}_2$. The following examples affirm the two cases of capitulation:

$d\ (=2pq)$	238	782	1022	1246	1358
$2pq$	$2\cdot17\cdot7$	$2\cdot17\cdot23$	$2\cdot73\cdot7$	$2\cdot89\cdot7$	$2\cdot97\cdot7$
$x+1$	108^2	28^2	32^2	21068856^2	1732^2
$\mathcal{H}_0\mathcal{O}_{\mathbb{K}_2}$	$[0,0,0]$	$[0,0,0]$	$[16,0,0]$	$[8,0,0]$	$[0,0,0]$
$\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[4,0,0]$	$[12,0,0]$	$[0,0,0]$	$[0,0,0]$	$[60,0,0]$
$\mathcal{H}_2\mathcal{O}_{\mathbb{K}_2}$	$[4,0,0]$	$[12,0,0]$	$[0,0,0]$	$[0,0,0]$	$[60,0,0]$
$\mathcal{H}_1\mathcal{H}_2\mathcal{O}_{\mathbb{K}_2}$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$
$\mathrm{Cl}(\mathbb{k})$	$(4,2,2)$	$(12,2,2)$	$(16,2,2)$	$(8,2,2)$	$(12,2,2)$
$\mathrm{Cl}(\mathbb{K}_2)$	$(8,2,2)$	$(24,6,2)$	$(32,8,2)$	$(16,4,2)$	$(120,2,2)$
$d\ (=2pq)$	374	534	1398	2118	2694
$2pq$	$2\cdot17\cdot11$	$2\cdot89\cdot3$	$2\cdot233\cdot3$	$2\cdot353\cdot3$	$2\cdot449\cdot3$
$x-1$	58^2	1918^2	2206^2	46^2	2095718^2
$\mathcal{H}_0\mathcal{O}_{\mathbb{K}_2}$	$[0,2]$	$[0,0]$	$[0,0]$	$[60,12]$	$[0,6,0]$
$\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[0,0]$	$[40,0]$	$[40,0]$	$[0,0]$	$[0,0,0]$
$\mathcal{H}_2\mathcal{O}_{\mathbb{K}_2}$	$[0,0]$	$[40,0]$	$[40,0]$	$[0,0]$	$[0,0,0]$
$\mathcal{H}_1\mathcal{H}_2\mathcal{O}_{\mathbb{K}_2}$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0,0]$
$\mathrm{Cl}(\mathbb{k})$	$(14,2,2)$	$(10,2,2)$	$(10,2,2)$	$(30,2,2)$	$(30,2,2)$
$\mathrm{Cl}(\mathbb{K}_2)$	$(28,4)$	$(80,2)$	$(80,2)$	$(120,24)$	$(60,12,3)$

(2) If $N(\varepsilon_{2p})=1$ and $x+1$, $x-1$ are not squares in $\mathbb{N}$, then the assumptions of Proposition 4.3 are satisfied, since $N(\varepsilon_{2p})=1$ yields that $p\equiv 1 \pmod 8$. Moreover, Lemma 2.5 implies that $E_{\mathbb{k}}=\langle \mathrm{i}, \varepsilon_{2pq}\rangle$.

(2.1) Assume $2p(x+1)$ is a square in $\mathbb{N}$, hence, according to Proposition 3.2, we have $E_{\mathbb{K}_2}=\langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_q}, \sqrt{\mathrm{i}\varepsilon_{2p}}, \sqrt{\mathrm{i}\varepsilon_{2pq}}\rangle$, and according to Theorem 5.2, there are two classes that capitulate in $\mathbb{K}_2$. So to prove the result, it suffices to show that $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_0\mathcal{H}_1$ do not capitulate in $\mathbb{K}_2$. If $\mathcal{H}_0$ or $\mathcal{H}_1$, $\mathcal{H}_0\mathcal{H}_1$ capitulate in $\mathbb{K}_2$, then there exists $\alpha\in\mathbb{K}_2$ such that $\mathcal{H}_0=(\alpha)$ or $\mathcal{H}_1=(\alpha)$, $\mathcal{H}_0\mathcal{H}_1=(\alpha)$, respectively, hence $(\alpha^2)=(1+\mathrm{i})$ or $(\alpha^2)=(\pi_1)$, $(\alpha^2)=((1+\mathrm{i})\pi_1)$. Consequently, $(1+\mathrm{i})\varepsilon=\alpha^2$ or $\alpha^2=\pi_1\varepsilon$, $\alpha^2=(1+\mathrm{i})\pi_1\varepsilon$ with some unit $\varepsilon\in\mathbb{K}_2$; note that ε can be taken as $\varepsilon=\mathrm{i}^a(\sqrt{\mathrm{i}\varepsilon_q})^b(\sqrt{\mathrm{i}\varepsilon_{2p}})^c(\sqrt{\mathrm{i}\varepsilon_{2pq}})^d$, where a, b, c and d are in $\{0,1\}$.

