SOME GENERALIZATIONS OF OLIVIER'S THEOREM

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Abstract. Let $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers. L. Olivier proved that if the sequence (a_n) is non-increasing, then $\lim\limits_{n\to\infty} na_n = 0$. In the present paper:

(a) We formulate and prove a necessary and sufficient condition for having $\lim\limits_{n\to\infty} na_n = 0$; Olivier's theorem is a consequence of our Theorem 2.1.

(b) We prove properties analogous to Olivier's property when the usual convergence is replaced by the $\mathcal{I}$-convergence, that is a convergence according to an ideal $\mathcal{I}$ of subsets of $\mathbb{N}$. Again, Olivier's theorem is a consequence of our Theorem 3.1, when one takes as $\mathcal{I}$ the ideal of all finite subsets of $\mathbb{N}$.

Keywords: convergent series; Olivier's theorem; ideal; $\mathcal{I}$-convergence; $\mathcal{I}$-monotonicity

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1. Introduction

During the last decades it was shown that several rigid mathematical concepts allow meaningful and applicable extensions. In this note we will deal with the concept of convergence of sequences with respect to a given ideal of subsets of positive integers, so-called $\mathcal{I}$-convergence. The standard convergence is a special case of such convergence with respect to the ideal of all finite subsets of $\mathbb{N} = \{1, 2, \ldots\}$. We will study the $\mathcal{I}$-convergence variant of a classical result in mathematical analysis, Olivier's theorem.

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The well known simple necessary condition for the convergence of series states that the limit of the sequence of its terms is zero. In the case when all terms of the series are positive and their sequence is non-increasing, some additional information about the speed of this convergence can be deduced. In 1827 Olivier [5] proved the following theorem. Note that it is also known as Abel-Pringsheim's theorem (see e.g. [1], Theorem 5.2.2).

Theorem 1.1 ([5]). *Let $(a_n)_{n\in\mathbb{N}}$ be a non-increasing sequence of positive numbers such that the corresponding series $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Then $\lim\limits_{n\to\infty} na_n = 0$.*

This result was later generalized and extended by several authors, see for example [2], [4], [6]. In this note we will extend and generalize this result in two steps. First we prove a necessary and sufficient condition for the series of positive terms $\sum\limits_{n\in\mathbb{N}} a_n$ to fulfil $\lim\limits_{n\to\infty} na_n = 0$. Then we will extend this result in terms of $\mathcal{I}$-convergence. Here we briefly mention some necessary definitions and properties related to this concept. An ideal $\mathcal{I}$ on $\mathbb{N}$ is any nonempty proper subclass of $2^{\mathbb{N}}$ which is closed with respect to subsets and finite unions, i.e.

$$A \in \mathcal{I}, \quad X \subset A \Rightarrow X \in \mathcal{I} \quad \text{and} \quad A \in \mathcal{I}, \quad B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}.$$

Usually ideals are used to express that the sets belonging to them are small in some sense. The class of all finite subsets of $\mathbb{N}$ forms the ideal usualy denoted by $\mathcal{I}_{\mathrm{f}}$. As further examples of ideals can serve either the class $\mathcal{I}_{\mathrm{d}}$ of all subsets of $\mathbb{N}$ having asymptotic density 0, or the class $\mathcal{I}_{\mathrm{c}}$ of all subsets of $\mathbb{N}$ such that the sum of their reciprocals converges. An ideal $\mathcal{I}$ is called *admissible* if it contains all finite subsets of $\mathbb{N}$, i.e. $\mathcal{I}_{\mathrm{f}} \subset \mathcal{I}$.

Definition 1.1. Let $\mathcal{I}$ be an admissible ideal and (a_n) be a sequence of real numbers. We say that (a_n) is $\mathcal{I}$-convergent to L if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N}\colon |a_n - L| \geqslant \varepsilon\}$ belongs to $\mathcal{I}$. We denote this by $\mathcal{I}\text{-}\lim\limits_{n\to\infty} a_n = L$.

