0-IDEALS IN 0-DISTRIBUTIVE POSETS

Khalid A. Mokbel, Hodaidah

Received March 27, 2014. First published October 13, 2016.
Communicated by Radomír Halaš

Abstract. The concept of a 0-ideal in 0-distributive posets is introduced. Several properties of 0-ideals in 0-distributive posets are established. Further, the interrelationships between 0-ideals and α-ideals in 0-distributive posets are investigated. Moreover, a characterization of prime ideals to be 0-ideals in 0-distributive posets is obtained in terms of non-dense ideals. It is shown that every 0-ideal of a 0-distributive meet semilattice is semiprime. Several counterexamples are discussed.

Keywords: 0-distributive poset; 0-ideal; α-ideal; prime ideal; semiprime ideal; dense ideal

MSC 2010: 06A06, 06A75

1. Introduction

Ideals play a major role in the theory of lattices, in particular distributive lattices. This fact gives the reason why some mathematicians have tried to study some types of ideals and establish their properties. Cornish [1] introduced the concept of 0-ideals in distributive lattices and obtained their properties in [2] using congruences. Jayaram [6] generalized the concept of 0-ideals in semilattices and studied their properties in [7] in the case of quasicomplemented 0-distributive semilattices.

In this paper we introduce the concept of 0-ideals for more general structures, namely the posets. In Section 2 of this paper, we will show that many of the classical results of the lattice theory can be extended to posets. In particular, we investigate the interrelationships between 0-ideals and α-ideals in 0-distributive posets. In Section 3, we establish the relations between 0-ideals and prime ideals and also between 0-ideals and semiprime ideals.

We begin with the necessary concepts and terminology. For undefined notation and terminology the reader is referred to Grätzer [3].

DOI: 10.21136/MB.2016.0028-14

Let P be a poset and $A \subseteq P$. The set $A^{\rm u} = \{x \in P\colon\ x \geqslant a$ for every $a \in A\}$ is called the *upper cone* of A. Dually, we have the concept of the *lower cone* $A^{\rm l}$ of A. We shall write $A^{\rm ul}$ instead of $\{A^{\rm u}\}^{\rm l}$ and dually. The upper cone $\{a\}^{\rm u}$ is simply denoted by $a^{\rm u}$ and $\{a,b\}^{\rm u}$ is denoted by $(a,b)^{\rm u}$. Similar notation is used for lower cones. Further, for $A, B \subseteq P$, $\{A \cup B\}^{\rm u}$ is denoted by $\{A, B\}^{\rm u}$ and for $x \in P$, the set $\{A \cup \{x\}\}^{\rm u}$ is denoted by $\{A, x\}^{\rm u}$. Similar notation is used for lower cones. We note that $A \subseteq A^{\rm ul}$ and $A \subseteq A^{\rm lu}$. If $A \subseteq B$, then $B^{\rm l} \subseteq A^{\rm l}$ and $B^{\rm u} \subseteq A^{\rm u}$. Moreover, $A^{\rm lul} = A^{\rm l}$, $A^{\rm ulu} = A^{\rm u}$ and $\{a^u\}^{\rm l} = \{a\}^{\rm l} = a^{\rm l}$.

A nonempty subset I of a poset P is called an *ideal* if $a, b \in I$ implies $(a,b)^{\rm ul} \subseteq I$, see Halaš [4]. Dually, we have the concept of a *filter*. Given $a \in P$, the subset $a^{\rm l} = \{x \in P\colon\ x \leqslant a\}$ is an ideal of P generated by a, denoted by $(a]$; we shall call $(a]$ a *principal ideal*. Dually, a filter $[a) = a^{\rm u} = \{x \in P\colon\ x \leqslant a\}$ generated by a is called a *principal filter*. A nonempty subset Q of P is called an *up directed set*, if $Q \cap (x,y)^{\rm u} \neq \varphi$ for any $x, y \in Q$. Dually, we have the concept of a *down directed set*. If an ideal I (filter F) is an up (down) directed set of P, then it is called a *u-ideal* (*l-filter*). An ideal or filter is called *proper* if it does not coincide with P.