First, let us show that the unit ε is neither real nor purely imaginary. In fact, if it is real (same proof if it is purely imaginary), then putting $\alpha=\alpha_1+\mathrm{i}\alpha_2$, where $\alpha_j\in\mathbb{K}_2^+$, we get:

(2.1.1) If $(1+\mathrm{i})\varepsilon=\alpha^2$, then $\alpha_1^2-\alpha_2^2+2\mathrm{i}\alpha_1\alpha_2=\varepsilon(1+\mathrm{i})$, hence

$$\begin{cases} \alpha_1^2-\alpha_2^2=\varepsilon,\\ 2\alpha_1\alpha_2=\varepsilon, \end{cases}$$

thus $\alpha_1^2 - 2\alpha_2\alpha_1 - \alpha_2^2 = 0$; therefore $\alpha_1 = \alpha_2(1 \pm \sqrt{2})$ and $\sqrt{2} \in \mathbb{K}_2^+$ (for the case $\alpha^2 = \pi_1\varepsilon$, we get $\sqrt{p} \in \mathbb{K}_2^+$), which is absurd.

(2.1.2) If $(1+\mathrm{i})\pi_1\varepsilon = \alpha^2$, then $\alpha_1^2 - \alpha_2^2 + 2\mathrm{i}\alpha_1\alpha_2 = \varepsilon(1+\mathrm{i})\pi_1$, hence

$$\begin{cases} \alpha_1^2 - \alpha_2^2 = \varepsilon(e-4f), \\ 2\alpha_1\alpha_2 = \varepsilon(e+4f), \end{cases}$$

where $p = e^2 + 16f^2$, since $p \equiv 1 \pmod 8$. Thus

$$4\alpha_1^4 - 4\varepsilon(e-4f)\alpha_1^2 - \varepsilon^2(e+4f)^2 = 0,$$

from which we deduce that $\alpha_1^2 = \varepsilon[(e-4f) \pm \sqrt{2p}]/2$. As $\alpha_1 \in \mathbb{K}_2^+$, so putting $\alpha_1 = a + b\sqrt{2p}$, where a, b are in $\mathbb{Q}(\sqrt{q})$, we get the unsolvable equation (in $\mathbb{Q}(\sqrt{q})$)

$$16a^4 - 8\varepsilon(e-4f)a^2 + 2p\varepsilon^2 = 0,$$

since its reduced discriminant is $\Delta' = -16\varepsilon^2(e+4f)^2 < 0$.

To this end, as $(1+\mathrm{i})\varepsilon = \alpha^2$ (same proof for the other cases), applying the norm $N_{\mathbb{K}_2/\Bbbk}$ we get that $(1+\mathrm{i})^2 N_{\mathbb{K}_2/\Bbbk}(\varepsilon) = N_{\mathbb{K}_2/\Bbbk}(\alpha)^2$ with $N_{\mathbb{K}_2/\Bbbk}(\varepsilon) \in E_\Bbbk = \langle \mathrm{i}, \varepsilon_{2pq}\rangle$. Without loss of generality, one can take $N_{\mathbb{K}_2/\Bbbk}(\varepsilon) \in \{\pm 1, \pm\mathrm{i}, \pm\varepsilon_{2pq}, \pm\mathrm{i}\varepsilon_{2pq}\}$.

$\triangleright$ As $N_{\mathbb{K}_2/\Bbbk}(\varepsilon)$ is a square in $E_\Bbbk$, so $N_{\mathbb{K}_2/\Bbbk}(\varepsilon) \notin \{\pm\mathrm{i}, \pm\varepsilon_{2pq}, \pm\mathrm{i}\varepsilon_{2pq}\}$.

$\triangleright$ If $N_{\mathbb{K}_2/\Bbbk}(\varepsilon) = \pm 1$, then there exist a, b, c and d in $\{0,1\}$ such that $\varepsilon = \mathrm{i}^a(\sqrt{\mathrm{i}\varepsilon_q})^b \times (\sqrt{\mathrm{i}\varepsilon_{2p}})^c(\sqrt{\mathrm{i}\varepsilon_{2pq}})^d$ and $N_{\mathbb{K}_2/\Bbbk}(\varepsilon) = \pm 1$, hence, $(-1)^a\varepsilon_{2pq}^d\mathrm{i}^{b+c+d} = \pm 1$; so necessarily we must have $b = c$ and $d = 0$. Therefore $\varepsilon = \mathrm{i}^{a+b}(\sqrt{\varepsilon_q\varepsilon_{2p}})^b$, which contradicts the fact that ε is not real or purely imaginary.

The following examples clarify this: the first table gives examples of the ideals $\mathcal{I}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ which are not principal in $\Bbbk$, and gives the structures of the class groups of $\Bbbk$ and $\mathbb{K}_2$; whereas the second table gives the cases of capitulation of these ideals in $\mathbb{K}_2$.

