A TOPOLOGICAL DUALITY FOR THE F-CHAINS ASSOCIATED WITH THE LOGIC C_ω

Verónica Quiroga, Víctor Fernández, San Juan

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To the memory of Professor Manuel Fidel, gratefully

Abstract. In this paper we present a topological duality for a certain subclass of the F_ω-structures defined by M. M. Fidel, which conform to a non-standard semantics for the paraconsistent N. C. A. da Costa logic C_ω. Actually, the duality introduced here is focused on F_ω-structures whose supports are chains. For our purposes, we characterize every F_ω-chain by means of a new structure that we will call *down-covered chain* (DCC) here. This characterization will allow us to prove the dual equivalence between the category of F_ω-chains and a new category, whose objects are certain special topological spaces (together with a distinguished family of open sets) and whose morphisms are particular continuous functions.

Keywords: paraconsistent logic; algebraic logic; dualities for ordered structures

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1. Introduction and preliminaries

The algebraic-relational structures known as F*-structures* were defined by Fidel to obtain semantics for the paraconsistent da Costa logics C_ω and $\{C_n\}_{\{n\geqslant 1\}}$ (see [5]). After their definition in [7] and their application to other logics (see [8]), F-structures became forgotten in some sense. However, since the works of Odintsov (see [11]), they were taken into account in the last years, and applied to the definition of "non-standard" semantics for other logics, different from C_n. So, the study of such structures has been renewed in the last years. This work is part of that trend.

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F_ω-structures (that is, F-structures related to C_ω) are defined on the basis of a relatively pseudocomplemented lattice L, together with a family of sets. This family interprets the negation connective in such logics. In this work, we focus on obtaining a topological duality for a particular subclass of F_ω-structures: the class of F_ω-chains. This case is the object of our study for several reasons. First, an adequate axiomatics for the logic C_ω can be defined as being the "intersection of axioms" of the rest of the C_n-logics.[1] So, the construction of a duality for the particular subclass of the F_ω-chains would suggest a way to obtain dualities for the whole class of the F_ω-structures, and also for the classes of F_n-structures (that is, F-structures related to C_n), in a general way. On the other hand, it will be noted that the traditional techniques for dualities are developed for several algebraic structures (lattices, Heyting algebras, and so on), but they are not usually concerned with algebraic-relational structures. So, this work is an attempt to investigate this kind of structures, starting from the very simple class of F_ω-chains.

To reach our purposes, we proceed as follows: we take as a starting point the formalism for F_ω-structures employed in [13] (applied to F_ω-chains, in this case). Besides, we define certain simple structures (called down-covered chains, or DCC) and we show that F_ω-chains can be characterized by them. This allows us to identify F_ω-chains with DCC, which will simplify our study. We give a representation for such structures by means of special topological spaces with additional properties (called ChS-spaces) and, at the end of the paper, we prove the duality between the respective categories to be defined later. Besides that, we show some examples of F_ω-chains and their duals.

With respect to the basic concepts and the notation to be employed, we will use certain algebraic notions given mainly in [14], but with a simpler formalism (taken, mostly, of [6]). The concept of a *relatively pseudocomplemented lattice* (or RPL, for short) is useful for our work: recall here that a lattice $(L, \vee, \wedge)$ (with associated order $\leqslant$) is a RPL[2] if and only if, for every a, $b \in L$, there exists $\max\{x\colon\ x \wedge a \leqslant b\}$ (which is denoted by $a \to b$). Every arbitrary RPL has the greatest element $\mathbf{1_L}$, defined by $\mathbf{1_L} := a \to a$, for any $a \in L$, but it does not necessarily have the least element. In this context, a Heyting algebra is understood as a RPL with the least element $\mathbf{0_L}$. By the way, Heyting algebras are applied to the study of F_n-structures, but they are not specially considered in the case of F_ω-structures. On the other

[1] However, C_ω should not be understood as "the intersection of the C_n-logics", but as a "smaller logic", yet. See [2] for a detailed discussion on this subject, far away of the scope of our paper.

[2] Relative pseudocomplemented lattices are known by other names, such as implicative lattices, Brouwerian lattices, or generalized Heyting algebras. See [9] for detailed information.

hand, for every RPL L with the greatest element $\mathbf{1_L}$ we define, for every $x \in L$, the set $x^{\intercal} := \{y \in L \colon y \vee x = \mathbf{1_L}\}$ (this definition is the dual of the notion of an *annihilator relative to* $\mathbf{0_L}$, given by Mandelker in [10]). The set $x^{\intercal}$ is useful in the definition of F-structures, in a general way.

The set of natural (even, integer) numbers, which will be used in some examples, is denoted by $\mathbb{N}$ ($\mathbb{P}$, $\mathbb{Z}$). In addition, it is convenient to fix some notation about posets. Let $(X, \leqslant)$ be an arbitrary poset: the *up-set* (*down-set*) *generated by* A will be denoted as $\uparrow A$ ($\downarrow A$). In this context, the set $\uparrow\{x\}$ ($\downarrow\{x\}$) will be merely denoted by $\uparrow x$ ($\downarrow x$). Besides, every subset $U \subseteq X$ satisfying $\uparrow U = U$ ($\downarrow U = U$) will be called, simply, *up-set* (*down-set*). Finally, the set of all subsets of X is denoted by $\wp(X)$, while the set of all the up-sets (down-sets) of X will be indicated by $\wp_u(X)$ ($\wp_d(X)$).

For the definition of the topological dual of F_ω-chains we will make use of prime filters and ideals. We assume that the basic concepts about them are known by the reader. In this context, the set $\mathrm{Fi}(M)$ ($\mathrm{Id}(M)$) denotes the filter (ideal) generated by $M \subseteq L$. In addition, the set $X(L) = \{P \subseteq L \colon P \text{ is a prime filter}\}$ will be used along this paper. In the particular context of chains, as here, the following fact should be emphasized:

Remark 1.1. Recall the behavior of filters and ideals in chains: if L is a chain, then the filters (ideals) in L are, simply, its nonempty up-sets (down-sets). On the other hand, the prime filters (ideals) of L are, exactly, its nonempty up-sets (down-sets), except L. In addition, for every $M \subseteq L$, $\mathrm{Fi}(M) = \uparrow M = \{a \in L \colon \exists\, m \in M \colon m \leqslant a\}$ ($\mathrm{Id}(M) = \downarrow M = \{a \in L \colon \exists\, m \in M \colon m \geqslant a\}$).

With respect to general topological notations, the set of compact elements of a given topological space (X, τ) will be indicated by $\mathcal{K}_\tau$, and therefore the set of the *open and compact* subsets of X will be denoted by $\tau \cap \mathcal{K}_\tau$. This last family will be used troughout this paper. Finally we remark that, if there is no risk of confusion, the analysed structures will be denoted as their supports. So, a lattice $(L, \wedge, \vee)$ will be indicated simply as L, and so on. However, when we consider necessary to denote two different structures with the same support, they will be distinguished by giving proper names to the involved structures, different from their supports. This exception is also valid for the definition of structures obtained by the application of functors, in the categories to be defined at the end of this paper.

2. F_ω-STRUCTURES AND F_ω-CHAINS

In this section we define F_ω-chains, and characterize them by means of the so-called *down-covering sets*. For that recall first the definition of F_ω-structures, according to the formalism used in [13]:

Definition 2.1. An *F_ω-structure* is a pair $(L, \mathfrak{f})$, where $L = (L, \vee, \wedge, \rightarrow, \mathbf{1_L})$ is a RPL and $\mathfrak{f}\colon L \rightarrow \wp(L)$ a function satisfying, for any $x \in L$:

(F1) $\emptyset \subsetneq \mathfrak{f}(x) \subseteq x^{\mathsf{T}}$;

(F2) $\mathfrak{f}(y) \cap \downarrow x \neq \emptyset$ for every $y \in f(x)$.