A proper ideal I is called *prime* if $(a,b)^{\rm l} \subseteq I$ implies that $a \in I$ or $b \in I$, see Halaš and Rachůnek [5]. An ideal I of a poset P is called *semiprime* if $(a,b)^{\rm l} \subseteq I$ and $(a,c)^{\rm l} \subseteq I$ together imply $\{a, (b,c)^{\rm u}\}^{\rm l} \subseteq I$, see Kharat and Mokbel [9]. Dually, we have the concepts of a *prime filter* and *semiprime filter*.

A poset P with 0 is called 0-*distributive* if $(x,y)^{\rm l} = \{0\} = (x,z)^{\rm l}$ implies $\{x, (y,z)^{\rm u}\}^{\rm l} = \{0\}$ for $x, y, z \in P$, see Joshi and Waphare [8]. Evidently, a poset P with 0 is 0-distributive if and only if $(0]$ is a semiprime ideal.

For a nonempty subset A of a poset P with 0, define a subset $A^{\perp}$ of P as

$$A^{\perp} = \{z \in P\colon\ (a,z)^{\rm l} = \{0\} \text{ for all } a \in A\};$$

if $A = \{a\}$, then we write $a^{\perp}$ instead of $\{a\}^{\perp}$. We note that $A \subseteq A^{\perp\perp}$ and $a \in a^{\perp\perp}$. Further, $A^{\perp} = \bigcap\limits_{a \in A} a^{\perp}$ and $A \cap A^{\perp} = \{0\}$. Moreover, if $A \subseteq B$ then $B^{\perp} \subseteq A^{\perp}$.

An ideal I of a poset P is said to be an *α-ideal* if $x^{\perp\perp} \subseteq I$ for all $x \in I$, see Mokbel [10]. An ideal I of a poset P is said to be dense if $I^{\perp} = \{0\}$.

For a nonempty subset A of a poset P with 0, consider the set

$$0(A) = \{x \in P\colon\ (a,x)^{\rm l} = \{0\} \text{ for some } a \in A\}.$$

A proper ideal I of a poset P with 0 is said to be a 0-*ideal* if $I = 0(F)$ for some proper filter F of P.

Note that, for a given proper filter F of a poset P, if $0(F)$ is a 0-ideal, then $0(F) \cap F = \varphi$. In fact, if there exists $x \in P$ such that $x \in 0(F) \cap F$, then exists

$y \in F$ such that $(x,y)^{\mathrm{l}} = \{0\}$. Since $x, y \in F$, we have $(x,y)^{\mathrm{lu}} = \{0\}^{\mathrm{u}} = P \subseteq F$, a contradiction to the fact that F is a proper filter.

If P is a lattice then each of the above concepts coincides with the corresponding concept for lattices. Throughout this paper, P denotes a poset with 0.

2. 0-ideals and α-ideals

In this section, we will study the relation between 0-ideals and α-ideals in posets. We begin by proving the following result in a general poset.

Theorem 2.1. *Every 0-ideal of a poset is an α-ideal.*

Proof. Let I be a 0-ideal of a poset P. Then there exists a proper filter F such that $I = 0(F)$. Let us show that I is an α-ideal. To this aim, let $x \in I$ and $a \in x^{\perp\perp}$. We have to show that $a \in I$. Since $x \in I = 0(F)$, there exists an element $y \in F$ such that $(x,y)^{\mathrm{l}} = \{0\}$, that is $y \in x^{\perp}$. Now, because $a \in x^{\perp\perp}$ and $y \in x^{\perp}$, we have $(a,y)^{\mathrm{l}} = \{0\}$. Hence by definition of $0(F)$, we have $a \in 0(F) = I$. □

Remark 2.2. The converse of Theorem 2.1 does not hold in general. Let $\mathbb{N}$ be the set of natural numbers. Consider the set $P = \{\varphi\} \cup \{\mathbb{N}\} \cup \{X\colon\ X$ is a finite subset of $\mathbb{N}\}$. It is easy to observe that P is a poset under set inclusion. Let $I = P - \{\mathbb{N}\}$. Then I is an α-ideal but not a 0-ideal. Indeed, $\{\mathbb{N}\}$ is the only filter disjoint with I and $0(\{\mathbb{N}\}) = \{\varphi\}$. In Theorem 2.13 of this paper, we answer the question "Under which conditions, the converse of Theorem 2.1 will be true?". Before that, let us extrapolate some properties of 0-ideals.