$d\ (=2pq)$	582	646	2822	5654	8854	10806
$2pq$	$2\cdot 97\cdot 3$	$2\cdot 17\cdot 19$	$2\cdot 17\cdot 83$	$2\cdot 257\cdot 11$	$2\cdot 233\cdot 19$	$2\cdot 1801\cdot 3$
$2p(x+1)$	194^2	102^2	850^2	178358^2	9786^2	258569570^2
$\mathcal{I}$	$[0,1,1]$	$[4,0,0]$	$[12,1,0]$	$[28,1,0]$	$[0,0,1]$	$[0,1,1]$
$\mathcal{I}^2$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$
$\mathcal{H}_0$	$[4,1,1]$	$[0,0,1]$	$[0,0,1]$	$[0,0,1]$	$[60,0,1]$	$[24,1,0]$
$\mathcal{H}_1$	$[4,0,0]$	$[4,2,0]$	$[12,0,0]$	$[28,0,0]$	$[60,0,0]$	$[24,0,0]$
$\mathrm{Cl}(\Bbbk)$	$(8,2,2)$	$(8,4,2)$	$(24,2,2)$	$(56,2,2)$	$(120,2,2)$	$(48,2,2)$
$\mathrm{Cl}(\mathbb{K}_2)$	$(80,4,2)$	$(8,8,2,2)$	$(48,12,2)$	$(224,8,4)$	$(120,8,2,2)$	$(48,48,6,2)$

$d\ (=2pq)$	582	646	2822	5654	8854	10806
$2pq$	$2\cdot97\cdot3$	$2\cdot17\cdot19$	$2\cdot17\cdot83$	$2\cdot257\cdot11$	$2\cdot233\cdot19$	$2\cdot1801\cdot3$
$\mathcal{H}_0\mathcal{O}_{\mathbb{K}_2}$	$[0,2,0]$	$[0,4,1,1]$	$[0,6,0]$	$[0,4,0]$	$[60,4,1,1]$	$[24,24,0,1]$
$\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[40,2,0]$	$[4,4,1,1]$	$[24,6,0]$	$[112,0,0]$	$[0,4,0,0]$	$[24,24,0,0]$
$\mathcal{H}_0\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[40,0,0]$	$[4,0,0,0]$	$[24,0,0]$	$[112,4,0]$	$[60,0,1,1]$	$[0,0,0,1]$
$\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[40,2,0]$	$[0,0,0,0]$	$[0,6,0]$	$[112,0,0]$	$[60,0,1,1]$	$[0,0,0,0]$
$\mathcal{H}_1\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[0,0,0]$	$[0,0,1,1]$	$[24,0,0]$	$[0,0,0]$	$[60,4,1,1]$	$[24,24,0,0]$
$\mathcal{H}_0\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[40,0,0]$	$[4,0,1,1]$	$[0,0,0]$	$[112,4,0]$	$[0,4,0,0]$	$[24,24,0,1]$
$\mathcal{H}_0\mathcal{H}_1\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[0,2,0]$	$[0,4,0,0]$	$[24,6,0]$	$[0,4,0]$	$[0,0,0,0]$	$[0,0,0,1]$

(2.2) Assume $p(x\pm1)$ is a square in $\mathbb{N}$, hence, according to Proposition 3.2, we have $E_{\mathbb{K}_2}=\langle \mathrm{i}, \sqrt{\mathrm{i}\varepsilon_q}, \sqrt{\mathrm{i}\varepsilon_{2p}}, \sqrt{\varepsilon_{2pq}}\rangle$. Thus proceeding as in the case (2.1) we prove that $\mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_0\mathcal{H}_1$ do not capitulate in $\mathbb{K}_2$. The following examples illustrate these results.

(2.2.1) First case: $p(x+1)$ is a square in $\mathbb{N}$. The first table gives examples of the ideals $\mathcal{I}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ which are not principal in $\Bbbk$, and gives the structures of the class groups of $\Bbbk$ and $\mathbb{K}_2$; whereas the second table gives the cases of capitulation of these ideals in $\mathbb{K}_2$.

$d\ (=2pq)$	3358	3502	6014	9118
$2pq$	$2\cdot73\cdot23$	$2\cdot17\cdot103$	$2\cdot97\cdot31$	$2\cdot97\cdot47$
$p(x+1)$	217248^2	447916^2	388^2	11181384^2
$\mathcal{I}$	$[4,0,0]$	$[2,2,0]$	$[12,0,0]$	$[4,0,0]$
$\mathcal{I}^2$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$
$\mathcal{H}_0$	$[0,2,1]$	$[2,0,1]$	$[0,4,1]$	$[4,0,1]$
$\mathcal{H}_1$	$[0,2,0]$	$[0,2,0]$	$[12,4,0]$	$[0,2,0]$
$\mathrm{Cl}(\Bbbk)$	$(8,4,2)$	$(4,4,2)$	$(24,8,2)$	$(8,4,2)$
$\mathrm{Cl}(\mathbb{K}_2)$	$(96,8,2,2)$	$(20,4,2,2,2)$	$(240,24,2,2)$	$(20,20,4,2,2)$