In addition, an F_ω-structure is an *F_ω-chain* if and only if the RPL L is a chain (with the greatest element $\mathbf{1_L}$, obviously).

Definition 2.2. A pair (L, A) is a *down-covered chain* (or DCC for short) if and only if L is a chain with the greatest element $\mathbf{1_L}$ and the set $A \subseteq L$ satisfies the following property:

(C) For every $x \in L$, $A \cap \downarrow x \neq \emptyset$ (or, equivalently, $\uparrow A = L$).

The set A will be called a *down-covering set of* L.[3]

Note that in every DCC (L, A), we have $A \neq \emptyset$. Some examples of down-covered chains are given below.

Example 2.3. Let $(\mathbb{Z}, \leqslant)$ be the chain of integers with its usual order. Consider $L := \mathbb{Z} \setminus \{0\}$ and the order $\leqslant_{L_1}$ defined on L as follows:

$$x \leqslant_{L_1} y \quad \text{if and only if} \begin{cases} x \in \mathbb{Z}^+,\ y \in \mathbb{Z}^+ & \text{and } y \leqslant x, \text{ or} \\ x \in \mathbb{Z}^+,\ y \in \mathbb{Z}^-, & \text{or} \\ x \in \mathbb{Z}^-,\ y \in \mathbb{Z}^- & \text{and } x \leqslant y. \end{cases}$$

Obviously, $\mathbf{L_1} := (L, \leqslant_{L_1})$ is a chain whose greatest element is $\mathbf{1_{L_1}} = -1$. Now, if we consider the sets $O := \{z \in L\colon z \text{ is odd}\}$ and $E := \{z \in L\colon z \text{ is even}\}$, it is easy to see that both pairs, $(\mathbf{L_1}, O)$ and $(\mathbf{L_1}, E)$, are down-covered chains. As a generalization of E take, for every $n \in \mathbb{N}$, the set $D_n = \{z \in L\colon n \text{ divides } z\}$. Obviously, all the pairs $(\mathbf{L_1}, D_n)$ are DCC, as well.

The previous example was built on the basis of a chain without the least element $\mathbf{0_L}$. An example *with the least element* is the following:

[3] Of course, the definition of DCC can be generalized to chains without the greatest element. Anyway, along this paper, we will assume that all the down-covered chains analysed here have the greatest element $\mathbf{1_L}$.

Example 2.4. Again, consider $(\mathbb{Z}, \leqslant)$ and L as above. Let $\leqslant_{L_2}$ be the order defined on L by:

$$x \leqslant_{L_2} y \quad \text{if and only if} \begin{cases} x \in \mathbb{Z}^+,\ y \in \mathbb{Z}^+ & \text{and } x \leqslant y, \text{ or} \\ x \in \mathbb{Z}^+,\ y \in \mathbb{Z}^-, & \text{or} \\ x \in \mathbb{Z}^-,\ y \in \mathbb{Z}^- & \text{and } x \leqslant y. \end{cases}$$

As before, the pair $\mathbf{L_2} := (L, \leqslant_{L_2})$ is a chain with $\mathbf{1_{L_2}} = -1$ *and the least element* $\mathbf{0_{L_2}} = 1$. Considering now the set O of Example 2.3, we have that $(\mathbf{L_2}, O)$ is a DCC, but $(\mathbf{L_2}, E)$ *is not* a DCC. Besides that, if we define (for every $n \in \mathbb{N}$) the set $[1, n] := \{z \in \mathbb{Z} \colon 1 \leqslant z \leqslant n\}$, we have that all the pairs $(\mathbf{L_2}, [1, n])$ are DCC, while no pair of the form $(\mathbf{L_1}, [1, n])$ is a DCC.[4]

We will return to these examples later.

The characterization of F_ω-chains by means of down-covered chains is very simple, and it is indebted to the following results:

Proposition 2.5. *Let $(L, \mathfrak{f})$ be an F_ω-chain. Then, $(L, \mathfrak{f}(\mathbf{1_L}))$ is a DCC.*

Proof. Let us consider an F_ω-chain $(L, \mathfrak{f})$, and let us prove that $\mathfrak{f}(\mathbf{1_L})$ satisfies (C), in Definition 2.2. For that, suppose $x \in L$: if $x = \mathbf{1_L}$, since $\mathfrak{f}(\mathbf{1_L}) \neq \emptyset$, there is $a \in \mathfrak{f}(\mathbf{1_L})$ and $a \leqslant \mathbf{1_L}$, obviously. If $x \neq \mathbf{1_L}$, since L is a chain and $\emptyset \subsetneq \mathfrak{f}(x) \subseteq x^{\intercal} = \{\mathbf{1_L}\}$, we have that $\mathbf{1_L} \in \mathfrak{f}(x)$. By (F2) of Definition 2.1, there is $a \in \mathfrak{f}(\mathbf{1_L})$, with $a \leqslant x$. Thus, $\mathfrak{f}(\mathbf{1_L})$ is a down-covering set of L. □

Proposition 2.6. *Let (L, A) be a down-covering chain, and define $\mathfrak{f} \colon L \to \wp(L)$ as follows: $\mathfrak{f}(\mathbf{1_L}) := A$; $\mathfrak{f}(x) := \{\mathbf{1_L}\}$ for every $x \in L \setminus \{\mathbf{1_L}\}$. Then $(L, \mathfrak{f})$ is an F_ω-chain.*

Proof. Suppose (L, A) and $\mathfrak{f}$ as was indicated above, and consider $x \in L$: taking into account the definition of $\mathfrak{f}$, it is clear that (F1) of Definition 2.1 is satisfied (because $A \neq \emptyset$). Now, suppose that $y \in \mathfrak{f}(x)$: if $x \neq \mathbf{1_L}$, then $y = \mathbf{1_L}$, by the definition of $\mathfrak{f}$. From (C), $\mathfrak{f}(y) \cap {\downarrow} x = A \cap {\downarrow} x \neq \emptyset$. If $x = \mathbf{1_L}$, then ${\downarrow} x = L$, and so $\mathfrak{f}(y) \cap {\downarrow} x = \mathfrak{f}(y) \neq \emptyset$, by the definition of $\mathfrak{f}$. So, (F2) is satisfied, too. □

It is easy to prove that Propositions 2.5 and 2.6 entail:

Proposition 2.7. *There exists a bijective correspondence between F_ω-chains of the form $(L, \mathfrak{f})$ and down-covered chains of the form (L, A).*

So, from now on, every F_ω-chain will be considered as a DCC of the form (L, A).

[4] It should be clear that the *sets* $[1, n]$ are defined taking into account merely the usual order $\leqslant$ on $\mathbb{Z}$. On the other hand, when these (already defined) sets are analysed (w.r.t. property (C)) in the context of $\mathbf{L_1}$ ($\mathbf{L_2}$), the considered order is $\leqslant_{L_1}$ ($\leqslant_{L_2}$).

3. Topological representation of F_ω-chains

In this section, we will define a new kind of structure, which will be based on a particular topological space (this structure will be called a Ch_ω-space along this paper). In addition, we will prove that any F_ω-chain can be represented by means of a convenient Ch_ω-space. Our definition is motivated mainly by the approach developed in [1].

