EXISTENCE OF INFINITELY MANY WEAK SOLUTIONS FOR SOME QUASILINEAR $\vec{p}(x)$-ELLIPTIC NEUMANN PROBLEMS

Ahmed Ahmed, Taghi Ahmedatt, Fez, Hassane Hjiaj, Tetouan
Abdelfattah Touzani, Fez

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Abstract. We consider the following quasilinear Neumann boundary-value problem of the type

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) + b(x)|u|^{p_0(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove the existence of infinitely many weak solutions for our equation in the anisotropic variable exponent Sobolev spaces and we give some examples.

Keywords: Neumann problem; quasilinear elliptic equation; weak solution; variational principle; anisotropic variable exponent Sobolev space

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, with boundary $\partial\Omega$ of class C^1, and let γ be the outward unit normal vector on $\partial\Omega$.

Zhao, Zhao and Xie have studied in [15] the quasilinear boundary value problem

$$\begin{cases} -\text{div } a(x, \nabla u) + |u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

They have proved the existence of nontrivial weak solutions for this problem under some assumptions on the Carathéodory function $a(x,\xi)$ and growth conditions on

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the function $f(x,s)$ with $\lambda > 0$. In the case of Neumann elliptic equations, Anello and Cordaro have considered in [1] the following p-Laplacian problem

$$(1.1) \qquad \begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega. \end{cases}$$

They have shown the existence and multiplicity of weak solutions for this problem under some growth conditions on functions $f(\cdot)$ and $g(\cdot)$.

In the framework of variable exponent, Fan and Ji have treated in [7] the problem

$$(1.2) \qquad \begin{cases} -\Delta_{p(x)} u + \lambda(x)|u|^{p(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda(\cdot) \in L^\infty(\Omega)$ is a positive function such that $\lambda^- = \operatorname*{ess\,inf}_{x\in\Omega} \lambda(x) > 0$. They have proved the existence of infinitely many weak solutions $W^{1,p(\cdot)}(\Omega)$ by applying the critical point theorem obtained by Ricceri in [13], which is a consequence of a more general result of variational principle.

In the recent years, the anisotropic variable exponent Sobolev spaces have attracted the attention of many mathematicians, physicists and engineers. The impulse for this mainly came from their important applications in modeling real-world problems in electrorheological and magnetorheological fluids (see for example [14]). In [12], Mihăilescu and Morosanu have studied the boundary value problem of the type

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a_i(x,t)$ are Carathéodory functions for $i = 1, \ldots, N$, and the function $f(x,s)$ on the right-hand side satisfies some suitable growth conditions (see also [2], [3], [4], [5], [8], [10]).

Our aim is to prove the existence of infinitely many weak solutions to the anisotropic quasilinear $p(x)$-elliptic problem

$$(1.3) \qquad \begin{cases} \displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) + b(x)|u|^{p_0(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that $a_i\colon \Omega \times \mathbb{R} \mapsto \mathbb{R}$ are Carathéodory functions for $i = 1, \ldots, N$, satisfying some assumptions. In the particular case when we take $a_i(x,s) = |s|^{p_i(x)-2}s$, we obtain the so-called $\vec{p}(\cdot)$-Laplace operator defined by

$$\Delta_{\vec{p}(x)} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\Big). \tag{1.4}$$

This paper is organized as follows: In Section 2, we present some necessary preliminary knowledge on the anisotropic variable exponent Sobolev spaces and we recall some classical existence result. We introduce in Section 3 some assumptions on the Carathéodory functions $a_i(x,\xi_i)$ and the two functions $f(x,s)$ and $g(x,s)$ for which there are solutions for our problem. In Section 4, we prove the existence of infinitely many weak solutions for our Neumann elliptic problem, followed by giving some interesting examples.

2. Preliminaries

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geqslant 1)$. We define

$$\mathcal{C}_+(\Omega) = \{\text{measurable function } p(\cdot)\colon\ \overline{\Omega} \longmapsto \mathbb{R} \text{ such that } 1 < p^- \leqslant p^+ < \infty\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x)\colon\ x \in \Omega\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x)\colon\ x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u\colon\ \Omega \longmapsto \mathbb{R}$ for which the convex modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\,\mathrm{d}x,$$

is finite. Then

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0\colon\ \varrho_{p(\cdot)}\Big(\frac{u}{\lambda}\Big) \leqslant 1\Big\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$ called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable and reflexive Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Finally, we have the generalized Hölder's type inequality:

$$\Big|\int_\Omega uv\,\mathrm{d}x\Big| \leqslant \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\varrho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1 (see [11]). *If $u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

(i) $\|u\|_{p(\cdot)} > 1 \Longrightarrow \|u\|_{p(\cdot)}^{p^-} < \varrho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+}$,

(ii) $\|u\|_{p(\cdot)} < 1 \Longrightarrow \|u\|_{p(\cdot)}^{p^+} < \varrho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^-}$.

The Sobolev space with variable exponent is defined as

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space.

R e m a r k 2.1. Recall that the definition of these spaces requires only the measurability of the exponent $p(x)$. In this work, we do not need to use Sobolev and Poincaré inequality. Note that the sharp Sobolev inequality is proved for $p(x)$-log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(x)$ (see [6], [9]).

Now, we present the anisotropic variable exponent Sobolev space, used for the study of the main problem.
Let $p_0(x), p_1(x), \ldots, p_N(x)$ be $N+1$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\vec{p}(x) = (p_0(x), \ldots, p_N(x)), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \ldots, N.$$

We define

$$\underline{p} = \min\{p_i^-,\ i = 0, 1, \ldots, N\} \quad (\text{then } \underline{p} > 1), \tag{2.2}$$

and

$$\overline{p}^+ = \max\{p_i^+,\ i = 0, 1, \ldots, N\}. \tag{2.3}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_0(x)}(\Omega) \text{ and } D^i u \in L^{p_i(x)}(\Omega) \text{ for } i = 1, 2, \ldots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \sum_{i=0}^{N} \|D^i u\|_{p_i(\cdot)}. \tag{2.4}$$

The space $\left(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{1,\vec{p}(\cdot)}\right)$ is separable and reflexive Banach space (cf. [3], [10]).