Note that if this limit exists, then it is unique and also that the standard concept of convergence of sequences is exactly the $\mathcal{I}_{\mathrm{f}}$-convergence. Also note that for admissible ideals $\mathcal{I} \subset \mathcal{J}$ the implication

$$\mathcal{I}\text{-}\lim_{n\to\infty} a_n = L \quad \Rightarrow \quad \mathcal{J}\text{-}\lim_{n\to\infty} a_n = L \tag{1.1}$$

holds. For any ideal $\mathcal{I}$ the *dual filter* is the class

$$\mathcal{F}(\mathcal{I}) = \{\mathbb{N} \setminus A\colon\ A \in \mathcal{I}\}.$$

Opposite to ideals, filters express that the sets belonging to them are big in some sense. There is another concept of convergence with respect to an ideal referring to its dual filter.

Definition 1.2. Let $\mathcal{I}$ be an admissible ideal and (a_n) a sequence of real numbers. We say that (a_n) is $\mathcal{I}^*$-convergent to L if there is a set $K = \{k_1 < k_2 < \ldots\}$ in the dual filter $\mathcal{F}(\mathcal{I})$ such that $\lim\limits_{n\to\infty} a_{k_n} = L$. We denote this fact by $\mathcal{I}^*\text{-}\lim\limits_{n\to\infty} a_n = L$.

Note that $\mathcal{I}^*$-convergence is a stronger concept than $\mathcal{I}$-convergence, i.e.

$$\mathcal{I}^*\text{-}\lim_{n\to\infty} a_n = L \quad \Rightarrow \quad \mathcal{I}\text{-}\lim_{n\to\infty} a_n = L,$$

but for many ideals both concepts coincide. These ideals are exactly those fulfilling the following (AP) property (see [3]).

Definition 1.3. We say that an admissible ideal $\mathcal{I}$ satisfies the (AP) condition if for every sequence of mutually disjoint sets (I_n) from $\mathcal{I}$ there exists a sequence (J_n) of sets $J_n \in \mathcal{I}$ such that

(i) each set $J_n \triangle I_n$, $n = 1, 2, \ldots$, is finite and

(ii) $\bigcup\limits_{n=1}^{\infty} J_n \in \mathcal{I}$.

For more information on $\mathcal{I}$-convergence refer to [3].

2. Standard convergence

There is a natural question: How much can the condition of monotonicity of the sequence (a_n) in Olivier's theorem be weakened so that the conclusion $\lim\limits_{n\to\infty} na_n = 0$, at least in some weaker sense, still holds? Of course, one can suppose that the monotonicity condition holds except for a finite number of terms, but this represents a very cheap extension of the original result.

Let $\mathbf{a} = (a_n)$ be a real sequence with positive terms such that the corresponding series $\sum\limits_{n=1}^{\infty} a_n$ converges. Let $S = \sum\limits_{n=1}^{\infty} a_n$ and, as usual, we denote by S_n the n-th partial sum, i.e. $S_n = \sum\limits_{k=1}^{n} a_k$. For a given $n \in \mathbb{N}$ let

$$P_n = P_n(\mathbf{a}) = \{k \in \{1, 2, \ldots, n\} \colon a_k \geqslant a_n\},$$

$$R_n = R_n(\mathbf{a}) = \{k \in \{1, 2, \ldots, n\} \colon a_k < a_n\}$$

and

$$p_n = \sum_{k\in P_n} (a_k - a_n), \quad \text{and} \quad r_n = \sum_{k\in R_n} (a_n - a_k).$$

Notice that both p_n and r_n are nonnegative. Thus, we have

$$S_n = p_n - r_n + na_n \quad \text{and} \quad na_n - r_n > 0. \tag{2.1}$$

Lemma 2.1. *For every convergent series* $\sum\limits_{n=1}^{\infty} a_n$ *with positive terms we have*

$$\lim_{n\to\infty} p_n = \sum_{n=1}^{\infty} a_n = S. \tag{2.2}$$

P r o o f. First, notice that, as $p_n < S_n < S$, the sequence (p_n) is bounded by S. Thus, to prove (2.2) it is sufficient to prove that for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $p_n > S - 2\varepsilon$. So, choose an $\varepsilon > 0$ and let $n_1 \in \mathbb{N}$ be such that $S_{n_1} > S - \varepsilon$. Let $n_0 > n_1$ be an integer such that $a_n < \varepsilon/n_1$ and $a_n \leqslant \min\{a_1, a_2, \ldots, a_{n_1}\}$, i.e. $\{1, 2, \ldots, n_1\} \subset P_n$, holds for all $n > n_0$. Then we have for $n > n_0$

$$p_n = \sum_{k\in P_n} (a_k - a_n) \geqslant \sum_{k=1}^{n_1} (a_k - a_n) = S_{n_1} - n_1 a_n > S - \varepsilon - \varepsilon,$$

which completes the proof. □

Using (2.1) and (2.2) we deduce the following theorem.