Definition 3.1. A *Ch_ω-space* is a system $(X, \tau, \mathcal{S})$ such that:

(S1) (X, τ) is a T_0-space.
(S2) The family $\tau \cap \mathcal{K}_\tau$ is a basis of τ.
(S3) $(\tau, \subseteq)$ is a chain.
(S4) For every family $\{S_i\}_{i\in I} \cup \{T_j\}_{j\in J} \subseteq \tau \cap \mathcal{K}_\tau \setminus \{\emptyset\}$, with $I \neq \emptyset \neq J$, such that $\bigcap\limits_{i\in I} S_i \subseteq \bigcup\limits_{j\in J} T_j$, there are $i_0 \in I$ and $j_0 \in J$ verifying $S_{i_0} \subseteq T_{j_0}$.
(S5) $\mathcal{S} \subseteq \tau \cap \mathcal{K}_\tau$ and, in addition, $\bigcup\limits_{S\in\mathcal{S}} S = X$.

Conditions (S1)–(S4) are variants of similar requirements given in [1] to obtain a Stone duality for distributive lattices. In particular, (S3) is a generalization of the definition of the well known Sierpinski topology. On the other hand, (S4) makes sense because we are dealing with chains, as we shall see. Besides that, condition (S5) can be understood as the topological dual to the notion of down-covering set, as the following result shows:

Proposition 3.2. *A structure $(X, \tau, \mathcal{S})$ is a Ch_ω-space if and only if it satisfies conditions* (S1)–(S4) *of the previous definition, together with the additional condition:*
(S5′) *$\mathcal{S} \subseteq \tau \cap \mathcal{K}_\tau$ and for every $B \in \tau \cap \mathcal{K}_\tau$ there is $S \in \mathcal{S}$ such that $B \subseteq S$.*

P r o o f. Starting from a *Ch_ω-space* $(X, \tau, \mathcal{S})$ (according to Definition 3.1), let us prove (S5′). For that, let $B \in \tau \cap \mathcal{K}_\tau$. By (S5), $B \subseteq \bigcup\limits_{S\in\mathcal{S}} S$ and, since $B \in \mathcal{K}_\tau$, there exists $\{S_1, \ldots, S_n\} \subseteq \mathcal{S}$ such that $\bigcup\limits_{i=1}^{n} S_i = S_0 \in \mathcal{S}$ (because $\mathcal{S} \subseteq \tau \cap \mathcal{K}_\tau$, and by (S3)). On the other hand, suppose that $(X, \tau, \mathcal{S})$ satisfies (S1)–(S4) and (S5′). Since $X \in \tau$, by (S2), $X = \bigcup\limits_{i\in I} B_i$ for some I, with $B_i \in \tau \cap \mathcal{K}_\tau$ for every $i \in I$. By (S5′), for every $i \in I$ there is $S_i \in \mathcal{S}$ such that $B_i \subseteq S_i$. Thus, $X \subseteq \bigcup\limits_{i\in I} S_i \subseteq \bigcup\limits_{S\in\mathcal{S}} S \subseteq X$. □

From this last result, by a *Ch_ω-space* we mean a system satisfying (S1)–(S4), (S5′).

E x a m p l e 3.3. Consider $(\mathbb{N}, \tau_{\mathbb{P}})$, with $\tau_{\mathbb{P}} = \{\emptyset, \mathbb{N}\} \cup \{[1, i]\}_{i\in\mathbb{P}}$ (here $[1, i] := \{j \in \mathbb{N} \colon 1 \leqslant j \leqslant i\}$). Now, the following facts are valid: first of all, it is obvious that $(\mathbb{N}, \tau_{\mathbb{P}})$ is a T_0-topological space, and $(\tau_{\mathbb{P}}, \subseteq)$ is a chain. Besides that, note that $\tau_{\mathbb{P}} \cap \kappa_{\tau_{\mathbb{P}}} = \tau_{\mathbb{P}} \setminus \{\mathbb{N}\}$. From this, (S2) can be proved, too. In addition, property (S4) in Definition 3.1 can be verified, recalling that $\mathbb{N}$ is a well-ordered set, and that $\tau_{\mathbb{P}} \cap \kappa_{\tau_{\mathbb{P}}} = \tau_{\mathbb{P}} \setminus \{\mathbb{N}\}$, again. Thus, $(\mathbb{N}, \tau_{\mathbb{P}})$ can be a suitable underlying structure for a Ch_ω-space. An example of such space can be the system $(\mathbb{N}, \tau_{\mathbb{P}}, \mathcal{S}_4)$, where $\mathcal{S}_4 = \{[1,i]\}_{\{i=4k,\, k\in\mathbb{N}\}} = \{[1,4],[1,8],[1,12],\ldots\}$. Or, more generally (having fixed any natural number $t = 2n$, with $n \in \mathbb{N}$), the structures $(\mathbb{N}, \tau_{\mathbb{P}}, \mathcal{S}_t)$, with $\mathcal{S}_t = \{[1,i]\}_{\{i=tk, k\in\mathbb{N}\}}$ are all Ch_ω-spaces.

To prove that the Ch_ω-spaces represent in an adequate way the down-covered chains, we will need:

Definition 3.4. For every chain $(L, \leqslant)$ with the greatest element $\mathbf{1_L}$, we define $\varphi_L\colon L \to \wp(X(L))$ by: $\varphi_L(x) := \{P \in X(L)\colon\ x \in P\}$ for every $x \in L$.

The following well-known result is valid by the application of the prime filter theorem for distributive lattices (in particular, applicable to every chain):

Proposition 3.5. *For every $x,\ y \in X$, it holds that $x \leqslant y$ if and only if $\varphi_L(x) \subseteq \varphi_L(y)$.*

Corollary 3.6. *If L is a chain with $\mathbf{1_L}$, then the posets $(\{\varphi_L(x)\}_{x\in L}, \subseteq)$ and $(\{(\varphi_L(x))^{\mathrm{c}}\}_{x\in L}, \subseteq)$ are both chains (here, $(\varphi_L(x))^{\mathrm{c}} := X(L) \setminus \varphi_L(x)$). In addition, φ_L is an order-isomorphism between $(L, \leqslant)$ and $(\{\varphi_L(x)\}_{x\in L}, \subseteq)$, while $(L, \leqslant)$ is anti-isomorphic to $(\{(\varphi_L(x))^{\mathrm{c}}\}_{x\in L}, \subseteq)$.*

As it is well known, the function φ_L relative to an F_ω-chain (L, A) plays two roles. It will be used for the construction of an F_ω-chain *isomorphic to* (L, A) (in a certain sense to be explicited later), as we shall see. On the other hand, it is applied to the definition of a Ch_ω-space which is actually *the dual* of (L, A). This space will be built in the sequel.

Definition 3.7. Let (L, A) be an F_ω-chain. The *dual of* (L, A) is the system $\mathbb{X}(L) := (X(L), \tau_L, \mathcal{S}_A)$, such that:

(a) τ_L is the topology that has the family $\mathcal{B}_L$ as a basis, with $\mathcal{B}_L := \{(\varphi_L(x))^{\mathrm{c}}\colon x \in L\}$. (Note that $\emptyset \in \mathcal{B}_L$, because $\mathbf{1_L} \in L$. In addition, note that $(\varphi_L(x))^{\mathrm{c}} \neq \emptyset$ for every $x \neq \mathbf{1_L}$, because of the prime filter theorem.)

(b) $\mathcal{S}_A := \{(\varphi_L(x))^{\mathrm{c}}\colon\ x \in A\}$.