Lemma 2.1. *Let Ω be a bounded open set in $\mathbb{R}^N$, then the following embeddings are compact:*
▷ *if $\underline{p} < N$ then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [\underline{p}, \underline{p}^*[$, where $1/\underline{p}^* = 1/\underline{p} - 1/N$,*
▷ *if $\underline{p} = N$ then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [\underline{p}, \infty[$,*
▷ *if $\underline{p} > N$ then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.*

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and from the classical embedding theorems of the Sobolev spaces.

Now, we introduce the following theorem, which will be essential to establish the existence of weak solutions for our main problem.

Theorem 2.1 (see [7], Theorem 2.2)**.** *Let X be a reflexive real Banach space and let $\Phi, \Psi \colon X \longmapsto \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is (strongly) continuous and satisfies $\lim\limits_{\|u\|_X \to \infty} \Psi(u) = \infty$. For each $\varrho > \inf\limits_X \Psi$ put*

$$\varphi(\varrho) = \inf_{u \in \Psi^{-1}(]-\infty,\varrho[)} \frac{\Phi(u) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,\varrho[))_w}} \Phi(v)}{\varrho - \Psi(u)}, \tag{2.5}$$

where $\overline{(\Psi^{-1}(]-\infty,\varrho[))_w}$ is the closure of $\Psi^{-1}(]-\infty,\varrho[)$ for the weak topology. Then the following conclusions hold:

(a) *If there exist $\varrho_0 > \inf\limits_X \Psi$ and $u_0 \in X$ such that*

$$\Psi(u_0) < \varrho_0 \tag{2.6}$$

and

$$\Phi(u_0) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,\varrho_0[))_w}} \Phi(v) < \varrho_0 - \Psi(u_0), \tag{2.7}$$

then the restriction of $\Psi + \Phi$ to $\Psi^{-1}(]-\infty,\varrho_0[)$ has a global minimum.

(b) *If there exists a sequence $\{r_n\} \subset \left(\inf\limits_X \Psi, \infty\right)$ with $r_n \to \infty$ and a sequence $\{u_n\} \subset X$ such that for each n*

$$\Psi(u_n) < r_n \tag{2.8}$$

and

$$\Phi(u_n) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,r_n[))_w}} \Phi(v) < r_n - \Psi(u_n), \tag{2.9}$$

and in addition

$$\liminf_{\|u\|\to\infty} \Psi(u) + \Phi(u) = -\infty, \tag{2.10}$$

then there exists a sequence $\{v_n\}_n$ *of local minima of* $\Psi + \Phi$ *such that* $\Psi(v_n) \to \infty$ *as* $n \to \infty$.

(c) *If there exists a sequence* $\{r_n\} \subset \big(\inf_X \Psi, \infty\big)$ *with* $r_n \to \inf_X \Psi$ *and a sequence* $\{u_n\} \subset X$ *such that for each* n *conditions* (2.8) *and* (2.9) *are satisfied, and in addition*

(2.11) *every global minimizer of* Ψ *is not a local minimizer of* $\Phi + \Psi$,

then there exists a sequence $\{v_n\}$ *of pairwise distinct local minimizers of* $\Phi + \Psi$ *such that* $\lim_{n\to\infty} \Psi(v_n) = \inf_X \Psi$ *and* $\{v_n\}$ *weakly converges to a global minimizer of* Ψ.

3. Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geqslant 1)$ with boundary $\partial\Omega$ of class C^1, and let γ be the outward unit normal vector on $\partial\Omega$. We assume that

$$\underline{p} > N. \tag{3.1}$$

Proposition 3.1. *Since* $W^{1,\vec{p}(\cdot)}(\Omega)$ *is continuously embedded in* $W^{1,\underline{p}}(\Omega)$ *and* $W^{1,\underline{p}}(\Omega)$ *is compactly embedded in* $C^0(\overline{\Omega})$ *(the space of continuous functions), thus the embedding of* $W^{1,\vec{p}(\cdot)}(\Omega)$ *in* $C^0(\overline{\Omega})$ *is continuous and compact, we set*

$$C_0 = \sup_{u \in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{1,\vec{p}(\cdot)}}. \tag{3.2}$$

We consider the quasilinear $\vec{p}(\cdot)$-elliptic problem of the type

$$\begin{cases} Au + b(x)|u|^{p_0(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega, \end{cases}$$

where A is a Leray-Lions operator acted from $W^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}'(\cdot)}(\Omega)$ defined by the formula

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big),$$

where $a_i\colon \Omega\times\mathbb{R}\longmapsto\mathbb{R}$ are Carathéodory functions which satisfy the following assumptions:

(A1) The growth condition:

$$|a_i(x,s)|\leqslant\gamma_i(d_i(x)+|s|^{p_i(x)-1})\quad\text{for } i=1,\dots,N,$$

where $d_i(\cdot)$ is a nonnegative function in $L^{p_i'(\cdot)}(\Omega)$ and $\gamma_i>0$.