Theorem 2.1. *For every convergent series* $\sum\limits_{n=1}^{\infty} a_n$ *with positive terms we have*

$$\lim_{n\to\infty} (na_n - r_n) = 0. \tag{2.3}$$

An immediate consequence of Theorem 2.1 is the following corollary.

Corollary 2.1. *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be a convergent series of positive terms. Then*

$$\lim_{n\to\infty} r_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} na_n = 0. \tag{2.4}$$

In particular, (2.4) generalizes the classical Olivier's theorem, in which $r_n = 0$ for every $n \in \mathbb{N}$.

We will conclude this section with another observation based on Theorem 2.1. For a sequence $\mathbf{a} = (a_n)$ let us denote

$$M = M(\mathbf{a}) = \{n \in \mathbb{N} \colon\ P_n = \{1, 2, \ldots, n\}\},$$

i.e. $n \in M$ if and only if $a_n = \min\{a_1, a_2, \dots, a_n\}$. Note that by the convergence of $\sum_{n=1}^{\infty} a_n$ we have that $\lim_{n\to\infty} a_n = 0$ and consequently the set M is infinite. As $r_n = 0$ if and only if $n \in M$, Theorem 2.1 immediately yields the following proposition.

Proposition 2.1. *Let $\mathbf{a} = (a_n)$ be a sequence of positive real numbers such that the corresponding series $\sum_{n=1}^{\infty} a_n$ converges. Then*

$$\lim_{m \in M(\mathbf{a})} m a_m = 0. \tag{2.5}$$

3. $\mathcal{I}$-monotonicity and Olivier's theorem

In this section we focus on a generalization of Olivier's theorem in terms of $\mathcal{I}$-convergence. To do so, we need some concept of $\mathcal{I}$-non-increasing sequences.

3.1. $\mathcal{I}$-monotonicity. Let $\mathcal{I}$ be an admissible ideal of subsets of $\mathbb{N}$. Perhaps, the most natural way to define the concept of $\mathcal{I}$-non-increasing sequences is the following one.

Definition 3.1. Let $\mathcal{I}$ be an admissible ideal of subsets of $\mathbb{N}$ and let (a_n) be a sequence of positive real numbers. We say that the sequence (a_n) is $\mathcal{I}^*$-non-increasing (in symbols: (a_n) $\mathcal{I}^*\!\downarrow$) if there is a set $K = \{k_1 < k_2 < \dots\}$ in the dual filter $\mathcal{F}(\mathcal{I})$ such that the sequence (a_{k_n}) is non-increasing.

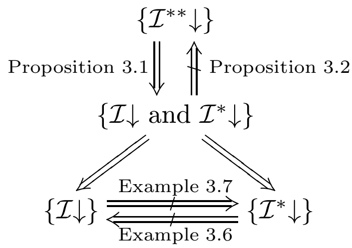
Here is also a stronger version.

Definition 3.2. Let $\mathcal{I}$ be an admissible ideal of subsets of $\mathbb{N}$ and let $\mathbf{a} = (a_n)$ be a sequence of positive real numbers. We say that the sequence $\mathbf{a}$ is $\mathcal{I}^{**}$-non-increasing (in symbols: $\mathbf{a}$ $\mathcal{I}^{**}\!\downarrow$) if $M(\mathbf{a}) \in \mathcal{F}(\mathcal{I})$.

These "star" definitions refer to the existence of a set in the dual filter, instead of assuming that all sets fulfilling a particular property belong to the original ideal. When searching for a non-star version of the definition, the following one can be considered.

Definition 3.3. Let $\mathcal{I}$ be an admissible ideal of subsets of $\mathbb{N}$ and let $\mathbf{a} = (a_n)$ be a sequence of positive real numbers. We say that the sequence $\mathbf{a}$ is $\mathcal{I}$-non-increasing (in symbols: $\mathbf{a}$ $\mathcal{I}\!\downarrow$) if $\mathcal{I}\text{-}\lim_{n\to\infty} r_n = 0$, i.e. if and only if for every $\varepsilon > 0$ the set $T_\varepsilon(\mathbf{a}) = \{n \in \mathbb{N} \colon r_n \geqslant \varepsilon\}$ belongs to $\mathcal{I}$.