Remark 3.8. It should be noted that the previous definition makes sense. That is, that the family $\mathcal{B}_L$ given above can be a basis of a suitable topology. Actually, suppose $P \in X(L)$. Since $P \neq L$, there is $x \in L$ such that $x \notin P$, and so

$$(*) \qquad P \in (\varphi_L(x))^{\mathrm{c}}.$$

Now, let $P \in (\varphi_L(a))^{\mathrm{c}} \cap (\varphi_L(b))^{\mathrm{c}}$, with $a, b \in L$. We can suppose that $a \leqslant b$, because L is a chain. Thus,

$$(**) \qquad P \in (\varphi_L(b))^{\mathrm{c}} = (\varphi_L(a))^{\mathrm{c}} \cap (\varphi_L(b))^{\mathrm{c}}.$$

From $(*)$ and $(**)$ we get that $\mathcal{B}_L$ is a base for a topology on $X(L)$.

Now, to prove that $\mathbb{X}(L)$ is actually a Ch_ω-space, we will need the following technical result:

Proposition 3.9. *Given* $\mathbb{X}(L) := (X(L), \tau_L, \mathcal{S}_A)$ *as in Definition* 3.7, *it holds that* $\mathcal{B}_L = \tau_L \cap \mathcal{K}_{\tau_L}$.

Proof. Obviously, $\mathcal{B}_L \subseteq \tau_L$. Besides that, let $(\varphi_L(a))^{\mathrm{c}}$ be in $\mathcal{B}_L$ and suppose that

$$(\star) \qquad (\varphi_L(a))^{\mathrm{c}} \subseteq \bigcup_{x \in J} (\varphi_L(x))^{\mathrm{c}} \quad \text{for a certain set } J \subseteq L.$$

In addition, suppose for the moment that $x \nleqslant a$ for every $x \in J$. Since L is a chain, $a < x$ for every $x \in J$. On the one hand, if we consider $\mathrm{Fi}(J)$ and $\mathrm{Id}(\{a\})$ (recall Section 1), we get $\mathrm{Fi}(J) \cap \mathrm{Id}(\{a\}) = \emptyset$. From this and the prime filter theorem, there is $P \in X(L)$ such that $\mathrm{Fi}(J) \subseteq P$, $P \cap \mathrm{Id}(a) = \emptyset$, which implies $J \subseteq P$ and $a \notin P$. So, $P \in (\varphi_L(a))^{\mathrm{c}}$ and, from $(\star)$, $x_0 \notin P$ for some $x_0 \in J$, which is absurd. Thus, there must exist $t_0 \in J$ such that $t_0 \leqslant a$ and so $(\varphi_L(a))^{\mathrm{c}} \subseteq (\varphi_L(t_0))^{\mathrm{c}}$. From this, it follows that $(\varphi_L(a))^{\mathrm{c}} \in \mathcal{K}_{\tau_L}$. On the other hand, if $U \in \tau_L \cap \mathcal{K}_{\tau_L}$, we get that $U = \bigcup\limits_{i=1}^{n} (\varphi_L(x_i))^{\mathrm{c}}$, with $\{x_1, \ldots, x_n\} \subseteq L$. By Corollary 3.6 (and recalling that L is a chain) we conclude that $U = (\varphi_L(x_{i_0}))^{\mathrm{c}}$ for some $i_0 \in \{1, \ldots, n\}$. Hence, $\tau_L \cap \mathcal{K}_{\tau_L} \subseteq \mathcal{B}_L$. □

Theorem 3.10. *The system* $\mathbb{X}(L) := (X(L), \tau_L, \mathcal{S}_A)$ *is a* Ch_ω*-space.*

Proof. Let (L, A) be an F_ω-chain. We will prove that $\mathbb{X}(L)$ satisfies conditions (S1)–(S4); (S5′), established in Proposition 3.2. Firstly, suppose P, $Q \in X(L)$, with $P \neq Q$. Without losing generality, suppose that there is $x \in P \setminus Q$. Then, there exists $(\varphi_L(x))^{\mathrm{c}} \in \mathcal{B}_L \subseteq \tau_L$ such that $P \notin (\varphi_L(x))^{\mathrm{c}}$, $Q \in (\varphi_L(x))^{\mathrm{c}}$. Hence, $\mathbb{X}(L)$ is a T_0-space. Besides that, (S2) is valid by Definition 3.7 and Proposition 3.9. Also, it is valid that $(\mathcal{B}_L, \subseteq)$ is a chain, by Corollary 3.6. This allows to prove condition (S3). To demonstrate (S4) we can start, using the characterization given in Proposition 3.9, from two nonempty families $\{(\varphi_L(x))^{\mathrm{c}}\}_{x \in S}$ and $\{(\varphi_L(y))^{\mathrm{c}}\}_{y \in T}$, where $(\varphi_L(x))^{\mathrm{c}} \neq \emptyset \neq (\varphi_L(y))^{\mathrm{c}}$, for every $x \in S$, for every $y \in T$, with $S \cup T \subseteq L$. Now, suppose that

$$(\star) \qquad \bigcap_{x \in S} (\varphi_L(x))^{\mathrm{c}} \subseteq \bigcup_{y \in T} (\varphi_L(y))^{\mathrm{c}}.$$

In addition, for the moment, suppose that

$$(*) \qquad \mathrm{Id}(S) \cap \mathrm{Fi}(T) = \emptyset.$$

By the prime filter theorem, this implies that there is $P \in X(L)$ such that $S \cap P = \emptyset$ and $T \subseteq P$. Then, there is $P \in \bigcap\limits_{x \in S} (\varphi_L(x))^{\mathrm{c}} \setminus \bigcup\limits_{y \in T} (\varphi_L(t))^{\mathrm{c}}$, contradicting $(\star)$. Hence, $(*)$ cannot be valid and so there is $z \in \mathrm{Id}(S) \cup \mathrm{Fi}(T)$. Then, there are $x_1, \ldots, x_m \in S$ and $y_1, \ldots, y_n \in T$, where $y_1 \wedge \ldots \wedge y_n \leqslant z \leqslant x_1 \vee \ldots \vee x_m$. Now, since L is a chain, there are $x_0 \in S$, $y_0 \in T$ such that $y_0 \leqslant z \leqslant x_0$, and therefore $(\varphi_L(x_0))^{\mathrm{c}} \subseteq (\varphi_L(y_0))^{\mathrm{c}}$, with $(\varphi_L(x_0))^{\mathrm{c}} \in S$, $(\varphi_L(y_0))^{\mathrm{c}} \in T$. Finally, for (S5$'$), suppose $(\varphi_L(x))^{\mathrm{c}} \in \mathcal{B}_L$. Since (L, A) is an F_ω-chain, we have that there exists $a \in A$ such that $a \leqslant x$. From Corollary 3.6, we obtain that $(\varphi_L(x))^{\mathrm{c}} \subseteq (\varphi_L(a))^{\mathrm{c}} \in \mathcal{S}_A$. $\square$

Now we will represent every Ch_ω-space by means of a convenient F_ω-chain. For that, we need the following definition:

Definition 3.11. The *dual of a Ch_ω-space* $(X, \tau, \mathcal{S})$ is $\mathbb{L}(X) := (L(X), A_{\mathcal{S}})$, where $L(X) := \{U \colon U^{\mathrm{c}} \in \tau \cap \mathcal{K}_\tau\}$ (ordered by inclusion), and $A_{\mathcal{S}} := \{S \colon S^{\mathrm{c}} \in \mathcal{S}\}$.