Lemma 2.3. *Let F be a proper l-filter of a 0-distributive poset P. Then $0(F)$ is a 0-ideal.*

Proof. Let $x, y \in 0(F)$. To show that $0(F)$ is an ideal, we have to show that $(x,y)^{\mathrm{ul}} \subseteq 0(F)$. Since $x, y \in 0(F)$, there exist $f_1, f_2 \in F$ such that $(x, f_1)^{\mathrm{l}} = \{0\} = (y, f_2)^{\mathrm{l}}$. Since F is an l-filter and $f_1, f_2 \in F$, there exists an element $f \in (f_1, f_2)^{\mathrm{l}} \cap F$. Evidently $(x,f)^{\mathrm{l}} = \{0\} = (y,f)^{\mathrm{l}}$. By 0-distributivity, $\{f, (x,y)^{\mathrm{u}}\}^{\mathrm{l}} = \{0\}$. Let $z \in (x,y)^{\mathrm{ul}}$. So $(f,z)^{\mathrm{l}} \subseteq \{f, (x,y)^{\mathrm{u}}\}^{\mathrm{l}} = \{0\}$ which gives $(f,z)^{\mathrm{l}} = \{0\}$. This implies $z \in 0(F)$, as $f \in F$. Thus $(x,y)^{\mathrm{ul}} \subseteq 0(F)$. Therefore $0(F)$ is an ideal. Now, we claim that $0(F)$ is a proper ideal. On the contrary, suppose that $0(F) = P$. Then clearly $F \subset 0(F)$. So for any $a \in F$ there exists $b \in F$ such that $(a,b)^{\mathrm{l}} = \{0\}$. As $a, b \in F$ and F is a filter, we get that $P = \{0\}^{\mathrm{u}} = (a,b)^{\mathrm{lu}} \subseteq F$, a contradiction with the properness of F. □

R e m a r k 2.4. Note that the condition on a filter being an l-filter is necessary in the statement of Lemma 2.3. Indeed, consider the 0-distributive poset P depicted in Figure 1. Observe that the set $F = \{a, b, c, 1\}$ is a proper filter but not an l-filter and $0(F) = \{0\} \cup \{x_i\} \cup \{y_i\} \cup \{z_i\}$, where $i = 1, 2, \ldots$ But $0(F)$ is not an ideal, as $x_1, y_1 \in 0(F)$ and $(x_1, y_1)^{\mathrm{ul}} = a^{\mathrm{l}} \nsubseteq 0(F)$.

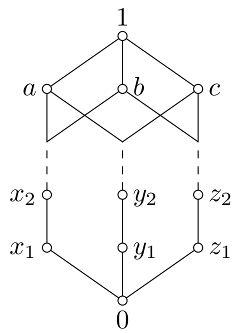


Figure 1.

Lemma 2.5 (Joshi and Waphare [8]). *A poset P is 0-distributive if and only if $x^{\perp}$ is an ideal for every $x \in P$.*

Theorem 2.6. *Let x be a nonzero element of a 0-distributive poset P. Then $x^{\perp}$ is a 0-ideal.*

P r o o f. Let x be a nonzero element of P, that means $x^{\perp} \neq P$. So $x^{\perp}$ is a proper ideal by Lemma 2.5. We claim that $x^{\perp} = 0([x))$. Suppose that $a \in x^{\perp}$. Then clearly, $(a, x)^{\mathrm{l}} = \{0\}$ and $x \in [x)$. Thus $a \in 0([x))$, and hence $x^{\perp} \subseteq 0([x))$. For the converse inclusion, let $a \in 0([x))$. Then there exists $z \in [x)$ such that $(a, z)^{\mathrm{l}} = \{0\}$. Since $x \leqslant z$, we obtain $(a, x)^{\mathrm{l}} = \{0\}$. This implies $a \in x^{\perp}$, and hence $0([x)) \subseteq x^{\perp}$. Combining both the inclusions, we get $x^{\perp} = 0([x))$. Thus $x^{\perp}$ is a 0-ideal. $\square$

Lemma 2.7. *Let P be a poset with 0. The following statements for $b, x, y \in P$ are equivalent:* (1) $b \in (x, y)^{\mathrm{l}\perp}$, (2) $(b, x, y)^{\mathrm{l}} = \{0\}$, (3) $(b, x)^{\mathrm{l}} \subseteq y^{\perp}$.