(A2) The coercivity condition: there exist two constants $\alpha,\beta>0$ such that

$$\alpha|s|^{p_i(x)}\leqslant a_i(x,s)s\leqslant\beta A_i(x,s),$$

where the function $A_i\colon \Omega\times\mathbb{R}\longmapsto\mathbb{R}$ is defined by

$$A_i(x,s)=\int_0^s a_i(x,t)\,\mathrm{d}t.$$

(A3) The monotonicity condition:

$$(a_i(x,s)-a_i(x,t))(s-t)>0.$$

Clearly as a consequence of (A2) and the continuity of the function $a_i(x,s)$ with respect to s, we have

$$a_i(x,0)=0.$$

The Carathéodory functions $f,g\colon \Omega\times\mathbb{R}\longmapsto\mathbb{R}$ satisfy condition

$$\sup_{|t|\leqslant r}|f(x,t)|\in L^1(\Omega)\quad\text{and}\quad\sup_{|t|\leqslant r}|g(x,t)|\in L^1(\Omega)\quad\text{for any } r>0,\tag{3.3}$$

and we set

$$F(x,t)=\int_0^t f(x,s)\,\mathrm{d}s\quad\text{and}\quad G(x,t)=\int_0^t g(x,s)\,\mathrm{d}s.\tag{3.4}$$

The function $b(\cdot)\in L^\infty(\Omega)$ and there exists a constant $b_0>0$ such that $b(x)\geqslant b_0$ a.e. in Ω.

We introduce the functionals $J(\cdot),\Psi(\cdot),\Phi(\cdot)\colon W^{1,\vec{p}(\cdot)}(\Omega)\longmapsto\mathbb{R}$ by

$$J(u)=\sum_{i=1}^N\int_\Omega A_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\,\mathrm{d}x+\int_\Omega\frac{b(x)}{p_0(x)}|u|^{p_0(x)}\,\mathrm{d}x,\tag{3.5}$$

and

$$\Psi(u)=J(u)-\int_\Omega G(x,u)\,\mathrm{d}x\quad\text{and}\quad\Phi(u)=-\int_\Omega F(x,u)\,\mathrm{d}x.\tag{3.6}$$

4. Main results

Definition 4.1. A function $F(x,t)$ satisfies condition (S) if for each compact subset E of $\mathbb{R}$ there exists $\xi \in E$ such that

$$F(x,\xi)=\sup_{t\in E}F(x,t) \quad \text{for a.e. } x\in\Omega. \tag{4.1}$$

Definition 4.2. A measurable function $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is called a weak solution of problem (1.3) if

$$\sum_{i=1}^{N}\int_{\Omega} a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}\,\mathrm{d}x+\int_{\Omega} b(x)|u|^{p_0(x)-2}uv\,\mathrm{d}x=\int_{\Omega} f(x,u)v\,\mathrm{d}x+\int_{\Omega} g(x,u)v\,\mathrm{d}x$$

for any $v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

It is clear that $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution of (1.3) if and only if u is a critical point of the functional $\Psi+\Phi$.

We take u_0 and u_n in Theorem 2.1 as the constant value functions ξ_0 and ξ_n. We consider the assumption

$$\liminf_{|\xi|\to\infty}\int_{\Omega}\Big(\frac{b(x)}{p_0(x)}|\xi|^{p_0(x)}-G(x,\xi)-F(x,\xi)\Big)\,\mathrm{d}x=-\infty, \tag{4.2}$$

and the condition

$$\int_{\Omega}\frac{b(x)}{p_0(x)}|\xi|^{p_0(x)}\,\mathrm{d}x-\int_{\Omega}G(x,\xi)\,\mathrm{d}x\leqslant d_1|\xi|^{p_0^+}+d_2 \quad \forall\,\xi\in\mathbb{R}, \tag{4.3}$$

where d_1 and d_2 are two positive constants.

We assume that

(G1) There exists $M>0$ and two measurable functions $\delta(\cdot),\theta(\cdot)\in L^1(\Omega)$ with $\|\delta(\cdot)\|_{L^1(\Omega)}>0$ such that

$$G(x,t)\leqslant\frac{C_1\delta(x)}{p_0^+C_0^{\underline{p}}\|\delta(\cdot)\|_{L^1(\Omega)}}|t|^{\underline{p}}+\theta(x) \quad \text{a.e. in } \Omega \text{ for any } |t|\geqslant M,$$

with $C_1<p_0^+/(N+1)^{\underline{p}-1}\min\{\alpha/\beta,b_0/p_0^+\}$.

(G2) There exist $M>0$, $\varepsilon\in]0,1]$ and $\theta'(\cdot)\in L^1(\Omega)$ such that

$$G(x,t)\leqslant\frac{(1-\varepsilon)b(x)}{p_0(x)}|t|^{p_0(x)}+\theta'(x) \quad \text{a.e. in } \Omega \text{ for any } |t|\geqslant M.$$

It is easy to see that condition (4.3) is satisfied if assumption (G1) or (G2) hold true.

Our two main results are as follows.

Theorem 4.1. *Assume that* (A1)–(A3) *hold true and the Carathéodory functions f and g satisfy* (3.3), (4.2), *with $F(x,\cdot)$ satisfying condition* (S), *and $G(x,\cdot)$ satisfying* (G1) *or* (G2).

Suppose that $\{y_n\}$ and $\{z_n\}$ are two positive sequences such that

$$\lim_{n\to\infty} z_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{y_n^{p_0^+}}{z_n^{\underline{p}}} = 0. \tag{4.4}$$

If there exists a positive function $h(\cdot) \in L^1(\Omega)$ with $\|h(\cdot)\|_{L^1(\Omega)} \neq 0$ such that for each n we have

$$F(x,y_n) + \frac{h(x)}{\|h(\cdot)\|_{L^1(\Omega)}}\Big(d_0\Big(\frac{z_n}{C_0}\Big)^{\underline{p}} - d_1 y_n^{p_0^+} - d_2\Big) \geqslant \sup_{t\in[y_n,z_n]} F(x,t) \quad \text{a.e. in } \Omega, \tag{4.5}$$

$$\begin{aligned} F(x,-y_n) + \frac{h(x)}{\|h(\cdot)\|_{L^1(\Omega)}}\Big(d_0\Big(\frac{z_n}{C_0}\Big)^{\underline{p}} - d_1 y_n^{p_0^+} - d_2\Big) \\ \geqslant \sup_{t\in[-z_n,-y_n]} F(x,t) \quad \text{a.e. in } \Omega, \end{aligned} \tag{4.6}$$

where d_0 is a positive constant, and the inequalities (4.5)–(4.6) *are strict on a subset of Ω with positive measure. Then there exists a sequence $\{v_n\}$ of local minima of $\Psi+\Phi$ such that $\lim_{n\to\infty}\Psi(v_n)=\infty$. Consequently, problem* (1.3) *admits an unbounded sequence of weak solutions.*