Theorem 3.12. *For every Ch_ω-space $(X, \tau, \mathcal{S})$, its dual $\mathbb{L}(X)$, defined above, is an F_ω-chain.*

P r o o f. Let $(X, \tau, \mathcal{S})$ be a Ch_ω-space (satisfying, in this way, the conditions of Proposition 3.2). Condition (S3) entails that $\tau \cap \mathcal{K}_\tau$ is a chain, and therefore $(L(X), \subseteq)$ is a chain, too. In addition, since $\emptyset \in \tau \cap \mathcal{K}_\tau$, we have that $L(X)$ has the greatest element $\mathbf{1}_{\mathbf{L(X)}} = X$. So, we just need to prove that $(L(X), A_{\mathcal{S}})$ satisfies condition (C) of Definition 2.2. But it is valid, by the definition of $A_{\mathcal{S}}$, and (S5$'$), applied to $(X, \tau, \mathcal{S})$. $\square$

With respect to the representation obtained in Theorems 3.10 and 3.12, note that the existence of the least element $\mathbf{0_L}$ in L is equivalent to the fact that $X(L) \in \mathcal{K}_\tau$, which also implies that $\emptyset \in L(X(L))$.

We will conclude this section by applying the obtained representation results.

E x a m p l e 3.13. Bearing in mind the DCC $(\mathbf{L_1}, O)$ from Example 2.3, note the following facts: first of all, $X(L_1) = \{\uparrow\{i\}\}_{i \in \mathbb{Z}} \cup \{\mathbb{Z}^-\}$ (here $\uparrow\{i\}$ is defined following the order $\leqslant_{L_1}$). Now, taking into account Definition 3.7, we have that $\mathcal{B}_{L_1} =$

$\{G_n\}_{n\in\mathbb{N}} \cup \{H_m\}_{m\in\mathbb{N}}$, where $G_n := \{\uparrow\{-k\}\}_{k\in\mathbb{N},k<n}$, while $H_m := \{\uparrow\{-i\}\}_{i\in\mathbb{N}} \cup \{\mathbb{Z}^-\} \cup \{\uparrow\{k\}\}_{k\in\mathbb{N},k<m}$. Note, in addition, that $\bigcup\limits_{n\in\mathbb{N}} G_n \in \tau_{\mathcal{B}_{L_1}}$, but $\bigcup\limits_{n\in\mathbb{N}} G_n \notin \mathcal{B}_{L_1}$. With respect to $\mathcal{S}_{O_{\mathbf{L_1}}}$, it is easy to see that it satisfies (S5) in Definition 3.1. Finally, note that $X(\mathbf{L_1})$ is not a compact space, as was already indicated.

The previous example developed the dual of a given F_ω-chain. With respect to the reciprocal representation, consider the next example.

Example 3.14. Considering Example 3.3 of the Ch_ω-space $(\mathbb{N}, \tau_{\mathbb{P}}, \mathcal{S}_4)$, we have that $\mathbb{L}(\mathbb{N}) := (L(\mathbb{N}), A_{\mathcal{S}_4})$, with $L(\mathbb{N}) = \{\uparrow 3, \uparrow 5, \uparrow 7, \ldots\}$, and $A_{\mathcal{S}_4} = \{\uparrow 5, \uparrow 9, \ldots\}$. This is another example of an F_ω-chain without the least element. Of course, we can consider now the Ch_ω-space $\mathbb{X}(\mathbb{L}(\mathbb{N}))$, which will be isomorphic to $(\mathbb{N}, \tau_{\mathbb{P}}, \mathcal{S}_4)$, in the sense developed in the next section.

4. Duality for F_ω-chains

In the previous section we have associated with every F_ω-chain a particular Ch_ω-space, and vice versa. This process is the basic step for the definition of two categories (of F_ω-chains and Ch_ω-spaces, respectively), which will be dually equivalent. The formal definition of such categories is as follows:

Definition 4.1. The category $\mathbf{FCH}_\omega$ is given by the following conditions:
(1) The $\mathbf{FCH}_\omega$-objects are F_ω-chains.
(2) Given two F_ω-chains (L_1, A_1) and (L_2, A_2), a mapping $h\colon L_1 \to L_2$ is an $[L_1, L_2]_{\mathbf{FCH}_\omega}$-morphism if and only if h is a RPL-homomorphism and, in addition,
   (2a) for every $y \in L_2$, there exists $x \in L_1$ such that $h(x) \leqslant y$,
   (2b) $h(A_1) \subseteq A_2$.

The class of the $[L_1, L_2]_{\mathbf{FCH}_\omega}$-morphisms is denoted by $\mathbf{FCH}_\omega[L_1, L_2]$.

Remark 4.2. It is easy to prove that $\mathbf{FCH}_\omega$ is a category, where the composition of morphisms is the set-theoretic composition of maps. In addition, h is an $[L_1, L_2]_{\mathbf{FCH}_\omega}$-morphism if and only if h is monotone, $h(\mathbf{1_{L_1}}) = \mathbf{1_{L_2}}$ and condition (2) in the previous definition is satisfied. Finally, note that h is an $[L_1, L_2]_{\mathbf{FCH}_\omega}$-*isomorphism* if and only if h is isotone and $h(A_1) = A_2$.

The following is a technical result about $\mathbf{FCH}_\omega$, to be used later.

Proposition 4.3. *If h is an $[L_1, L_2]_{\mathbf{FCH}_\omega}$-morphism, then $P \in X(L_2)$ implies $h^{-1}(P) \in X(L_1)$.*

P r o o f. Consider h and P as indicated. Obviously, $h^{-1}(P)$ is a non-void up-set of L_1. Now, suppose that $h^{-1}(P) = L_1$. Since P is proper, there exists $y \in L_2 \setminus P$. By condition (2) of Definition 4.1, there is $x \in L_1$ such that $h(x) \leqslant y$. Now, since $h(x) \in P$, we have $y \in P$, which is absurd. This implies that $h^{-1}(P) \neq L_1$. By Remark 1.1, we have that $h^{-1}(P) \in X(L_1)$. $\square$

Note that, if (L_1, A_1) and (L_2, A_2) are F_ω-chains and $h\colon L_1 \to L_2$ is merely a RPL-morphism, it does not hold that $P \in X(L_2)$ implies $h^{-1}(P) \in X(L_1)$. As a counterexample, consider the DCC whose respective supports are the real intervals $L_1 = (0,1]$ and $L_2 = [0,1]$, and $h = inc$ (the inclusion morphism). Obviously, h is a RPL-morphism. Now, $(0,1] \in X(L_2)$ but $h^{-1}((0,1]) = (0,1] \notin X(L_1)$.

We define the category of Ch_ω-spaces in the sequel.

Definition 4.4. The category $\mathbf{ChS}_\omega$ is given by the following conditions:

(1) The $\mathbf{ChS}_\omega$-objects are Ch_ω-spaces.

(2) Given two Ch_ω-spaces $(X_1, \tau_1, \mathcal{S}_1)$ and $(X_2, \tau_2, \mathcal{S}_2)$, $f\colon X_1 \to X_2$ is an $[X_1, X_2]_{\mathbf{ChS}_\omega}$-morphism if and only if it satisfies:
   (2a) $f^{-1}(T) \in \tau_1 \cap \mathcal{K}_{\tau_1}$ for every $T \in \tau_2 \cap \mathcal{K}_{\tau_2}$,
   (2b) $f^{-1}(T) \in \mathcal{S}_1$ for every $T \in \mathcal{S}_2$.