P r o o f. (1) $\Rightarrow$ (2). Suppose $b \in (x, y)^{\mathrm{l}\perp}$. Let $z \in (b, x, y)^{\mathrm{l}}$. Clearly, $z \leqslant b$ and $z \in (x, y)^{\mathrm{l}}$. Since $z \in (x, y)^{\mathrm{l}}$ and $b \in (x, y)^{\mathrm{l}\perp}$, we get $(b, z)^{\mathrm{l}} = \{0\}$. But $z \leqslant b$, therefore $z = 0$. Thus $(b, x, y)^{\mathrm{l}} = \{0\}$.

(2) $\Rightarrow$ (3). Suppose that $(b, x, y)^{\mathrm{l}} = \{0\}$. Let $z \in (b, x)^{\mathrm{l}}$. Then $(z, y)^{\mathrm{l}} \subseteq (b, x, y)^{\mathrm{l}} = \{0\}$. Hence $z \in y^{\perp}$.

(3) $\Rightarrow$ (1). Let $(b, x)^{\mathrm{l}} \subseteq y^{\perp}$ and $z \in (x, y)^{\mathrm{l}}$. To prove that $b \in (x, y)^{\mathrm{l}\perp}$ it is sufficient to show that $(b, z)^{\mathrm{l}} = \{0\}$. Since $z \leqslant x$ and $(b, x)^{\mathrm{l}} \subseteq y^{\perp}$, we have $(b, z)^{\mathrm{l}} \subseteq y^{\perp}$. Further, if $a \in (b, z)^{\mathrm{l}}$, then $a \in y^{\perp}$ and $a \leqslant z \leqslant y$. Consequently, $a = 0$ and hence $(b, z)^{\mathrm{l}} = \{0\}$ as required. $\square$

Lemma 2.8. *Let x and y be elements of a poset P with 0. Then*

$$x^{\perp\perp} \cap y^{\perp\perp} = (x,y)^{\mathrm{l}\perp\perp}.$$

Proof. It is enough to show that $x^{\perp\perp} \cap y^{\perp\perp} \subseteq (x,y)^{\mathrm{l}\perp\perp}$, as the converse inclusion is always true. Suppose that $a \in x^{\perp\perp} \cap y^{\perp\perp}$ and $b \in (x,y)^{\mathrm{l}\perp}$. We have to show that $(a,b)^{\mathrm{l}} = \{0\}$. Evidently $a \in x^{\perp\perp}$ and $a \in y^{\perp\perp}$, so we have $x^{\perp} \subseteq a^{\perp}$ and $y^{\perp} \subseteq a^{\perp}$. Now, since $b \in (x,y)^{\mathrm{l}\perp}$, by Lemma 2.7 we have $(b,x)^{\mathrm{l}} \subseteq y^{\perp}$. This implies $(b,x)^{\mathrm{l}} \subseteq a^{\perp}$. Again by the assertion of Lemma 2.7, $(b,x)^{\mathrm{l}} \subseteq a^{\perp}$ implies $(a,b)^{\mathrm{l}} \subseteq x^{\perp}$. Hence $(a,b)^{\mathrm{l}} \subseteq x^{\perp} \subseteq a^{\perp}$. Now, it is clear that $(a,b)^{\mathrm{l}} = \{0\}$. Indeed, if $z \in (a,b)^{\mathrm{l}}$, then $z \in (a] \cap a^{\perp} = \{0\}$. Therefore $z = 0$, as we need. □

For an ideal I of a poset P, let I' and $I_{\perp}$ denote the following subsets of P:

$$I' = \{x \in P\colon\ z^{\perp} \subseteq x^{\perp} \text{ for some } z \in I\},$$
$$I_{\perp} = \{x \in P\colon\ z^{\perp} \subseteq x^{\perp\perp} \text{ for some } z \in I\}.$$

In the next result, we establish some properties of $I_{\perp}$.