Theorem 4.2. *Suppose that* (A1)–(A3) *hold true, and the function $G(x,t)$ satisfy*

$$\forall\, t \in \mathbb{R} \quad G(x,t) \leqslant 0 \quad \text{a.e. in } \Omega, \tag{4.7}$$

and there exist two positive constants M and ε such that

$$-G(x,t) \leqslant M|t|^{p_0^-} \quad \text{for } t \leqslant \varepsilon \text{ and a.e. in } \Omega. \tag{4.8}$$

The functional $F(x,\cdot)$ satisfies condition (S) *with $F(x,0)=0$ and*

$$\limsup_{|\xi|\to 0} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x + \int_\Omega G(x,\xi)\,\mathrm{d}x}{|\xi|^{\underline{p}}} > \int_\Omega \frac{b(x)}{p_0(x)}\,\mathrm{d}x. \tag{4.9}$$

Suppose that there exist two positive sequences $\{y_n\}$ *and* $\{z_n\}$ *such that*

$$\lim_{n\to\infty} z_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{y_n^{p_0^-}}{z_n^{\overline{p}^+}} = 0, \tag{4.10}$$

and there exists a positive function $h(\cdot) \in L^1(\Omega)$ *with* $\|h(\cdot)\|_{L^1(\Omega)} \neq 0$ *such that for each* n *we have*

$$F(x, y_n) + \frac{h(x)}{\|h(\cdot)\|_{L^1(\Omega)}}\Big(d_4\Big(\frac{z_n}{C_0}\Big)^{\overline{p}^+} - d_3 y_n^{p_0^-}\Big) \geqslant \sup_{t\in[y_n,z_n]} F(x,t) \quad \textit{a.e. in } \Omega, \tag{4.11}$$

$$F(x, -y_n) + \frac{h(x)}{\|h(\cdot)\|_{L^1(\Omega)}}\Big(d_4\Big(\frac{z_n}{C_0}\Big)^{\overline{p}^+} - d_3 y_n^{p_0^-}\Big) \geqslant \sup_{t\in[-z_n,-y_n]} F(x,t) \quad \textit{a.e. in } \Omega, \tag{4.12}$$

and the inequalities (4.11) *and* (4.12) *are strict on a subset of* Ω *with positive measure, where*

$$d_3 = \int_\Omega \frac{b(x)}{p_0(x)}\,\mathrm{d}x + M|\Omega| \quad \textit{and} \quad d_4 = \frac{1}{(N+1)^{\overline{p}^+-1}} \min\Big\{\frac{\alpha}{\beta}, \frac{b_0}{p_0^+}\Big\}.$$

Then there exists a sequence $\{v_n\}$ *of pairwise distinct local minima of* $\Psi + \Phi$ *such that* $v_n \to 0$ *in* $W^{1,\vec{p}(\cdot)}(\Omega)$*. Consequently, problem* (1.3) *admits a sequence of nonzero weak solutions which converges to* 0 *in* $W^{1,\vec{p}(\cdot)}(\Omega)$.

Proof of Theorem 4.1.
Step 1: Technical Lemma.

Lemma 4.1 (see [4], Lemma 1)**.** *The functionals* $\Psi(\cdot)$ *and* $\Phi(\cdot)$ *are well-defined on* $W^{1,\vec{p}(\cdot)}(\Omega)$*. In addition,* $\Psi(\cdot)$ *and* $\Phi(\cdot)$ *are of class* $C^1(W^{1,\vec{p}(\cdot)}(\Omega), \mathbb{R})$ *and*

$$\langle \Psi'(u), v\rangle = \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \int_\Omega b(x)|u|^{p_0(x)-2}uv\,\mathrm{d}x - \int_\Omega g(x,u)v\,\mathrm{d}x,$$

$$\langle \Phi'(u), v\rangle = -\int_\Omega f(x,u)v\,\mathrm{d}x$$

for all $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

Under assumptions (A1)–(A3) and (3.3), the functionals $\Psi(\cdot)$ and $\Phi(\cdot)$ are sequentially weakly lower semicontinuous (see [4], Lemma 3).

Step 2: Coerciveness of the operator $\Psi(\cdot)$.

Proposition 4.1. *Assume that* $G(x,t)$ *satisfies* (G1) *or* (G2)*. Then the functional* $\Psi(\cdot)$ *is coercive, i.e.*

$$\Psi(u) \to \infty \quad \textit{as} \quad \|u\|_{1,\vec{p}(\cdot)} \to \infty \quad \textit{for } u \in W^{1,\vec{p}(\cdot)}(\Omega).$$

Proof. (i) Assume that condition (G1) is satisfied,

$$G(x,t) \leqslant \frac{C_1\delta(x)}{p_0^+ C_0^{\underline{p}} \|\delta(\cdot)\|_{L^1(\Omega)}} |t|^{\underline{p}} + \theta(x) \quad \text{a.e. in } \Omega \text{ for any } |t| \geqslant M.$$