R e m a r k 4.5. As in $\mathbf{FCH}_\omega$, it can be proved here that $\mathbf{ChS}_\omega$ is a category, with the composition of morphisms given by the set-theoretic composition of maps. Besides, every $[X_1, X_2]_{\mathbf{ChS}_\omega}$-morphism is also a continuous map. Finally, f is an $[X_1, X_2]_{\mathbf{ChS}_\omega}$-isomorphism if and only if it satisfies $\tau_2 \cap \mathcal{K}_{\tau_2} = \{f(T)\colon T \in \tau_1 \cap \mathcal{K}_{\tau_1}\}$ and $\mathcal{S}_2 = \{f(T)\colon T \in \mathcal{S}_1\}$.

Definition 4.6. Let $L_1 = (L_1, A_1)$, $L_2 = (L_2, A_2)$ be two F_ω-chains. For every $[L_1, L_2]_{\mathbf{FCH}_\omega}$-morphism h we define:

(a) $f_h\colon X(L_2) \to X(L_1)$, where $f_h(P) := h^{-1}(P)$.

Conversely, every $[X_1, X_2]_{\mathbf{ChS}_\omega}$-morphism f induces the map

(b) $h_f\colon L(X_2) \to L(X_1)$, defined as: $h_f(U) = f^{-1}(U)$.

Lemma 4.7. *Let $L_i = (L_i, A_i)$, $i = 1, 2$ be two F_ω-chains. If h is an $[L_1, L_2]_{\mathbf{FCH}_\omega}$-morphism, then the function f_h given in Definition* 4.6 (a) *satisfies:*

(a) $f_h^{-1}(\varphi_{L_1}(x)) = \varphi_{L_2}(h(x))$ *for every* $x \in L_1$,

(b) f_h *is an* $[X(L_2), X(L_1)]_{\mathbf{ChS}_\omega}$*-morphism.*

P r o o f. Note first that $f_h\colon X(L_2) \to X(L_1)$ is well defined, because of Proposition 4.3. Now, to prove (a), suppose (L_i, A_i), $i = 1, 2$, h and f_h as above, and take $x \in L_1$. Then, for every $P \in X(L_2)$ we have that $P \in f_h^{-1}(\varphi_{L_1}(x))$ if and only if $h^{-1}(P) \in \varphi_{L_1}(x)$, if and only if $x \in h^{-1}(P)$, if and only if $P \in \varphi_{L_2}(h(x))$. For (b), let $T \in \tau_{L_1} \cap \mathcal{K}_{\tau_{L_1}}$. By Proposition 3.9, $T = (\varphi_{L_1}(x))^{\mathrm{c}}$ for some $x \in L_1$. By (a), we have $f_h^{-1}(T) = (f_h^{-1}(\varphi_{L_1}(x)))^{\mathrm{c}} = (\varphi_{L_2}(h(x)))^{\mathrm{c}} \in \mathcal{B}_{L_2} = \tau_{L_2} \cap \mathcal{K}_{\tau_{L_2}}$. Moreover, if $T = (\varphi_{L_1}(x))^{\mathrm{c}} \in \mathcal{S}_{L_1}$, then $x \in A_1$. So, $h(x) \in A_2$ and, from this, $f_h^{-1}(T) \in \mathcal{S}_{L_2}$. Thus, conditions (2a) and (2b) of Definition 4.4 are satisfied, too. □

As a counterpart of the previous result, we have:

Lemma 4.8. *Let $(X_i, \tau_i, \mathcal{S}_i)$, $i = 1, 2$ be two Ch_ω-spaces. For every $[X_1, X_2]_{\mathbf{ChS}_\omega}$-morphism f, the map h_f given in condition* (2) *of Definition* 4.6 *is an* $[L(X_2), L(X_1)]_{\mathbf{FCH}_\omega}$*-morphism.*

Proof. We will prove our claim taking into account Remark 4.2. For that, suppose $f \colon X_1 \to X_2$, with $(X_i, \tau_i, \mathcal{S}_i)$, $i = 1, 2$ as indicated above. By condition (2a) of Definition 4.4, h_f is well defined. Moreover, h_f is obviously monotonic and $h_f(X_2) = X_1$. Now, if $U \in L(X_1)$, then $U^{\mathrm{c}} \in \tau_1 \cap \mathcal{K}_{\tau_1}$. This implies that

$$(\star) \qquad f(U^{\mathrm{c}}) \in \mathcal{K}_{\tau_2} \quad \text{(since } f \text{ is continuous).}$$

On the other hand, recalling that $\tau_2 \cap \mathcal{K}_{\tau_2}$ is a basis of τ_2, we have that $f(U^{\mathrm{c}}) \subseteq X_2 = \bigcup_{B \in I} B$ for some $I \subseteq \tau_2 \cap \mathcal{K}_{\tau_2}$. Using $(\star)$, we have that there exists a family $\{B_1, \ldots, B_n\} \in I$, with $f(U^{\mathrm{c}}) \subseteq \bigcup_{i=1}^{n} B_i$. So, $U^{\mathrm{c}} \subseteq f^{-1}\Bigl(\bigcup_{i=1}^{n} B_i\Bigr)$, and then $f^{-1}\Bigl(\bigcap_{i=1}^{n} B_i^{\mathrm{c}}\Bigr) \subseteq U$. Now, since $\{B_1, \ldots, B_n\}$ is a finite chain (because of Definition 3.1), we have that $\bigcap_{i=1}^{n} B_i^{\mathrm{c}} = B_{i_0}^{\mathrm{c}}$ for some $B_{i_0} \in \{B_1, \ldots, B_n\}$. Hence, there exists $V := B_{i_0}^{\mathrm{c}} \in L(X_2)$ such that $h_f(V) = f^{-1}(B_{i_0}^{\mathrm{c}}) \subseteq U$. This proves condition (2a) in Definition 4.1. Finally, take $T \in h_f(A_{\mathcal{S}_2})$. Then, there is $S \in A_{\mathcal{S}_2}$ such that $f^{-1}(S) = T$. Hence, $f^{-1}(S^{\mathrm{c}}) \in \mathcal{S}_1$, by Definition 3.11. (Because $S^{\mathrm{c}} \in \mathcal{S}_2$). Thus, $T \in A_{\mathcal{S}_1}$. This proves condition (2b) in Definition 4.1, which concludes the proof. □

The next result follows straightforwardly:

Proposition 4.9. *The following properties are satisfied in* $\mathbf{FCH}_\omega$ *and* $\mathbf{ChS}_\omega$*:*

(1a) *If (L, A) is an F_ω-chain and Id_L is the identity function on L, then Id_L is the identity morphism in the category $\mathbf{FCH}_\omega$, while f_{Id_L} is the identity morphism (w.r.t. $X(L)$) in the category $\mathbf{ChS}_\omega$.*

(1b) *If $h \in [L_1, L_2]_{\mathbf{FCH}}$ and $t \in [L_2, L_3]_{\mathbf{FCH}}$, then $f_{t \circ h} = f_h \circ f_t$.*

(2a) *If $(X, \tau, \mathcal{S})$ is a Ch_ω-space and Id_X is the identity function on X, then Id_X is the identity morphism in $\mathbf{ChS}_\omega$, and h_{Id_X} is the identity on $\mathbf{FCH}_\omega$ (w.r.t. $L(X)$).*

(2b) *If f is a $[X_1, X_2]_{\mathbf{ChS}_\omega}$-morphism and $g \in [X_2, X_3]_{\mathbf{ChS}_\omega}$, then $h_{g \circ f} = h_f \circ h_g$.*

From Theorem 3.10, Lemma 4.7, and Proposition 4.9, we obtain:

Theorem 4.10. *The correspondence* $\mathbb{X}\colon \mathbf{FCH}_\omega \to \mathbf{ChS}_\omega$, *defined by*
▷ $\mathbb{X}(L) = (X(L), \tau_L, \mathcal{S}_A)$, *if* $L = (L, A)$ *is an* F_ω*-chain,*
▷ $\mathbb{X}(h) = f_h$, *if* h *is a morphism in* $\mathbf{FCH}_\omega$
is a contravariant functor.