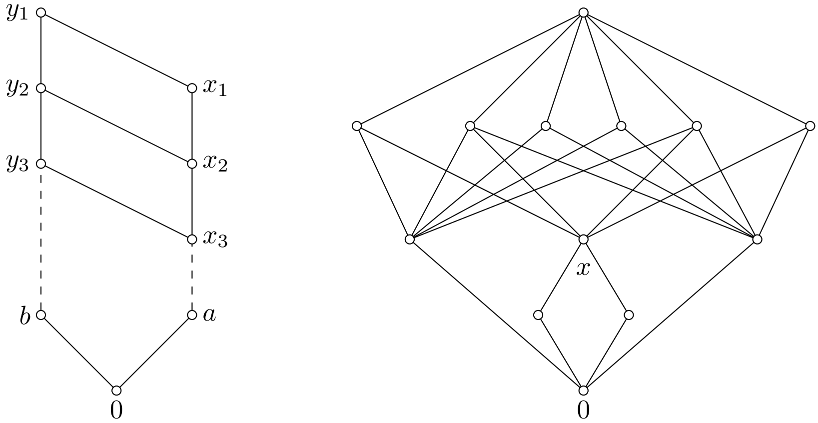
Lemma 2.9. *Let I be a proper u-ideal of a 0-distributive poset P. Then $I_{\perp}$ is a filter. Moreover, if I is an α-ideal, then $I_{\perp}$ is a proper filter.*

Proof. Let I be a proper u-ideal of P. We show that $I_{\perp}$ is a filter. For this assume that $x, y \in I_{\perp}$. We have to show that $(x,y)^{\mathrm{lu}} \subseteq I_{\perp}$. Since $x, y \in I_{\perp}$, there exist $z_1, z_2 \in I$ such that $z_1^{\perp} \subseteq x^{\perp\perp}$ and $z_2^{\perp} \subseteq y^{\perp\perp}$, and thus $z_1^{\perp} \cap z_2^{\perp} \subseteq x^{\perp\perp} \cap y^{\perp\perp}$. This implies $z_1^{\perp} \cap z_2^{\perp} \subseteq (x,y)^{\mathrm{l}\perp\perp}$ by Lemma 2.8. Since I is a u-ideal and $z_1, z_2 \in I$, there exists an element $z \in P$ such that $z \in (z_1,z_2)^{\mathrm{u}} \cap I$. Now, $z \in (z_1,z_2)^{\mathrm{u}}$ gives $z^{\perp} \subseteq z_1^{\perp} \cap z_2^{\perp}$, hence $z^{\perp} \subseteq (x,y)^{\mathrm{l}\perp\perp}$. Now, let $a \in (x,y)^{\mathrm{lu}}$. Then clearly $(x,y)^{\mathrm{l}} \subseteq a^{\mathrm{l}}$, thus $(x,y)^{\mathrm{l}\perp\perp} \subseteq a^{\perp\perp}$. This implies $z^{\perp} \subseteq (x,y)^{\mathrm{l}\perp\perp} \subseteq a^{\perp\perp}$. Since $z^{\perp} \subseteq a^{\perp\perp}$ and $z \in I$ by the definition of $I_{\perp}$, we get $a \in I_{\perp}$. Consequently, $(x,y)^{\mathrm{lu}} \subseteq I_{\perp}$.

Further, let I be an α-ideal. We claim that $I_{\perp} \neq P$. Suppose on the contrary that $I_{\perp} = P$. Observe that $0 \in I_{\perp}$. Hence by the definition of $I_{\perp}$, there exists $z \in I$ such that $z^{\perp} \subseteq 0^{\perp\perp} = \{0\}$, that is, $z^{\perp} = \{0\}$. Since I is an α-ideal and $z \in I$, we have $P = \{0\}^{\perp} = z^{\perp\perp} \subseteq I$, a contradiction to the fact that I is a proper ideal. □

Remark 2.10. (1) In Lemma 2.9, the condition on I of being a u-ideal is necessary. For example in the 0-distributive poset P depicted in Figure 2, the ideal $I = \{0, a, b\}$ is not a u-ideal and $I_{\perp} = \{y_i\} \cup \{x_i\} \cup \{a, b\}$, where $i = 1, 2, \ldots$, is not a filter. In fact, $a, b \in I_{\perp}$ but $(a,b)^{\mathrm{lu}} = P \nsubseteq I_{\perp}$.