It is natural to ask if there are some relations among the above defined concepts. We prove hereon that the situation is as the following diagram shows.



The only general relation among the above concepts is presented in the following proposition.

Proposition 3.1. *Let a sequence of positive terms* (a_n) *be* $\mathcal{I}^{**}\!\downarrow$. *Then it is also both* $\mathcal{I}^{*}\!\downarrow$ *and* $\mathcal{I}\!\downarrow$.

The proof of the above statement follows immediately from the corresponding definitions and is omitted. The next proposition says that in no admissible ideal, except $\mathcal{I}_{\mathrm{f}}$, the opposite implications hold.

Proposition 3.2. *Let an admissible ideal* $\mathcal{I}$ *contain an infinite set:* $\mathcal{I} \supsetneq \mathcal{I}_{\mathrm{f}}$. *Then there exists a convergent series of positive numbers such that the corresponding sequence is both* $\mathcal{I}\!\downarrow$ *and* $\mathcal{I}^{*}\!\downarrow$ *but is not* $\mathcal{I}^{**}\!\downarrow$.

P r o o f. Let $I = \{1 = i_1 < i_2 < \ldots\}$ be an infinite set in $\mathcal{I}$. We will construct the required sequence $\mathbf{a} = (a_n)$ by induction. In the first step let us choose a positive number a_1 such that

$$(i_2 - i_1)a_1 < \frac{1}{2}$$

and put

$$a_2 = a_3 = \ldots = a_{i_2-1} = 2a_1.$$

Now choose an $a_{i_2} < \frac{1}{2}\min\{a_1, \ldots, a_{i_2-1}\}$ such that

$$(i_3 - i_2)a_{i_2} < \frac{1}{2^2}$$

and put

$$a_{i_2+1} = a_{i_2+2} = \ldots = a_{i_3-1} = 2a_{i_2}.$$

Continuing this process, in a general step we choose an $a_{i_n} < \frac{1}{2}\min\{a_1, \ldots, a_{i_n-1}\}$ such that

$$(i_{n+1} - i_n)a_{i_n} < \frac{1}{2^n}$$

and put

$$a_{i_n+1} = a_{i_n+2} = \ldots = a_{i_{n+1}-1} = 2a_{i_n}.$$

For this sequence we have $r_n = 0$ if and only if $n \in I$, thus (a_n) is not $\mathcal{I}^{**}\downarrow$.

Denote $J = (\mathbb{N} \setminus I) \in \mathcal{F}(\mathcal{I})$. The subsequence $(a_n)_{n\in J}$ is non-increasing, thus (a_n) is $\mathcal{I}^*\downarrow$.

To show that (a_n) is also $\mathcal{I}\downarrow$ choose $n \in (i_k, i_{k+1})$ and calculate

$$r_n = \sum_{j\leqslant n, a_n > a_j} (a_n - a_j) = a_n - a_{i_k} = a_{i_k} < \frac{1}{2^k} \to 0$$

as $n \to \infty$, so $(r_n)_{n\in J}$ tends to 0. Consequently, $\mathcal{I}\text{-}\lim_{n\to\infty} r_n = 0$ and $\mathbf{a}$ is $\mathcal{I}\downarrow$.

Finally, the series $\sum_{n=1}^{\infty} a_n$ converges:

$$\sum_{n=1}^{\infty} a_n = \sum_{n\in I} a_n + \sum_{n\in J} a_n < \sum_{m=1}^{\infty} \frac{1}{2^m} + \sum_{m=1}^{\infty} \frac{1}{2^{m-1}} = 3.$$

□

We conclude this subsection with two examples providing the fact that $\mathcal{I}\downarrow$ and $\mathcal{I}^*\downarrow$ are independent for general $\mathcal{I}$. Notice that in a special case $\mathcal{I} = \mathcal{I}_{\mathrm{f}}$ the implication $\mathcal{I}_{\mathrm{f}}^*\downarrow \Rightarrow \mathcal{I}_{\mathrm{f}}\downarrow$ holds.