In a similar way (applying Theorem 3.12, Lemma 4.8 and Proposition 4.9 in this case) we have:

Theorem 4.11. *The correspondence* $\mathbb{L}\colon \mathbf{ChS}_\omega \to \mathbf{FCH}_\omega$ *given by*
▷ $\mathbb{L}(X) = (L(X), A_{\mathcal{S}})$, *if* $X = (X, \tau, \mathcal{S})$ *is a* Ch_ω*-space,*
▷ $\mathbb{L}(f) = h_f$, *if* f *is a morphism in* $\mathbf{ChS}_\omega$
is a contravariant functor.

To complete our proof of duality we need to prove the natural (dual) equivalence between the functors $\mathbb{X}$ and $\mathbb{L}$. For that realize that, from Theorems 3.10 and 3.12, we obtain, for every F_ω-chain (L, A), a new one: $\mathbb{L}(\mathbb{X}(L)) = (L(X(L)), A_{\mathcal{S}_A})$. As expected, we have:

Proposition 4.12. *The* F_ω*-chains* (L, A) *and* $(L(X(L)), A_{\mathcal{S}_A})$ *are isomorphic in the category* $\mathbf{FCH}_\omega$.

P r o o f. By Definitions 3.7 and 3.11, and the representation results already proven, we have that $(L(X(L)), A_{\mathcal{S}_A})$ is an F_ω-chain, where $L(X(L)) = \{\varphi_L(x)\}_{x\in L}$. Then, $\varphi_L\colon L \to L(X(L))$ (indicated in Definition 3.4) is an order-isomorphism, by Corollary 3.6. Moreover, it is clear that $A_{\mathcal{S}_A} = \{\varphi_L(x)\}_{x\in A} = \varphi_L(A)$ and that $\varphi_L(\mathbf{1_L}) = X(L)$. By Remark 4.2, φ_L is an $[L, L(X(L))]_{\mathbf{FCH}_\omega}$-isomorphism. $\square$

For the reciprocal result, we will need the following definition:

Definition 4.13. For every Ch_ω-space $(X, \tau, \mathcal{S})$, the map $\varepsilon_X\colon X \to X(L(X))$ is given by: $\varepsilon_X(w) := \{U \in L(X)\colon w \in U\}$, for every $w \in X$.

Proposition 4.14. *The function* $\varepsilon_X\colon X \to X(L(X))$ *given in the previous definition is a bijection.*

P r o o f. For that, bearing in mind Definitions 3.7 and 3.11, we have:
(a) ε_X is well defined: $\varepsilon_X(w) \neq \emptyset$ for every $w \in X$, because $X \in \varepsilon_X(w)$. Moreover, it is clear that $\varepsilon_X(w)$ is an up-set of $L(X)$. Besides that, there is $B \in \tau \cap \mathcal{K}_\tau$ such that $w \in B$, by (S2) in Definition 3.1. So, $B^{\mathrm{c}} \in L(X) \setminus \varepsilon_X(w)$. That is, $\varepsilon_X(w)$ is a proper set. These facts imply that $\varepsilon_X(w) \in X(L(X))$, because $L(X)$ is a chain.

(b) ε_X is onto: let $E \in X(L(X))$. Let us define $\mathbf{V} := \{A \in L(X)\colon\ A \neq X,\ A \in E\}$ and $\mathbf{W} := \{B \in L(X)\colon\ B \notin E\}$. From these definitions, we have that $A \neq X$ for every $A \in \mathbf{V}$, $B \neq X$ for every $B \in \mathbf{W}$, and (since $E \in X(L(X))$),

$$(*) \qquad \mathbf{V} \neq \emptyset \neq \mathbf{W}.$$

Besides that, it is routine to prove the following fact, using Definition 4.13: for every $w \in X$, $E = \varepsilon_X(w)$ if and only if $w \in \bigcap\limits_{A \in \mathbf{V}} A \setminus \bigcup\limits_{B \in \mathbf{W}} B$. Then, to demonstrate that ε_X is onto, we only need to prove

$$(\star) \qquad \bigcap_{A \in \mathbf{V}} A \setminus \bigcup_{B \in \mathbf{W}} B \neq \emptyset.$$

Suppose that it is not the case. This implies that $\bigcap\limits_{A \in \mathbf{V}} A \subseteq \bigcup\limits_{B \in \mathbf{W}} B$, and therefore $\bigcap\limits_{B \in \mathbf{W}} B^{\mathrm{c}} \subseteq \bigcup\limits_{A \in \mathbf{V}} A^{\mathrm{c}}$. Using this fact, $(*)$ and condition (S4) of Definition 3.1, there is a pair $\{B_0^{\mathrm{c}}, A_0^{\mathrm{c}}\} \subseteq \tau \cap \mathcal{K}_\tau$ such that $B_0^{\mathrm{c}} \subseteq A_0^{\mathrm{c}}$, and $A_0 \in E$, $B_0 \notin E$. But this implies that $A_0 \subseteq B_0$ and, thus, $B_0 \in E$, which is absurd. So, $(\star)$ is valid, as it was desired.
(c) ε_X is injective: this holds because (X, τ) is a T_0-space. $\square$

In addition, the following technical result will be useful:

Proposition 4.15. *Let $(X, \tau, \mathcal{S})$ be a Ch_ω-space. The following facts are valid for every $E \subseteq X(L(X))$, and every $B \subseteq X$:*
(a) *If $E = (\varphi_{L(X)}(U))^{\mathrm{c}}$ for some $U \in L(X)$, then $\varepsilon_X^{-1}(E) = U^{\mathrm{c}}$.*
(b) *If $B \in \tau \cap \mathcal{K}_\tau$, then $\varepsilon_X(B) = (\varphi_{L(X)}(B^{\mathrm{c}}))^{\mathrm{c}}$.*

Proof. To prove (a) suppose $E = (\varphi_{L(X)}(U))^{\mathrm{c}}$, with $U \in L(X)$. Then, it is easy to prove that $w \in \varepsilon_X^{-1}(E)$ if and only if $w \in U^{\mathrm{c}}$. For the proof of (b), suppose $E \in \varepsilon_X(B)$. Then, there exists $w \in B$ such that $E = \varepsilon_X(w) = \{U \in L(X)\colon\ w \in U\}$. Since $B^{\mathrm{c}} \in L(X)$ and $w \notin B^{\mathrm{c}}$, we have that $B^{\mathrm{c}} \notin E$. So, $E \in (\varphi_{L(X)}(B^{\mathrm{c}}))^{\mathrm{c}}$. On the other hand, suppose $E \in (\varphi_{L(X)}(B^{\mathrm{c}}))^{\mathrm{c}}$. From the definition of $\varphi_{L(X)}$, $B^{\mathrm{c}} \notin E$. Since ε_X is surjective (see Proposition 4.14), there is $w \in X$ such that $E = \varepsilon_X(w)$. We will prove that $w \in B$. In fact, if not, we would have $w \in B^{\mathrm{c}}$, which implies $B^{\mathrm{c}} \in \varepsilon_X(w) = E$, a contradiction. Thus, $E \in \varepsilon_X(B)$. $\square$