$d\ (=2pq)$	3358	3502	6014	9118
$2pq$	$2\cdot73\cdot23$	$2\cdot17\cdot103$	$2\cdot97\cdot31$	$2\cdot97\cdot47$
$\mathcal{H}_0\mathcal{O}_{\mathbb{K}_2}$	$[48,4,0,0]$	$[0,0,1,0,0]$	$[120,12,0,0]$	$[10,10,2,1,0]$
$\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[48,0,0,0]$	$[0,2,0,0,0]$	$[120,0,0,0]$	$[10,10,2,0,0]$
$\mathcal{H}_0\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[0,4,0,0]$	$[0,2,1,0,0]$	$[0,12,0,0]$	$[0,0,0,1,0]$
$\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[0,4,0,0]$	$[0,2,0,0,0]$	$[120,12,0,0]$	$[0,0,0,0,0]$
$\mathcal{H}_1\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[48,4,0,0]$	$[0,0,0,0,0]$	$[0,12,0,0]$	$[10,10,2,0,0]$
$\mathcal{H}_0\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[48,0,0,0]$	$[0,2,1,0,0]$	$[0,0,0,0]$	$[10,10,2,1,0]$
$\mathcal{H}_0\mathcal{H}_1\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[0,0,0,0]$	$[0,0,1,0,0]$	$[120,0,0,0]$	$[0,0,0,1,0]$

(2.2.2) Second case: $p(x-1)$ is a square in $\mathbb{N}$. The first table gives examples of the ideals $\mathcal{I}$, $\mathcal{H}_0$ and $\mathcal{H}_1$ which are not principal in $\Bbbk$, and gives the structures of the class groups of $\Bbbk$ and $\mathbb{K}_2$; whereas the second table gives the cases of capitulation of these ideals in $\mathbb{K}_2$.

$d\ (=2pq)$	438	2022	2598	5622
$2pq$	$2\cdot 73\cdot 3$	$2\cdot 337\cdot 3$	$2\cdot 433\cdot 3$	$2\cdot 937\cdot 3$
$p(x-1)$	21316	454276	749956	3511876
$\mathcal{I}$	$[0,1,1]$	$[6,1,0]$	$[6,1,1]$	$[0,2,1]$
$\mathcal{I}^2$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$
$\mathcal{H}_0$	$[2,1,1]$	$[0,0,1]$	$[0,1,1]$	$[0,0,1]$
$\mathcal{H}_1$	$[2,0,0]$	$[6,0,0]$	$[6,0,0]$	$[8,2,0]$
$\mathrm{Cl}(\Bbbk)$	$(4,2,2)$	$(12,2,2)$	$(12,2,2)$	$(16,4,2)$
$\mathrm{Cl}(\mathbb{K}_2)$	$(32,2,2,2)$	$(48,24,2)$	$(132,4,4)$	$(224,8,4)$
$d\ (=2pq)$	438	2022	2598	5622
$2pq$	$2\cdot 73\cdot 3$	$2\cdot 337\cdot 3$	$2\cdot 433\cdot 3$	$2\cdot 937\cdot 3$
$\mathcal{H}_0\mathcal{O}_{\mathbb{K}_2}$	$[16,1,1,1]$	$[24,12,0]$	$[66,2,0]$	$[112,4,0]$
$\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[0,1,1,1]$	$[0,12,0]$	$[0,2,2]$	$[112,0,0]$
$\mathcal{H}_0\mathcal{H}_1\mathcal{O}_{\mathbb{K}_2}$	$[16,0,0,0]$	$[24,0,0]$	$[66,0,2]$	$[0,4,0]$
$\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[0,1,1,1]$	$[24,12,0]$	$[66,0,2]$	$[0,0,0]$
$\mathcal{H}_1\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[0,0,0,0]$	$[24,0,0]$	$[66,2,0]$	$[112,0,0]$
$\mathcal{H}_0\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[16,0,0,0]$	$[0,0,0]$	$[0,2,2]$	$[112,4,0]$
$\mathcal{H}_0\mathcal{H}_1\mathcal{I}\mathcal{O}_{\mathbb{K}_2}$	$[16,1,1,1]$	$[0,12,0]$	$[0,0,0]$	$[0,4,0]$

(3) Suppose that $N(\varepsilon_{2p})=-1$. Let us prove that $\mathcal{H}_0\mathcal{H}_1$ and $\mathcal{H}_0\mathcal{H}_2$ capitulate in $\mathbb{K}_2$. Put $\varepsilon_{2p}=a+b\sqrt{2p}$, then $a^2+1=2b^2p$, hence by the decomposition uniqueness in $\mathbb{Z}[\mathrm{i}]$ there exist b_1 and b_2 in $\mathbb{Z}[\mathrm{i}]$ such that

$$\begin{cases} a\pm\mathrm{i}=b_1^2(1+\mathrm{i})\pi_1, \\ a\mp\mathrm{i}=b_2^2(1-\mathrm{i})\pi_2, \end{cases} \quad\text{or}\quad \begin{cases} a\pm\mathrm{i}=\mathrm{i}(1+\mathrm{i})b_1^2\pi_1, \\ a\mp\mathrm{i}=-\mathrm{i}(1-\mathrm{i})b_2^2\pi_2, \end{cases} \quad \text{with } b=b_1b_2.$$