Example 3.1. There is an admissible ideal $\mathcal{I}$ and a real sequence $\mathbf{a} = (a_n)$, $a_n > 0$, $\sum_{n=1}^{\infty} a_n < \infty$, which is $\mathcal{I}^*\downarrow$ but not $\mathcal{I}\downarrow$.

Indeed, let $\mathcal{I}$ be an admissible ideal such that the set $K = \{2^k \colon k \in \mathbb{N}\}$ belongs to its dual filter $\mathcal{F}(\mathcal{I})$. For $k \in \mathbb{N}$ let us define

$$a_n = \begin{cases} \dfrac{1}{2^{k+1}} & \text{if } 2^{k-1} < n < 2^k; \\ \dfrac{1}{2^k} & \text{if } n = 2^k. \end{cases}$$

Then the subsequence $(a_n)_{n\in K}$ is non-increasing, thus (a_n) is $\mathcal{I}^*\downarrow$. On the other hand, for each $n = 2^k \in K$ we have

$$r_{2^k} = \sum_{j<2^k,\, a_j<a_{2^k}} (a_{2^k} - a_j) = \sum_{2^{k-1}<j<2^k} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = \frac{2^{k-1}-1}{2^{k+1}} \to \frac{1}{4}$$

as $k \to \infty$ verifying that (a_n) is not $\mathcal{I}\downarrow$.

E x a m p l e 3.2. There is an admissible ideal $\mathcal{J}$ and a real sequence (b_n) with $b_n > 0$, $\sum\limits_{n=1}^{\infty} b_n < \infty$, and such that (b_n) is $\mathcal{J}\!\downarrow$ but not $\mathcal{J}^*\!\downarrow$.

Let $L = \{2m\colon\ m \in \mathbb{N}\}$ and $L^{\mathrm{c}} = \mathbb{N} \setminus L$. Define an admissible ideal $\mathcal{J}$ by

$$\mathcal{J} = \{A \cup B\colon\ A \subset L^{\mathrm{c}},\ B \text{ is finite}\} = \langle \mathcal{I}_{\mathrm{f}}, L^{c} \rangle\,,$$

that is $\mathcal{J}$ is the smallest admissible ideal containing L^{c}. Denote by $\mathcal{F}(\mathcal{J})$ its dual filter and notice that $C \in \mathcal{F}(\mathcal{J})$ if and only if C contains almost all members of L. The sequence b_n is defined as follows:

$$b_n = \begin{cases} \dfrac{1}{n^2} & \text{if } n \in L^{\mathrm{c}}; \\ \dfrac{1}{n^2} & \text{if } n = 2m,\ m \text{ is odd}; \\ \dfrac{1}{2^n} & \text{if } n = 2m,\ m \text{ is even}. \end{cases}$$

Evidently $b_n > 0$, $\sum\limits_{n=1}^{\infty} b_n < \infty$, $nb_n \to 0$, so by Corollary 2.1 we have $r_n \to 0$ and (b_n) is $\mathcal{J}\!\downarrow$.

Let us prove that (b_n) is not $\mathcal{J}^*\!\downarrow$. Suppose the contrary, that is suppose that there is some $T \in \mathcal{F}(\mathcal{J})$ such that $(b_n)_{n\in T}$ is non-increasing. Then T contains almost all members of L, say $T \supset L'$, $L \setminus L'$ is finite and, by assumption, $(b_n)_{n\in L'}$ is non-increasing. On the other hand, for all odd $m \geqslant 3$ we have

$$\frac{1}{(2m)^2} > \frac{1}{2^{2(m+1)}} < \frac{1}{(2(m+2))^2},$$

a contradiction.

3.2. Ideal variants of Olivier's theorem. We have the following "$\mathcal{I}$-Olivier" theorem.

Theorem 3.1. *Let $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series with positive terms and $\mathcal{I}$ be an admissible ideal of subsets of $\mathbb{N}$. Then*

$$(a_n)\ \mathcal{I}\!\downarrow \quad \Leftrightarrow \quad \mathcal{I}\text{-}\lim_{n\to\infty} na_n = 0.$$

P r o o f. Since $\lim\limits_{n\to\infty}(na_n - r_n) = 0$ by Theorem 2.1, we have $\mathcal{I}\text{-}\lim\limits_{n\to\infty}(na_n - r_n) = 0$. The condition $(a_n)\ \mathcal{I}\!\downarrow$ means that $\mathcal{I}\text{-}\lim\limits_{n\to\infty} r_n = 0$; the result follows. □

Applying Theorem 3.1 and Proposition 3.1 to a convergent series with positive terms yields that if (a_n) is $\mathcal{I}^{**}\!\downarrow$, then $\mathcal{I}\text{-}\lim_{n\to\infty} na_n = 0$. But we obtain even more.