Corollary 4.16. *With the previous conditions, we have:*
(a) *For every $B \in \tau \cap \mathcal{K}_\tau$ it holds that $\varepsilon_X(B) \in \tau_{L(X)} \cap \mathcal{K}_{\tau_{L(X)}}$.*
(b) *For every $E \in \tau_{L(X)} \cap \mathcal{K}_{\tau_{L(X)}}$, $\varepsilon_X^{-1}(E) \in \tau \cap \mathcal{K}_\tau$.*

P r o o f. Let $B \in \tau \cap \mathcal{K}_\tau$. Then $B^c \in L(X)$ and so, from Definition 3.7, Proposition 3.9, and Proposition 4.15 (b), we have that $\varepsilon_X(B) \in \tau_{L(X)} \cap \mathcal{K}_{\tau_{L(X)}}$. That is, (a) is valid. To prove (b), suppose $E \in \tau_{L(X)} \cap \mathcal{K}_{\tau_{L(X)}}$. Considering Proposition 3.9 we have that $E = (\varphi_{L(X)}(U))^c$ for some $U \in L(X)$, and then $U^c \in \tau \cap \mathcal{K}_\tau$. Now, apply Proposition 4.15 (a). $\square$

Also, from Proposition 4.15, and by Definitions 3.7 and 3.11, it holds:

Corollary 4.17. *Considering the Ch_ω-spaces $(X,\tau,\mathcal{S})$ and $(X(L(X)),\tau_{L(X)},\mathcal{S}_{A_\mathcal{S}})$, we have that $\mathcal{S}_{A_\mathcal{S}} = \{\varepsilon_X(B) \colon B \in \mathcal{S}\}$. Moreover, $\mathcal{S} = \{\varepsilon_X^{-1}(E) \colon E \in \mathcal{S}_{A_\mathcal{S}}\}$.*

Therefore, considering Definition 4.4, Proposition 4.14, and Corollaries 4.16 and 4.17, we have:

Theorem 4.18. *Every Ch_ω-space $X = (X, \tau, \mathcal{S})$ is isomorphic (in the context of $\mathbf{ChS}_\omega$) to $\mathbb{X}(\mathbb{L}(X)) = (X(L(X)), \tau_{L(X)}, \mathcal{S}_{A_\mathcal{S}})$.*

Finally, it is possible to prove:

Lemma 4.19. *The family $\Phi := \{\varphi_L \colon L \text{ is a } \mathbf{FCH}_\omega\text{-object}\}$ is a natural equivalence from the identity functor in $\mathbf{FCH}_\omega$ to $\mathbb{L} \circ \mathbb{X}$.*

P r o o f. Let $L_i = (L_i, A_i)$, $i = 1, 2$ be two F_ω-chains. By Proposition 4.12, we have that Φ is a family of $\mathbf{FCH}_\omega$-isomorphisms. Now, taking into account Definition 4.6, Lemma 4.7 (a) and Proposition 4.9 we have that, for every $[L_1, L_2]_{\mathbf{FCH}_\omega}$-morphism h, $(\mathbb{L} \circ \mathbb{X})(h) \circ \varphi_{L_1} = \varphi_{L_2} \circ \mathrm{Id}_{\mathbf{FCH}_\omega}(h)$. $\square$

In addition, it is easy to prove the following:

Lemma 4.20. *The family $\Lambda := \{\varepsilon_X \colon X \text{ is a } \mathbf{ChS}\text{-object}\}$ is a natural equivalence from the identity functor in $\mathbf{ChS}_\omega$ to $\mathbb{X} \circ \mathbb{L}$.*

P r o o f. Let $X_i = (X_i, \tau_i, \mathcal{S}_i)$, $i = 1, 2$ be two Ch_ω-spaces. We will prove that $(\mathbb{X} \circ \mathbb{L})(f) \circ \varepsilon_{X_1} = \varepsilon_{X_2} \circ \mathrm{Id}_{\mathbf{ChS}_\omega}(f)$, for every $[X_1, X_2]_{\mathbf{ChS}_\omega}$-morphism f. For that, consider $h_f \colon L(X_2) \to L(X_1)$ as in Definition 4.6 (b). Now, for every $w \in X_1$, $U \in h_f^{-1}(\varepsilon_{X_1}(w))$ implies $f^{-1}(U) \in \varepsilon_{X_1}(w)$, with $U \in L(X_2)$. Hence, $w \in f^{-1}(U)$ and so, $U \in \varepsilon_{X_2}(f(w))$. That is, $h_f^{-1}(\varepsilon_{X_1}(w)) \subseteq \varepsilon_{X_2}(f(w))$. On the other hand, if $U \in \varepsilon_{X_2}(f(w))$ $(\subseteq L(X_2))$, then $f(w) \in U$. Since $U^c \in \tau_2 \cap \mathcal{K}_{\tau_2}$ we get $f^{-1}(U^c) \in \tau_1 \cap \mathcal{K}_{\tau_1}$. Therefore, $f^{-1}(U) \in L(X_1)$ and so, $h_f(U) = f^{-1}(U) \in \varepsilon_{X_1}(w)$, i.e., $U \in h_f^{-1}(\varepsilon_{X_1}(w))$. That is, $\varepsilon_{X_2}(f(w)) \subseteq h_f^{-1}(\varepsilon_{X_1}(w))$. From this, Λ is a natural (dual) equivalence. $\square$

From the previous results, we conclude:

Theorem 4.21. *The categories* $\mathbf{FCH}_\omega$ *and* $\mathbf{ChS}_\omega$ *are dually equivalent.*

5. Concluding remarks

In this work we have shown a topological duality for a category of algebraic-relational structures, namely the F_ω-chains. The construction presented here can be done in a simple way because F_ω-chains are characterized by down-covered chains. By the way, it is clear that we can define a new category *isomorphic* to $\mathbf{FCH}_\omega$, whose objects are the "original" F_ω-chains (and not merely their associated DCC, which are, indeed, the $\mathbf{FCH}_\omega$-objects). Having in mind Definition 2.1, we get that the morphisms in this new category are naturally defined as the RPL-morphisms $h\colon L_1 \to L_2$ such that $h(\mathfrak{f}_1(x)) \subseteq \mathfrak{f}_2(h(x))$ for every $x \in L_1$. The details of the definition of this new category are left to the reader.

It should be remarked that this work is a first step toward a deeper result, which is the definition of a duality for F_ω-structures (and not merely F_ω-chains). For that, it would be convenient to characterize F_ω-structures by special sets, playing the same role as DCC in this paper. Besides that, the duality techniques should be adapted from those existent for RPL (or similar algebraic structures), since they are the supports of F_ω-structures. For example, some duality techniques developed for Heyting algebras (as Esakia duality) or for Hilbert algebras (see [3] or [4]) can be adapted for our purposes. Of course, in all these works the topological spaces involved are also ordered, as usual in the dualities based on the well-known Priestley one (see [12]).

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Authors' address: *Verónica Quiroga*, *Víctor Fernández*, Basic Sciences Institute, National University of San Juan, Av. José Ignacio de la Roza Oeste 230, San Juan 5400, Argentina, e-mail: `veronicaandreaquiroga@gmail.com`; `vlfernan@ffha.unsj.edu.ar`.