(2) The assertion of Lemma 2.9 is not true if we remove the condition that I is an α-ideal. For this, consider the three elements poset $P = \{0, a, 1\}$, where $0 < a < 1$.



Figures 2 and 3.

It can be easily seen that P is 0-distributive. Observe that the set $I = \{0, a\}$ is a proper u-ideal but not an α-ideal. Note that $I_\perp$ is a filter but not proper.

We say that a poset P satisfies the condition (Q) if the following assertion is true.

(Q) For any $x \in P$, there exists $y \in P$ such that $x^{\perp\perp} = y^\perp$.

Remark 2.11. The poset P depicted in Figure 3 is an example of a 0-distributive one which does not satisfy (Q). In fact, $x \in P$ but there is no element $y \in P$ for which $x^{\perp\perp} = y^\perp$.

Lemma 2.12 (Mokbel [10]). *Let I be a u-ideal of a 0-distributive poset P. Then I' is the smallest α-ideal containing I. Moreover, an ideal I of P is an α-ideal if and only if $I = I'$.*

Theorem 2.13. *Let I be a proper u-ideal of a 0-distributive poset P satisfying the condition (Q). If I is an α-ideal, then I is a 0-ideal.*

Proof. Let I be an α-ideal of P. By Lemma 2.9, $I_\perp$ is a proper filter. To show that I is a 0-ideal, it is enough to show that $I = 0(I_\perp)$. Let $x \in I$. Since $I = I'$ by Lemma 2.12, we have $x \in I'$. Hence there exists $z \in I$ such that $z^\perp \subseteq x^\perp$. Therefore $x^{\perp\perp} \subseteq z^{\perp\perp}$. Since $x \in P$, by (Q) there exists $y \in P$ such that $x^{\perp\perp} = y^\perp$. Using $x^{\perp\perp} \subseteq z^{\perp\perp}$, we get $y^\perp \subseteq z^{\perp\perp}$. This yields $z^\perp \subseteq y^{\perp\perp}$. Now, $z^\perp \subseteq y^{\perp\perp}$ and $z \in I$ together imply that $y \in I_\perp$. Since $y \in I_\perp$ and $x \in x^{\perp\perp} = y^\perp$, that is, $(x, y)^{\rm l} = \{0\}$, we have $x \in 0(I_\perp)$. Therefore $I \subseteq 0(I_\perp)$.

For the converse inclusion, let $x \in 0(I_\perp)$. Then there is an element $b \in I_\perp$ such that $(x, b)^{\rm l} = \{0\}$. This gives $b^{\perp\perp} \subseteq x^\perp$. Since $b \in I_\perp$, there exists an element $z \in I$ such that $z^\perp \subseteq b^{\perp\perp}$. This means that $z^\perp \subseteq b^{\perp\perp} \subseteq x^\perp$. Since $z^\perp \subseteq x^\perp$ and $z \in I$, we have $x \in I' = I$. Thus $0(I_\perp) \subseteq I$. By combining both the inclusions, we have $I = 0(I_\perp)$. □

R e m a r k 2.14. (1) The condition on I of being a u-ideal cannot be dropped in the statement of Theorem 2.13. The 0-distributive poset P shown in Figure 4 clearly satisfies the condition (Q). Now, consider the proper α-ideal $I = \{0, a, b\}$ which is not a u-ideal. Observe that there does not exist a filter F of P for which $I = 0(F)$.

(2) Also, consider the 0-distributive poset P depicted in Figure 3 which does not satisfy (Q). Observe that the proper u-ideal $I = (x]$ is an α-ideal but there does not exist a filter F in P for which $I = 0(F)$. Therefore the condition (Q) cannot be dropped out in Theorem 2.13.

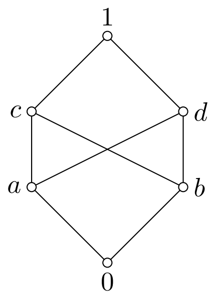


Figure 4.