In view of (A2) and since $C_1 < p_0^+(N+1)^{1-\underline{p}} \min\{\alpha/\beta, b_0/p_0^+\}$,

$$\begin{aligned}
(4.13)\qquad \Psi(u) &= \sum_{i=1}^N \int_\Omega A_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\,\mathrm{d}x + \int_\Omega \frac{b(x)}{p_0(x)} |u|^{p_0(x)}\,\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x \\
&\geqslant \frac{\alpha}{\beta} \sum_{i=1}^N \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}\,\mathrm{d}x + \frac{b_0}{p_0^+} \int_\Omega |u|^{p_0(x)}\,\mathrm{d}x \\
&\quad - \frac{C_1}{p_0^+\, C_0^{\underline{p}}} \int_\Omega \frac{\delta(x)|u|^{\underline{p}}}{\|\delta(\cdot)\|_{L^1(\Omega)}}\,\mathrm{d}x - \int_\Omega \theta(x)\,\mathrm{d}x \\
&\geqslant \frac{\alpha}{\beta} \sum_{i=1}^N \Big(\Big\|\frac{\partial u}{\partial x_i}\Big\|_{p_i(\cdot)}^{\underline{p}} - 1\Big) \\
&\quad + \frac{b_0}{p_0^+}(\|u\|_{p_0(\cdot)}^{\underline{p}} - 1) - \frac{C_1}{p_0^+\, C_0^{\underline{p}}} \|u\|_{L^\infty(\Omega)}^{\underline{p}} - \|\theta(\cdot)\|_{L^1(\Omega)} \\
&\geqslant \frac{1}{(N+1)^{\underline{p}-1}} \min\Big\{\frac{\alpha}{\beta}, \frac{b_0}{p_0^+}\Big\} \Big(\sum_{i=1}^N \Big\|\frac{\partial u}{\partial x_i}\Big\|_{p_i(\cdot)} + \|u\|_{p_0(\cdot)}\Big)^{\underline{p}} \\
&\quad - \frac{C_1}{p_0^+} \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \|\theta(\cdot)\|_{L^1(\Omega)} - \frac{\alpha N}{\beta} - b_0 \\
&\geqslant \Big(\frac{1}{(N+1)^{\underline{p}-1}} \min\Big\{\frac{\alpha}{\beta}, \frac{b_0}{p_0^+}\Big\} - \frac{C_1}{p_0^+}\Big) \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} \\
&\quad - \|\theta(\cdot)\|_{L^1(\Omega)} - \frac{\alpha N}{\beta} - b_0.
\end{aligned}$$

(ii) Under condition (G2), we have

$$G(x,t) \leqslant \frac{(1-\varepsilon)b(x)}{p_0(x)} |t|^{p_0(x)} + \theta'(x), \quad \text{a.e. in } \Omega \text{ for any } |t| \geqslant M.$$

Thus

$$\begin{aligned}
(4.14)\qquad \Psi(u) &\geqslant \sum_{i=1}^N \int_\Omega A_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\,\mathrm{d}x + \int_\Omega \frac{b(x)}{p_0(x)} |u|^{p_0(x)}\,\mathrm{d}x \\
&\quad - \int_\Omega \frac{(1-\varepsilon)b(x)}{p_0(x)} |u|^{p_0(x)}\,\mathrm{d}x - \int_\Omega \theta'(x)\,\mathrm{d}x \\
&\geqslant \frac{\alpha}{\beta} \sum_{i=1}^N \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)}\,\mathrm{d}x + \varepsilon \frac{b_0}{p_0^+} \int_\Omega |u|^{p_0(x)}\,\mathrm{d}x - \|\theta'(\cdot)\|_{L^1(\Omega)} \\
&\geqslant \frac{1}{(N+1)^{\underline{p}-1}} \min\Big\{\frac{\alpha}{\beta}, \varepsilon \frac{b_0}{p_0^+}\Big\} \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \frac{N\alpha}{\beta} - b_0 - \|\theta'(\cdot)\|_{L^1(\Omega)}.
\end{aligned}$$

Thanks to (4.13) and (4.14), we conclude that $\Psi(\cdot)$ is coercive. Moreover, there exist two positive constants d_0 and σ_0 such that

$$\Psi(u) \geqslant d_0 \|u\|^{p}_{1,\vec{p}(\cdot)} \quad \text{for } \|u\|_{1,\vec{p}(\cdot)} \geqslant \sigma_0. \tag{4.15}$$

□

Step 3: A priori estimates. For $r > \inf\limits_{u \in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u)$ we define

$$K(r) = \inf\{\sigma > 0 \text{ such that } \Psi^{-1}(]-\infty, r[) \subset \overline{B(0,\sigma)}\}, \tag{4.16}$$

where $B(0,\sigma) = \{u \in W^{1,\vec{p}(\cdot)}(\Omega)\colon \|u\|_{1,\vec{p}(\cdot)} < \sigma\}$ and $\overline{B(0,\sigma)}$ denotes the closure of $B(0,\sigma)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ for the norm topology. We have that $\Psi\colon W^{1,\vec{p}(\cdot)}(\Omega) \longmapsto \mathbb{R}$ is coercive, then $0 < K(r) < \infty$ for each $r > \inf\limits_{u \in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u)$. In view of (4.15), we have

$$\Psi(u) < d_0 \|u\|^{p}_{1,\vec{p}(\cdot)} \implies \|u\|_{1,\vec{p}(\cdot)} < \sigma_0.$$

Thanks to (4.16), we have $\Psi^{-1}(]-\infty, r[) \subset \overline{B(0,K(r))}$, then $\overline{(\Psi^{-1}(]-\infty, r[))_w} \subset \overline{B(0,K(r))}$, and using (3.2), we get

$$\|u\|_{L^\infty(\Omega)} \leqslant C_0 \|u\|_{1,\vec{p}(\cdot)}, \text{ then } \overline{B(0,K(r))} \subset \{u \in C(\overline{\Omega})\colon \|u\|_{L^\infty(\Omega)} \leqslant C_0 K(r)\}.$$

It follows that

$$\inf\nolimits_{v \in \overline{(\Psi^{-1}(]-\infty,r[))_w}} \Phi(v) \geqslant \inf_{\|v\|_{1,\vec{p}(\cdot)} \leqslant K(r)} \Phi(v) \geqslant \inf_{\|v\|_{L^\infty(\Omega)} \leqslant C_0 K(r)} \Phi(v). \tag{4.17}$$

By taking u_0 and u_n as constant value functions ξ_0 and ξ_n in Theorem 2.1, and using (4.17), we conclude the following Theorem 4.3, that relies on Theorem 2.1.