Consequently, $\sqrt{\varepsilon_{2p}}=(b_1(1+\mathrm{i})\sqrt{(1\pm\mathrm{i})\pi_1}+b_2(1-\mathrm{i})\sqrt{(1\mp\mathrm{i})\pi_2})/2$, hence $(1\pm\mathrm{i})\times\pi_1\varepsilon_{2p}$ and $(1\mp\mathrm{i})\pi_2\varepsilon_{2p}$ are squares in $\mathbb{K}_2$. Thus $(\alpha^2)=((1\pm\mathrm{i})\pi_1)$ and $(\beta^2)=((1\mp\mathrm{i})\pi_2)$, with some α, β in $\mathbb{K}_2$. Therefore $\mathcal{H}_0\mathcal{H}_1=(\alpha)$ and $\mathcal{H}_0\mathcal{H}_2=(\beta)$, i.e. $\mathcal{H}_0\mathcal{H}_1$ and $\mathcal{H}_0\mathcal{H}_2$ capitulate in $\mathbb{K}_2$.

(3.1) If $x\pm1$ is a square in $\mathbb{N}$, then Lemma 4.1 yields that $\mathcal{H}_1\mathcal{H}_2$, $\mathcal{H}_0\mathcal{H}_1$ and $\mathcal{H}_0\mathcal{H}_2$ are not principal in $\Bbbk$, hence the result.

(3.2) If $x+1$ and $x-1$ are not squares in $\mathbb{N}$, then Lemma 4.1 yields that $[\mathcal{H}_0\mathcal{H}_1]=[\mathcal{H}_0\mathcal{H}_2]$, hence the result. $\square$

5.3. Capitulation in $\mathbb{K}_3$. Let $\mathbb{K}_3=\Bbbk(\sqrt{2})=\mathbb{Q}(\sqrt{2},\sqrt{pq},\mathrm{i})$ and put $\varepsilon_{pq}=a+b\sqrt{pq}$, $\varepsilon_{2pq}=x+y\sqrt{2pq}$. Let $Q_{\mathbb{K}_3}$ denote the unit index of $\mathbb{K}_3$.

Theorem 5.7. *Keep the notation and hypotheses previously mentioned.*

(1) *If both of $x\pm1$ and $a\pm1$ are squares in $\mathbb{N}$, then*
   (a) *if $Q_{\mathbb{K}_3}=2$, then $\kappa_{\mathbb{K}_3}=\langle[\mathcal{H}_0]\rangle$,*
   (b) *if $Q_{\mathbb{K}_3}=1$, then $\kappa_{\mathbb{K}_3}=\langle[\mathcal{H}_0],[\mathcal{H}_1\mathcal{H}_2]\rangle$.*

(2) *If $x \pm 1$ is a square in $\mathbb{N}$ and $a+1$, $a-1$ are not, then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$.*

(3) *If $a \pm 1$ is a square in $\mathbb{N}$ and $x+1$, $x-1$ are not, then there exists an unambiguous ideal $\mathcal{I}$ in $\Bbbk/\mathbb{Q}(\mathrm{i})$ of order 2 such that $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{I}] \rangle$ or $\langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{I}] \rangle$.*

(4) *If $x+1$, $x-1$, $a+1$ and $a-1$ are not squares in $\mathbb{N}$, then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$.*

Proof. As $N(\varepsilon_2) = -1$, we have $\sqrt{(1+\mathrm{i})\varepsilon_2} = (2 + (1+\mathrm{i})\sqrt{2})/2$. Hence there exists $\beta \in \mathbb{K}_3$ such that $\mathcal{H}_0^2 = (1+\mathrm{i}) = (\beta^2)$, therefore $\mathcal{H}_0$ capitulates in $\mathbb{K}_3$.

(1) Assume $x \pm 1$ and $a \pm 1$ are squares in $\mathbb{N}$.

(a) If $Q_{\mathbb{K}_3} = 2$, then by Theorem 5.2, $|\kappa_{\mathbb{K}_3}| = 2$, hence $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$.

(b) If $Q_{\mathbb{K}_3} = 1$, then by Theorem 5.2, $|\kappa_{\mathbb{K}_3}| = 4$. Since $a \pm 1$ is a square in $\mathbb{N}$, so Lemma 5.5 yields that $p \equiv 1 \pmod 8$. Therefore Proposition 4.3 implies that

$$\mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle.$$

Proceeding as in the proof of Theorem 5.6 (2), we show that $\mathcal{H}_1$ and $\mathcal{H}_2$ do not capitulate in $\mathbb{K}_3$. On the other hand, as $|\kappa_{\mathbb{K}_3}| = 4$ and $\kappa_{\mathbb{K}_3} \subseteq \mathrm{Am}(\Bbbk/\mathbb{Q}(\mathrm{i}))$, so necessarily $\mathcal{H}_1\mathcal{H}_2$ capitulate in $\mathbb{K}_3$. Finally, Lemma 4.1 yields that $\mathcal{H}_1\mathcal{H}_2$, $\mathcal{H}_0$ and $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2$ are not principal in $\Bbbk$. Thus the result.