An immediate consequence of Theorem 2.1 and Theorem 2.13 is

Corollary 2.15. *Let I be a proper u-ideal of a 0-distributive poset P satisfying the condition (Q). Then I is an α-ideal if and only if I is a 0-ideal.*

3. 0-ideals and primeness

Lemma 3.1. *Every non-dense prime ideal of a 0-distributive poset P is of the form $x^\perp$ for some nonzero x of P.*

P r o o f. Let I be a non-dense prime ideal of P, that is, $I^\perp \neq \{0\}$. Then there exists an element $x \in I^\perp$ such that $x \neq 0$. Using the fact that $I \cap I^\perp = \{0\}$, we get that $x \notin I$. We claim that $I = x^\perp$. Since $x \in I^\perp$, we obtain $I \subseteq I^{\perp\perp} \subseteq x^\perp$. Hence $I \subseteq x^\perp$. For the converse inclusion, suppose $z \in x^\perp$. We have $(z, x)^l = \{0\} \subseteq I$ and $x \notin I$; by primeness of I, we get $z \in I$. Thus $x^\perp \subseteq I$, as we need. □

By Theorem 2.6 and Lemma 3.1, the following corollary follows.

Corollary 3.2. *If a prime ideal I of a 0-distributive poset P is non-dense, then I is a 0-ideal.*

An element q of a poset P with 0 is called an atom if there is no $c \in P$ for which $0 < c < q$.

Lemma 3.3 (Kharat and Mokbel [9]). *Every l-filter of a finite poset P is principal.*

Theorem 3.4. *Let F be an l-filter of a finite 0-distributive poset P. Then $0(F)$ is a semiprime ideal.*

P r o o f. If $F = P$, then $0(F) = P$ is a semiprime ideal. Suppose that $F \neq P$. By Lemma 3.3, F is principal, say $F = [f)$. In view of Lemma 2.3, it is enough to show that $0([f))$ is semiprime. Suppose that $(x, y)^{\rm l} \subseteq 0([f))$ and $(x, z)^{\rm l} \subseteq 0([f))$. We have to show that $\{x, (y, z)^{\rm u}\}^{\rm l} \subseteq 0([f))$. Let $a \in \{x, (y, z)^{\rm u}\}^{\rm l}$. Suppose on the contrary that $a \notin 0([f))$. Therefore $(a, f)^{\rm l} \neq \{0\}$, and so there is a nonzero element $b \in P$ such that $b \in (a, f)^{\rm l}$. Since P is finite and $b \neq 0$, there exists an atom $q \in P$ such that $q \leqslant b$. Observe that $(q, y)^{\rm l} = \{0\}$. Indeed, if $(q, y)^{\rm l} \neq \{0\}$, then $q \leqslant y$. Since $q \leqslant a \leqslant x$ and $q \leqslant y$, we get that $q \in (x, y)^{\rm l} \subseteq 0([f))$, thus $(q, f)^{\rm l} = \{0\}$, a contradiction to the fact that $q \leqslant f$. Similarly, $(q, z)^{\rm l} = \{0\}$. Now, by 0-distributivity, we get $\{q, (y, z)^{\rm u}\}^{\rm l} = \{0\}$. Since $a \in (y, z)^{\rm ul}$, we have $(q, a)^{\rm l} = \{0\}$, a contradiction to the fact that $q \leqslant a$. Thus $a \in 0([f))$. $\square$

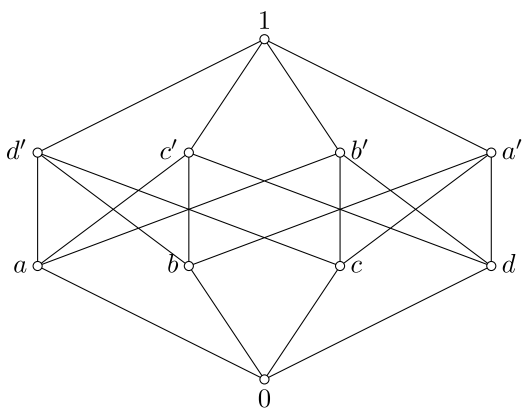


Figure 5.