Theorem 4.3. *Assume that* (A1)–(A3) *hold true, the Carathéodory functions f and g satisfy* (3.3), *and consider that $\Psi(\cdot)$ and $\Phi(\cdot)$ are defined as in* (3.6). *When $\Psi(\cdot)$ is coercive, then*

(a) *If there exist $\varrho_0 > \inf\limits_{u \in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u)$ and $\xi_0 \in \mathbb{R}$ such that*

$$\int_\Omega \frac{b(x)}{p_0(x)} |\xi_0|^{p_0(x)} \,\mathrm{d}x - \int_\Omega G(x,\xi_0) \,\mathrm{d}x := e_0 < \varrho_0 \tag{4.18}$$

and

$$\int_\Omega F(x,\xi_0) \,\mathrm{d}x + (\varrho_0 - e_0) > \sup_{v \in C(\overline{\Omega}), \|v\|_{L^\infty(\Omega)} \leqslant C_0 K(\varrho_0)} \int_\Omega F(x, v(x)) \,\mathrm{d}x, \tag{4.19}$$

then the restriction of $\Psi + \Phi$ to $\Psi^{-1}(]-\infty, \varrho_0[)$ has a global minimum.

(b) *If there exist a sequence* $\{r_n\} \subset \Bigl(\inf_{u\in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u), \infty\Bigr)$ *with* $\lim_{n\to\infty} r_n \to \infty$ *and a sequence* $\{\xi_n\} \subset \mathbb{R}$ *such that for each* n

$$\int_\Omega \frac{b(x)}{p_0(x)}|\xi_n|^{p_0(x)}\,\mathrm{d}x - \int_\Omega G(x,\xi_n)\,\mathrm{d}x := e_n < r_n \tag{4.20}$$

and

$$\int_\Omega F(x,\xi_n)\,\mathrm{d}x + (r_n - e_n) > \sup_{v\in C(\overline{\Omega}),\|v\|_{L^\infty(\Omega)}\leqslant C_0K(r_n)} \int_\Omega F(x,v(x))\,\mathrm{d}x, \tag{4.21}$$

and in addition (4.2) *holds, then there exists a sequence* $\{v_n\}$ *of local minima of* $\Psi+\Phi$ *such that* $\lim_{n\to\infty}\Psi(v_n)\to\infty$.

(c) *If there exist a sequence* $\{r_n\} \subset \Bigl(\inf_{u\in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u), \infty\Bigr)$ *with* $\lim_{n\to\infty} r_n = \inf_{u\in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u)$ *and a sequence* $\{\xi_n\}\subset\mathbb{R}$ *such that for each* n *conditions* (4.20) *and* (4.21) *are satisfied, and in addition, condition* (2.11) *is satisfied, then there exists a sequence* $\{v_n\}$ *of pairwise distinct local minima of* $\Psi+\Phi$ *such that* $\lim_{n\to\infty}\Psi(v_n) = \inf_{u\in W^{1,\vec{p}(\cdot)}(\Omega)}\Psi(u)$ *(i.e, the sequence* $\{v_n\}$ *converges weakly to the global minimizer of* $\Psi(\cdot)$*).*

P r o o f of Theorem 4.3. In view of (4.18), assume that $\varrho_0 > \inf_{u\in W^{1,\vec{p}(\cdot)}(\Omega)}\Psi(u)$ and $\xi_0\in\mathbb{R}$ such that

$$\int_\Omega \frac{b(x)}{p_0(x)}|\xi_0|^{p_0(x)}\,\mathrm{d}x - \int_\Omega G(x,\xi_0)\,\mathrm{d}x := e_0 < \varrho_0 \implies \Psi(\xi_0) < \varrho_0.$$

Then (2.6) holds. On the other hand, thanks to (4.19), we have

$$\int_\Omega F(x,\xi_0)\,\mathrm{d}x + (\varrho_0 - e_0) > \sup_{v\in C(\overline{\Omega}),\|v\|_{L^\infty(\Omega)}\leqslant C_0K(\varrho_0)} \int_\Omega F(x,v)\,\mathrm{d}x.$$

Then

$$\varrho_0 - \Psi(\xi_0) > -\int_\Omega F(x,\xi_0)\,\mathrm{d}x + \sup_{v\in C(\overline{\Omega}),\|v\|_{L^\infty(\Omega)}\leqslant C_0K(\varrho_0)} -\Phi(v).$$

Using (4.17), we deduce that

$$\varrho_0 - \Psi(\xi_0) > \Phi(\xi_0) - \inf\nolimits_{v\in\overline{(\Psi^{-1}(]-\infty,\varrho_0[))}_w}\Phi(v).$$

Therefore the hypotheses (2.6) and (2.7) of Theorem 2.1 (a) are satisfied. Thus, the restriction of $\Psi+\Phi$ to $\Psi^{-1}(]-\infty,\varrho_0[)$ has a global minimum.

Assuming that the hypotheses of Theorem 4.3 (b) and Theorem 4.3 (c) are satisfied, using the same steps, we can prove that the assumptions of Theorem 2.1 (b) and Theorem 2.1 (c) are satisfied, which concludes the proof of Theorem 4.3. □

For condition (4.19) in Theorem 4.3 (a) we state the following proposition.