(2) Assume $x \pm 1$ is a square in $\mathbb{N}$ and $a+1$, $a-1$ are not. As $\mathcal{H}_0$ capitulates in $\mathbb{K}_3$ and $|\kappa_{\mathbb{K}_3}| = 4$ (Theorem 5.2), it suffices to prove that $\mathcal{H}_1\mathcal{H}_2$ capitulates in $\mathbb{K}_3$. According to the proof of Proposition 3.3, $p\varepsilon_{pq}$ is a square in $\mathbb{K}_3$; hence there exists α in $\mathbb{K}_3$ such that $(p) = (\alpha^2)$, so $\mathcal{H}_1\mathcal{H}_2 = (\alpha)$. Thus the result.

(3) If $a \pm 1$ is a square in $\mathbb{N}$ and $x+1$, $x-1$ are not, then Lemma 5.5 implies that $p \equiv 1 \pmod 8$; hence the hypotheses of Proposition 4.3 are satisfied. On the other hand, from Lemma 4.1 we get $[\mathcal{H}_1] = [\mathcal{H}_2]$. Therefore, proceeding as in the proof of Theorem 5.6, we show that $\mathcal{H}_1$ does not capitulate in $\mathbb{K}_3$. The following examples clarify the two cases of capitulation:

$d\ (=2pq)$	582	2006	2454	2742
$2pq$	$2\cdot 97\cdot 3$	$2\cdot 17\cdot 59$	$2\cdot 409\cdot 3$	$2\cdot 457\cdot 3$
$\mathcal{H}_0\mathcal{O}_{\mathbb{K}_3}$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$	$[0,0,0]$
$\mathcal{H}_1\mathcal{O}_{\mathbb{K}_3}$	$[8,2,0]$	$[24,0,0]$	$[16,0,0]$	$[48,2,0]$
$\mathcal{I}\mathcal{O}_{\mathbb{K}_3}$	$[0,0,0]$	$[24,0,0]$	$[16,0,0]$	$[0,0,0]$
$\mathcal{H}_1\mathcal{I}\mathcal{O}_{\mathbb{K}_3}$	$[8,2,0]$	$[0,0,0]$	$[0,0,0]$	$[48,2,0]$
$\mathrm{Cl}(\Bbbk)$	$(8,2,2)$	$(24,2,2)$	$(16,2,2)$	$(16,2,2)$
$\mathrm{Cl}(\mathbb{K}_2)$	$(16,4,2)$	$(48,4,2)$	$(32,4,2)$	$(96,4,2)$

(4) Suppose that $x+1$, $x-1$, $a+1$ and $a-1$ are not squares in $\mathbb{N}$, then $|\kappa_{\mathbb{K}_3}| = 2$ (Theorem 5.2). Thus $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$. $\square$

From Theorems 5.3, 5.6 and 5.7 we deduce the following theorem.

Theorem 5.8. *Let $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$, where $p \equiv -q \equiv 1 \pmod 4$ are different primes, and $\Bbbk^{(*)}$ its genus field. Put $\varepsilon_{2pq} = x + y\sqrt{2pq}$ and $\varepsilon_{pq} = a + b\sqrt{pq}$.*

(1) *If $x \pm 1$ is a square in $\mathbb{N}$, then $\langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2]\rangle \subseteq \kappa_{\Bbbk^{(*)}}$.*
(2) *If $x+1$ and $x-1$ are not squares in $\mathbb{N}$, then*
  (a) *if $N(\varepsilon_{2p}) = 1$ or $a \pm 1$ is a square in $\mathbb{N}$, then there exists an unambiguous ideal $\mathcal{I}$ in $\Bbbk/\mathbb{Q}(\mathrm{i})$ of order 2 such that: $\langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{I}]\rangle \subseteq \kappa_{\Bbbk^{(*)}}$;*
  (b) *else $\langle[\mathcal{H}_0], [\mathcal{H}_1]\rangle \subseteq \kappa_{\Bbbk^{(*)}}$.*

Theorem 5.8 implies the following corollary:

Corollary 5.9. *Let $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$, where $p \equiv -q \equiv 1 \pmod 4$ are different primes. Let $\Bbbk^{(*)}$ be the genus field of $\Bbbk$ and $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i}))$ the group of the strongly ambiguous class of $\Bbbk/\mathbb{Q}(\mathrm{i})$, then $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) \subseteq \kappa_{\Bbbk^{(*)}}$.*

6. Application

Let $p \equiv -q \equiv 1 \pmod 4$ be different primes such that $p \equiv 1 \pmod 8$, $q \equiv 3 \pmod 8$ and $(p/q) = -1$. Hence, according to [3], $\mathrm{Cl}_2(\Bbbk)$ is of type $(2,2,2)$. Therefore, under these assumptions, $\mathrm{Cl}_2(\Bbbk) = \mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2]\rangle$ (see [6]). To continue we need the following result.