Theorem 3.2. *Let $\mathcal{I}$ be an admissible ideal of subsets of $\mathbb{N}$ and let $\mathbf{a} = (a_n)$ be a sequence of positive terms such that $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Then*

$$(a_n)\mathcal{I}^{**}\!\downarrow \quad \Rightarrow \quad \mathcal{I}^* - \lim_{n\to\infty} na_n = 0.$$

Proof. By hypothesis $M(\mathbf{a}) \in \mathcal{F}(\mathcal{I})$; by Proposition 2.1

$$\lim_{m\in M(\mathbf{a})} ma_m = 0; \tag{3.1}$$

hence $\mathcal{I}^*\text{-}\lim_{n\to\infty} na_n = 0$. □

Now there is a question: Can the assumption that (a_n) is $\mathcal{I}^{**}\!\downarrow$ be relaxed to $\mathcal{I}^*\!\downarrow$? In other words, is the "$\mathcal{I}^*$-Olivier" theorem true? The following example shows that the answer is, in general, no. We construct an ideal $\mathcal{I}$ and an $\mathcal{I}^*\!\downarrow$ sequence (a_n) with positive terms such that the corresponding series is convergent, but $\mathcal{I}^*\text{-}\lim_{n\to\infty} na_n = 0$ fails.

Example 3.3. Let S be the set of squares and $S^{\mathrm{c}} = \mathbb{N} \setminus S$ be the set of all positive integers not being squares. Let

$$\mathcal{I} = \{A \cup B\colon\ A \subset S^{\mathrm{c}} \text{ and } B \text{ is finite}\} = \langle \mathcal{I}_{\mathrm{f}}, S^{\mathrm{c}} \rangle.$$

Now define the sequence (a_n) as

$$a_n = \begin{cases} \dfrac{1}{n^2} & \text{if } n \in S^{\mathrm{c}}; \\ \dfrac{1}{n} & \text{if } n \in S. \end{cases}$$

Then $\sum\limits_{n=1}^{\infty} a_n$ converges, (a_n) is $\mathcal{I}^*\!\downarrow$ as the subsequence $(a_n)_{n\in S}$ is non-increasing and $S \in \mathcal{F}(\mathcal{I})$.

We are going to show that $\mathcal{I}^*\text{-}\lim_{n\to\infty} na_n = 0$ fails. Suppose the contrary, i.e. $\mathcal{I}^*\text{-}\lim_{n\to\infty} na_n = 0$. Then there is a set $K = \{k_1 < k_2 < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that $\lim_{n\to\infty} k_n a_{k_n} = 0$. On the other hand, by definition of $\mathcal{I}$, K contains infinitely many k_n's, for which $k_n a_{k_n} = 1$, a contradiction.

We see from the above example that the star version of Olivier's theorem does not hold in general. Thus, it would be interesting to characterize those ideals for which this takes place. To that purpose, let us say that an ideal $\mathcal{I}$ *satisfies* the $(*)$ *condition* if for every convergent series $\sum_{n=1}^{\infty} a_n$ with positive terms, the implication

$$(*) \qquad (a_n) \text{ is } \mathcal{I}^*\!\downarrow \quad \Rightarrow \quad \mathcal{I}^*\text{-}\lim_{n\to\infty} na_n = 0$$

holds. The next theorem says that an ideal $\mathcal{I}$ satisfies $(*)$ provided no set in $\mathcal{I}$ is too big.