Proposition 4.2. *Assume that* $\varrho_0 > \inf_{u\in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u)$ *and* $\xi_0 \in \mathbb{R}$ *such that* (4.18) *hold true. If there exists a positive function* $\alpha(\cdot) \in L^1(\Omega)$ *with* $\|\alpha(\cdot)\|_{L^1(\Omega)} \neq 0$ *such that*

$$F(x,\xi_0) + \frac{\alpha(x)}{\|\alpha(\cdot)\|_{L^1(\Omega)}}(\varrho_0 - e_0) \geqslant \sup_{|t|\leqslant C_0K(\varrho_0)} F(x,t) \quad \text{for a.e. } x \in \Omega, \tag{4.22}$$

and inequality (4.22) *is strict on a subset of* Ω *with positive measure, then* (4.19) *holds.*

P r o o f. Integrating (4.22) over Ω and noting that

$$\int_\Omega \sup_{|t|\leqslant C_0K(\varrho_0)} F(x,t)\,\mathrm{d}x \geqslant \sup_{v\in C(\overline{\Omega}),\|v\|_{L^\infty(\Omega)}\leqslant C_0K(\varrho_0)} \int_\Omega F(x,v(x))\,\mathrm{d}x,$$

we obtain (4.19). □

Step 4: Proof of statements (4.20) and (4.21).

Proposition 4.3. *Assume that* $\Psi(\cdot)$ *is coercive and* (4.15) *holds. Then for* $r \geqslant d_0\sigma_0^{\underline{p}}$,

$$K(r) \leqslant \Big(\frac{r}{d_0}\Big)^{1/\underline{p}}. \tag{4.23}$$

P r o o f. Let $r \geqslant d_0\sigma_0^{\underline{p}}$ and $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that $\Psi(u) < r$. When $\|u\|_{1,\vec{p}(\cdot)} \geqslant \sigma_0$, by (4.15) one has

$$r > \Psi(u) \geqslant d_0\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}},$$

which implies that $\|u\|_{1,\vec{p}(\cdot)} \leqslant (r/d_0)^{1/\underline{p}}$.

When $\|u\|_{1,\vec{p}(\cdot)} < \sigma_0$, it is clear that $\|u\|_{1,\vec{p}(\cdot)} \leqslant (r/d_0)^{1/\underline{p}}$. By the definition of $K(r)$, inequality (4.23) holds. □

Now, we set $r_n = d_0(z_n/C_0)^{\underline{p}}$, then $\lim_{n\to\infty} r_n \to \infty$, and thanks to (4.23) we obtain

$$K(r_n) \leqslant \frac{z_n}{C_0} \quad \text{and then} \quad C_0K(r_n) \leqslant z_n. \tag{4.24}$$

Since $F(x,\cdot)$ satisfies condition (S), for each n there exists $\xi_n \in [-y_n, y_n]$ such that

$$F(x,\xi_n) = \sup_{t\in[-y_n,y_n]} F(x,t) \quad \text{a.e. in } \Omega. \tag{4.25}$$

By (4.3), one has

$$e_n = \int_\Omega \frac{b(x)}{p_0(x)} |\xi_n|^{p_0(x)} \, \mathrm{d}x - \int_\Omega G(x, \xi_n) \, \mathrm{d}x \leqslant d_1 |\xi_n|^{p_0^+} + d_2 \leqslant d_1 |y_n|^{p_0^+} + d_2.$$

It follows from (4.4) that for n large enough

$$d_1 |y_n|^{p_0^+} + d_2 < d_0 \Big(\frac{z_n}{C_0}\Big)^{\underline{p}} = r_n,$$

and consequently $e_n < r_n$, that is (4.20) holds. Without loss of generality we may assume that (4.20) holds for all n.

On the other hand, thanks to (4.25), we obtain

$$F(x, \xi_n) + \frac{h(x)}{\|h(\cdot)\|_{L^1(\Omega)}} (r_n - e_n) \geqslant \sup_{|t| \leqslant y_n} F(x, t) \quad \text{a.e. in } \Omega.$$

Therefore, having in mind (4.5) and (4.6), we deduce that

$$F(x, \xi_n) + \frac{h(x)}{\|h(\cdot)\|_{L^1(\Omega)}} (r_n - e_n) \geqslant \sup_{|t| \leqslant z_n} F(x, t) \quad \text{a.e. in } \Omega, \tag{4.26}$$

and inequality (4.26) is strict on a subset of Ω with positive measure. Using (4.24) and Proposition 4.2, we obtain (4.21).

Therefore all hypotheses of Theorem 4.3 (b) are satisfied, so the proof of the Theorem 4.1 is concluded.

P r o o f of Theorem 4.2. Let us verify all the hypotheses of Theorem 4.3 (c). Using (A2) and (4.7) we have

$$\begin{aligned} \Psi(u) &= J(u) - \int_\Omega G(x, u) \, \mathrm{d}x \\ &\geqslant \sum_{i=1}^N \int_\Omega A_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \, \mathrm{d}x + \int_\Omega \frac{b(x)}{p_0(x)} |u|^{p_0(x)} \, \mathrm{d}x \\ &\geqslant \frac{1}{(N+1)^{\underline{p}-1}} \min\Big\{\frac{\alpha}{\beta}, \frac{b_0}{p^+}\Big\} \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \frac{\alpha N}{\beta} - b_0, \end{aligned}$$

then the functional $\Psi(\cdot)$ is coercive, and there exist positive constants d_3 and σ_1 such that

$$\Psi(u) \geqslant d_3 \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} \quad \text{for } \|u\|_{1,\vec{p}(\cdot)} \geqslant \sigma_1. \tag{4.27}$$

Using (4.7) and (4.8), we have $\inf\limits_{u\in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u) = \Psi(0) = 0$ and 0 is the unique global minimizer of $\Psi(\cdot)$. Thanks to (4.9), we obtain

$$\begin{aligned}\limsup_{|\xi|\to 0}\{\Psi(\xi)+\Phi(\xi)\} &= \limsup_{|\xi|\to 0}\Big\{\int_\Omega \frac{b(x)}{p_0(x)}|\xi|^{p_0(x)}\,\mathrm{d}x - \int_\Omega G(x,\xi)\,\mathrm{d}x - \int_\Omega F(x,\xi)\,\mathrm{d}x\Big\} \\ &\leqslant \limsup_{|\xi|\to 0}\Big\{\int_\Omega \frac{b(x)}{p_0(x)}|\xi|^{\underline{p}}\,\mathrm{d}x - \int_\Omega G(x,\xi)\,\mathrm{d}x - \int_\Omega F(x,\xi)\,\mathrm{d}x\Big\} \\ &< 0 = \Psi(0)+\Phi(0).\end{aligned}$$

Then 0 is not a local minimizer of the functional $\Psi+\Phi$, so (2.11) is satisfied.