Lemma 6.1. *Let $\varepsilon_{2pq} = x + y\sqrt{2pq}$ and $\varepsilon_{pq} = a + b\sqrt{pq}$ denote the fundamental units of $\mathbb{Q}(\sqrt{2pq})$ and $\mathbb{Q}(\sqrt{pq})$, respectively. Then*

(1) *$x - 1$ is a square in $\mathbb{N}$;*
(2) *$a - 1$ is a square in $\mathbb{N}$.*

P r o o f. (1) By Lemma 2.2 and according to the decomposition uniqueness in $\mathbb{Z}$, there are six cases to discuss: $x \pm 1$ or $p(x \pm 1)$ or $2p(x \pm 1)$ is a square in $\mathbb{N}$.

(a) If $x+1$ is a square in $\mathbb{N}$, then

$$\begin{cases} x + 1 = y_1^2, \\ x - 1 = 2pqy_2^2, \end{cases}$$

hence $1 = ((x+1)/q) = ((x-1+2)/q) = (2/q)$, which contradicts the fact that $(2/q) = -1$.

(b) If $p(x \pm 1)$ is a square in $\mathbb{N}$, then

$$\begin{cases} x \pm 1 = py_1^2, \\ x \mp 1 = 2qy_2^2, \end{cases}$$

hence $(2q/p) = ((x \mp 1)/p) = ((x \pm 1 \mp 2)/p) = (2/p)$, thus $(q/p) = 1$. This is false, since $(p/q) = -1$.

(c) If $2p(x+1)$ is a square in $\mathbb{N}$, then

$$\begin{cases} x+1 = py_1^2, \\ x-1 = 2qy_2^2, \end{cases}$$

hence $(2p/q) = ((x+1)/q) = ((x-1+2)/q) = (2/q)$, which leads to the contradiction $(q/p) = 1$.

(d) If $2p(x-1)$ is a square in $\mathbb{N}$, then

$$\begin{cases} x-1 = py_1^2, \\ x+1 = 2qy_2^2, \end{cases}$$

hence $(q/p) = ((x+1)/p) = ((x-1+2)/p) = (2/p) = 1$, which is false.

Consequently, the only case which is possible is: $x-1$ is a square in $\mathbb{N}$.

(2) Proceeding similarly, we show that $a-1$ is a square in $\mathbb{N}$. □

Theorem 6.2. *Let* $\mathbb{k} = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$, *where* $p \equiv -q \equiv 1 \pmod 4$ *are different primes satisfying the conditions* $p \equiv 1 \pmod 8$, $q \equiv 3 \pmod 8$ *and* $(p/q) = -1$. *Put* $\mathbb{K}_1 = \mathbb{k}(\sqrt{p})$, $\mathbb{K}_2 = \mathbb{k}(\sqrt{q})$ *and* $\mathbb{K}_3 = \mathbb{k}(\sqrt{2})$. *Let* $\mathbb{k}^{(*)}$ *denote the absolute genus field of* $\mathbb{k}$ *and* $(\mathbb{k}/\mathbb{Q}(\mathrm{i}))^*$ *its relative genus field over* $\mathbb{Q}(\mathrm{i})$.

(1) $\mathbb{k}^{(*)} \subsetneq (\mathbb{k}/\mathbb{Q}(\mathrm{i}))^*$.

(2) $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.

(3) *Denote by* ε_{2p} *the fundamental unit of* $\mathbb{Q}(\sqrt{2p})$.

   (a) *If* $N(\varepsilon_{2p}) = 1$, *then* $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$ *or* $\langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$.

   (b) *Else,* $\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$.

(4) *Denote by* $Q_{\mathbb{K}_3}$ *the unit index of* $\mathbb{K}_3$.

   (a) *If* $Q_{\mathbb{K}_3} = 1$, *then* $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$.

   (b) *If* $Q_{\mathbb{K}_3} = 2$, *then* $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$.

(5) $\kappa_{\mathbb{k}^{(*)}} = \mathrm{Am}_s(\mathbb{k}/\mathbb{Q}(\mathrm{i})) = \mathrm{Cl}_2(\mathbb{k})$.

Proof. (1) From Lemma 6.1, we have that $x-1$ is a square in $\mathbb{N}$. Then Proposition 4.3 yields the first assertion.

(2) From Lemma 6.1, we have that $x-1$ is a square in $\mathbb{N}$. Then Theorem 5.3 (1) yields the second assertion.

(3) From Lemma 6.1, we have that $x-1$ is a square in $\mathbb{N}$. Therefore

(a) if $N(\varepsilon_{2p}) = 1$, then Theorem 5.6 (1) yields the result;

(b) if $N(\varepsilon_{2p}) = -1$, then Theorem 5.6 (3) yields the result.

(4) As $x-1$ and $a-1$ are squares in $\mathbb{N}$ (Lemma 6.1), so Theorem 5.7 (1) yields the result.

(5) As $p \equiv 1 \pmod 8$, so from Proposition 4.3 we get $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \langle[\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2]\rangle$. Hence $\mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Cl}_2(\Bbbk)$. The assertions (2), (3) and (4) imply that $\kappa_{\Bbbk^{(*)}} = \mathrm{Am}_s(\Bbbk/\mathbb{Q}(\mathrm{i})) = \mathrm{Cl}_2(\Bbbk)$. □

Acknowledgement. We thank the unknown referee for his/her several helpful suggestions and for calling our attention to the missing details.