Before stating this theorem, let us recall that for a set $J = \{j_1 < j_2 < \ldots\} \subset \mathbb{N}$ the lower and upper asymptotic densities are defined and denoted, respectively, by

$$\underline{d}(J) = \liminf_{n\to\infty} \frac{n}{j_n} \quad \text{and} \quad \overline{d}(J) = \limsup_{n\to\infty} \frac{n}{j_n}.$$

Theorem 3.3. *Let $\mathcal{I}$ be an ideal such that $\overline{d}(J) < 1$ for every $J \in \mathcal{I}$. Then $\mathcal{I}$ satisfies $(*)$.*

Proof. Assuming (a_n) is $\mathcal{I}^*\!\downarrow$, we can choose $K = \{k_1 < k_2 < \ldots\} \in \mathcal{F}(\mathcal{I})$ such that (a_{k_n}) is non-increasing. Of course, $\sum_{n=1}^{\infty} a_{k_n} < \infty$, thus Olivier's theorem can be applied to the sequence (a_{k_n}) to get $\lim_{n\to\infty} na_{k_n} = 0$. By assumptions, $\alpha = \liminf_{n\to\infty} n/k_n = \underline{d}(K) = 1 - \overline{d}(\mathbb{N} \setminus K) > 0$. Thus, we have

$$\limsup_{n\to\infty} \frac{k_n}{n} \lim_{n\to\infty} na_{k_n} = \frac{1}{\alpha} 0 = 0,$$

which implies $\lim_{n\to\infty} k_n a_{k_n} = 0$, so $\mathcal{I}^*\text{-}\lim_{n\to\infty} na_n = 0$. □

4. Application

Lemma 2.1 proves a "nice" asymptotic behavior of p_n. On the other hand, both statement (2.4) and Theorem 3.1 say that it is not the case of the asymptotic behavior of r_n. To see this, it is sufficient to consider a sequence $\mathbf{a} = (a_n)$, $a_n > 0$ such that the series $\sum_{n=1}^{\infty} a_n$ is convergent and na_n does not tend to 0. Nevertheless, some information on the asymptotic behavior of r_n can be derived. The inequality in (2.1) implies $a_n > r_n/n$ and, consequently, we obtain

$$\sum_{n=1}^{\infty} \frac{r_n}{n} < S < \infty. \tag{4.1}$$

Recall (Definition 3.3) that for every $\varepsilon > 0$, $T_\varepsilon(\mathbf{a}) = \{n \in \mathbb{N} \colon r_n \geqslant \varepsilon\}$. Then (4.1) implies

$$S > \sum_{n=1}^{\infty} \frac{r_n}{n} \geqslant \sum_{n \in T_\varepsilon(\mathbf{a})} \frac{r_n}{n} \geqslant \varepsilon \sum_{n \in T_\varepsilon(\mathbf{a})} \frac{1}{n}$$

and, consequently, $T_\varepsilon(\mathbf{a}) \in \mathcal{I}_c = \Bigl\{X \subset \mathbb{N} \colon \sum\limits_{x \in X} 1/x < \infty\Bigr\}$. Thus, for every convergent series with positive terms we have

$$\mathcal{I}_c\text{-}\lim_{n\to\infty} r_n = 0. \tag{4.2}$$

Since $\mathcal{I}_c\text{-}\lim\limits_{n\to\infty} (na_n - r_n) = 0$, the following theorem (see [6]) holds.

Theorem 4.1 ([6]). *For every convergent series $\sum\limits_{n=1}^{\infty} a_n$ of positive terms and each ideal $\mathcal{I} \supset \mathcal{I}_c$ we have $\mathcal{I}\text{-}\lim\limits_{n\to\infty} na_n = 0$.*

Remark 4.1. It is an easy exercise to show that the ideal $\mathcal{I}_c$ satisfies the (AP) condition and, analogously to (1.1), for any pair of admissible ideals $\mathcal{I} \subset \mathcal{J}$ the implication

$$\mathcal{I}^*\text{-}\lim_{n\to\infty} a_n = L \quad \Rightarrow \quad \mathcal{J}^*\text{-}\lim_{n\to\infty} a_n = L$$

holds for every sequence (a_n). Consequently, in the previous theorem the $\mathcal{I}$-convergence can be substituted by the $\mathcal{I}^*$-convergence.

Theorem 4.2. *For every convergent series $\sum\limits_{n=1}^{\infty} a_n$ of positive terms and each ideal $\mathcal{I} \supset \mathcal{I}_c$ we have $\mathcal{I}^*\text{-}\lim\limits_{n\to\infty} na_n = 0$.*

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