R e m a r k 3.5. In the finite 0-distributive poset P depicted in Figure 5, consider the filter $F = \{1, a', b', c'\}$ which is not an l-filter. Observe that $0(F) = \{0, a, b, c\}$ is an ideal but not a semiprime one. In fact, $(d', b')^{\rm l} \subseteq 0(F)$ and $(d', c')^{\rm l} \subseteq 0(F)$, but $d'^{\rm l} = \{d', (b', c')^{\rm u}\}^{\rm l} \nsubseteq 0(F)$. Hence the condition of the filter being an l-filter is essential in Theorem 3.4.

However, in the case of meet semilattices we have

Theorem 3.6. *Every* 0*-ideal of a* 0*-distributive meet semilattice* S *is semiprime.*

P r o o f. Suppose that $0(F)$ is a 0-ideal of S. Let $x \wedge y \in 0(F)$ and $x \wedge z \in 0(F)$. We have to show that $\{x, (y,z)^{\rm u}\}^{\rm l} \subseteq 0(F)$. Since $x \wedge y$, $x \wedge z \in 0(F)$, there exist $f_1, f_2 \in F$ such that $(x \wedge y) \wedge f_1 = 0$ and $(x \wedge z) \wedge f_2 = 0$. Since S is a meet semilattice and $f_1, f_2 \in F$, hence $f_1 \wedge f_2$ exists, say $f_1 \wedge f_2 = f$, and $f \in F$. As $f \leqslant f_1$ and $(x \wedge y) \wedge f_1 = 0$, we get that $(x \wedge y) \wedge f = (x \wedge f) \wedge y = 0$. Similarly, $(x \wedge f) \wedge z = 0$. By 0-distributivity, we have $\{x \wedge f, (y,z)^{\rm u}\}^{\rm l} = \{0\}$. This implies $f^{\rm l} \cap \{x, (y,z)^{\rm u}\}^{\rm l} = \{0\}$. Now, let $a \in \{x, (y,z)^{\rm u}\}^{\rm l}$. Then we have $(f,a)^{\rm l} = \{0\}$. Observe that $(f,a)^{\rm l} = \{0\}$ and $f \in F$ together imply that $a \in 0(F)$. Therefore $\{x, (y,z)^{\rm u}\}^{\rm l} \subseteq 0(F)$ as required. □

References

[1] *W. H. Cornish*: Annulets and α-ideals in a distributive lattice. J. Aust. Math. Soc. *15* (1973), 70–77. zbl MR doi

[2] *W. H. Cornish*: 0-ideals, congruences, and sheaf representations of distributive lattices. Rev. Roum. Math. Pures Appl. *22* (1977), 1059–1067. zbl MR

[3] *G. Grätzer*: General Lattice Theory. Birkhäuser, Basel, 1998. zbl MR

[4] *R. Halaš*: Characterization of distributive sets by generalized annihilators. Arch. Math., Brno *30* (1994), 25–27. zbl MR

[5] *R. Halaš, R. J. Rachůnek*: Polars and prime ideals in ordered sets. Discuss. Math., Algebra Stoch. Methods *15* (1995), 43–59. zbl MR

[6] *C. Jayaram*: 0-ideals in semilattices. Math. Semin. Notes, Kobe Univ. *8* (1980), 309–319. zbl MR

[7] *C. Jayaram*: Quasicomplemented semilattices. Acta Math. Acad. Sci. Hung. *39* (1982), 39–47. zbl MR doi

[8] *V. V. Joshi, B. N. Waphare*: Characterizations of 0-distributive posets. Math. Bohem. *130* (2005), 73–80. zbl MR

[9] *V. S. Kharat, K. A. Mokbel*: Primeness and semiprimeness in posets. Math. Bohem. *134* (2009), 19–30. zbl MR

[10] *K. A. Mokbel*: α-ideals in 0-distributive posets. Math. Bohem. *140* (2015), 319–328. zbl MR

Author's address: *Khalid A. Mokbel*, Mathematics Department, Education Faculty, Hodaidah University, P. O. Box 3114, Hodaidah, Yemen, e-mail: `khalidalaghbari@yahoo.com`.