References

[1] *A. Azizi*: On the capitulation of the 2-class group of $\Bbbk = \mathbb{Q}(\sqrt{2pq}, \mathrm{i})$ where $p \equiv -q \equiv 1 \bmod 4$. Acta Arith. *94* (2000), 383–399. (In French.) zbl MR

[2] *A. Azizi*: Units of certain imaginary abelian number fields over $\mathbb{Q}$. Ann. Sci. Math. Qué. *23* (1999), 15–21. (In French.) zbl MR

[3] *A. Azizi, M. Taous*: Determination of the fields $\Bbbk = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ given the 2-class groups are of type $(2,4)$ or $(2,2,2)$. Rend. Ist. Mat. Univ. Trieste *40* (2008), 93–116. (In French.) zbl MR

[4] *A. Azizi, A. Zekhnini, M. Taous*: On the strongly ambiguous classes of $\Bbbk/\mathbb{Q}(\mathrm{i})$ where $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2}, \mathrm{i})$. Asian-Eur. J. Math. *7* (2014), Article ID 1450021, 26 pages. zbl MR doi

[5] *A. Azizi, A. Zekhnini, M. Taous*: Structure of $\mathrm{Gal}(\Bbbk_2^{(2)}/\Bbbk)$ for some fields $\Bbbk = \mathbb{Q}(\sqrt{2p_1p_2}, \mathrm{i})$ with $\mathrm{Cl}_2(\Bbbk) \simeq (2,2,2)$. Abh. Math. Semin. Univ. Hamb. *84* (2014), 203–231. zbl MR

[6] *A. Azizi, A. Zekhnini, M. Taous*: On the generators of the 2-class group of the field $\Bbbk = \mathbb{Q}(\sqrt{d}, \mathrm{i})$. Int. J. of Pure and Applied Math. *81* (2012), 773–784.

[7] *A. Azizi, A. Zekhnini, M. Taous*: On the unramified quadratic and biquadratic extensions of the field $\mathbb{Q}(\sqrt{d}, \mathrm{i})$. Int. J. Algebra *6* (2012), 1169–1173. zbl MR

[8] *C. Chevalley*: Sur la théorie du corps de classes dans les corps finis et les corps locaux. J. Fac. Sci., Univ. Tokyo, Sect. (1) *2* (1933), 365–476. (In French.) zbl

[9] *H. Furuya*: Principal ideal theorems in the genus field for absolutely Abelian extensions. J. Number Theory *9* (1977), 4–15. zbl MR doi

[10] *G. Gras*: Class Field Theory: From Theory to Practice. Springer Monographs in Mathematics, Springer, Berlin, 2003. zbl MR

[11] *H. Hasse*: On the class number of abelian number fields. Mathematische Lehrbücher und Monographien I, Akademie, Berlin, 1952. (In German.) zbl MR

[12] *F.-P. Heider, B. Schmithals*: Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen. J. Reine Angew. Math. *336* (1982), 1–25. (In German.) zbl MR

[13] *M. Hirabayashi, K.-I. Yoshino*: Unit indices of imaginary abelian number fields of type $(2,2,2)$. J. Number Theory *34* (1990), 346–361. zbl MR doi

[14] *T. Kubota*: Über den bizyklischen biquadratischen Zahlkörper. Nagoya Math. J. *10* (1956), 65–85. (In German.) zbl MR

[15] *F. Lemmermeyer*: The ambiguous class number formula revisited. J. Ramanujan Math. Soc. *28* (2013), 415–421. MR

[16] *S. Louboutin*: Hasse unit indices of dihedral octic CM-fields. Math. Nachr. *215* (2000), 107–113. zbl MR doi

[17] *T. M. McCall, C. J. Parry, R. Ranalli*: Imaginary bicyclic biquadratic fields with cyclic 2-class group. J. Number Theory *53* (1995), 88–99. zbl MR doi

[18] *P. J. Sime*: On the ideal class group of real biquadratic fields. Trans. Am. Math. Soc. *347* (1995), 4855–4876. zbl MR doi

[19] *F. Terada*: A principal ideal theorem in the genus field. Tohoku Math. J. (2) *23* (1971), 697–718. zbl MR doi
[20] *H. Wada*: On the class number and the unit group of certain algebraic number fields. J. Fac. Sci., Univ. Tokyo, Sect. (1) *13* (1966), 201–209. zbl MR

Authors' addresses: *Abdelmalek Azizi*, Department of Mathematics, Sciences Faculty, Mohammed First University, Boulevard Mohammed IV, B.P. 524, Oujda, 60000, Morocco, e-mail: `abdelmalekazizi@yahoo.fr`; *Abdelkader Zekhnini*, Department of Mathematics, Pluridisciplinary Faculty of Nador, Mohammed First University, B.P. 300, Selouane, Nador, 62700, Morocco, e-mail: `zekha1@yahoo.fr`; *Mohammed Taous*, Department of Mathematics, Sciences and Techniques Faculty, Moulay Ismail University, B.P. 509, Boutalamine, Errachidia, 52000, Morocco, e-mail: `taousm@hotmail.com`.