For $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that $\|u\|_{1,\vec{p}(\cdot)} \leqslant 1$ we have

$$\Psi(u) \geqslant \frac{1}{(N+1)^{\underline{p}-1}} \min\Big\{\frac{\alpha}{\beta}, \frac{b_0}{\overline{p}^+}\Big\}\|u\|^{\overline{p}^+}_{1,\vec{p}(\cdot)} \geqslant d_4\|u\|^{\overline{p}^+}_{1,\vec{p}(\cdot)}.$$

For $r > 0$ sufficiently small the condition $\Psi(u) < r$ implies that $\|u\|_{1,\vec{p}(\cdot)} < (r/d_4)^{1/\overline{p}^+}$, this shows that $K(r) \leqslant (r/d_4)^{1/\overline{p}^+}$.

Take $r_n = d_4(z_n/C_0)^{\overline{p}^+}$. Then

$$C_0K(r_n) \leqslant z_n. \tag{4.28}$$

In view of (4.8) there exists a sequence $(\xi_n)_n \subset \mathbb{R}$ with $\xi_n \in [-y_n, y_n]$ such that for each y_n sufficiently small

$$\begin{aligned}e_n &= \int_\Omega \frac{b(x)}{p_0(x)}|\xi_n|^{p_0(x)}\,\mathrm{d}x - \int_\Omega G(x,\xi_n)\,\mathrm{d}x \\ &\leqslant \Big(\int_\Omega \frac{b(x)}{p_0(x)}\,\mathrm{d}x + M|\Omega|\Big)|\xi_n|^{p_0^-} = d_3|\xi_n|^{p_0^-} \leqslant d_3|y_n|^{p_0^-}.\end{aligned} \tag{4.29}$$

It follows from (4.10) that for n large enough

$$d_3|y_n|^{p_0^-} < d_4\Big(\frac{z_n}{C_0}\Big)^{\overline{p}^+} = r_n.$$

Then (4.20) is obtained.

Noting that $F(x,\cdot)$ satisfies condition (S), and thanks to (4.11), (4.12) and (4.25), we can obtain that

$$F(x,\xi_n) + \frac{h(x)}{\|h(\cdot)\|_{L^1(\Omega)}}(r_n - e_n) \geqslant \sup_{|t|\leqslant z_n} F(x,t) \quad \text{a.e. in } \Omega, \tag{4.30}$$

and the inequality (4.30) is strict on a subset of Ω with positive measure. Thanks to (4.28) and Proposition 4.2, we obtain (4.21). Therefore all the hypotheses of Theorem 4.3 (c) are satisfied.

Consequently, there exists a sequence $\{v_n\}$ of pairwise distinct local minima of $\Psi+\Phi$ such that $\Psi(v_n)\to 0$, which implies $\|v_n\|_{1,\vec{p}(\cdot)}\to 0$, thus, the proof of Theorem 4.2 is complete. □

Proposition 4.4. *Assume that* (A1)–(A3) *and* (3.3) *are satisfied, the function* $G(x,\cdot)$ *satisfies* (G1) *or* (G2), *and let* $F(x,\cdot)$ *satisfy condition* (S).

If there exist two positive constants $\varrho_0>\inf_{u\in W^{1,\vec{p}(x)}(\Omega)}\Psi(u)$ *and* $\xi_0\in\mathbb{R}$ *such that*

$$e_0=\Psi(\xi_0)<\varrho_0 \tag{4.31}$$

and

$$\int_\Omega F(x,\xi_0)\,\mathrm{d}x+(\varrho_0-e_0)>\sup_{v\in C(\overline{\Omega}),\|v\|_{L^\infty(\Omega)}\leqslant C_0K(\varrho_0)}\int_\Omega F(x,v(x))\,\mathrm{d}x, \tag{4.32}$$

then the restriction of $\Psi+\Phi$ *to* $\Psi^{-1}(]-\infty,\varrho_0[)$ *has a global minimum. Consequently, problem* (1.3) *has at least one weak solution* $u\in W^{1,\vec{p}(\cdot)}(\Omega)$.

Proof. The proof of Proposition 4.4 can be deduced from Theorem 4.3 (a), and using the same steps in the proof of Theorem 4.1. □

Examples 4.1. Let $\underline{p}>N$. Taking $b(\cdot)\equiv 1$ and $g(x,\cdot)\equiv 0$ in problem (1.3), we obtain

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)+|u|^{p_0(x)-2}u=f(x,u) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial\gamma}=0 & \text{on } \partial\Omega.\end{cases} \tag{4.33}$$

We have $G(x,t)\equiv 0$, and the operator $\Psi(\cdot)$ is coercive, i.e.

$$\Psi(u)=J(u)\geqslant\frac{1}{(N+1)^{\underline{p}-1}}\min\Big\{\frac{\alpha}{\beta},\frac{1}{\overline{p}^+}\Big\}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}}-\frac{\alpha N}{\beta}-1\to\infty\quad\text{as } \|u\|_{1,\vec{p}(\cdot)}\to\infty,$$

with $d_0=d_4=(2(N+1)^{\underline{p}-1})^{-1}\min\{\alpha/\beta,1/\overline{p}^+\}$ and $d_1=d_2=|\Omega|/\underline{p}$, also we have

$$\Psi(u)\geqslant d_4\|u\|_{1,\vec{p}(\cdot)}^{\overline{p}^+}\quad\text{for } \|u\|_{1,\vec{p}(\cdot)}\leqslant 1.$$

(a) Taking $f(x,t) \equiv h(x) \in L^1(\Omega)$, we have $F(x,t) = h(x)t$ satisfying the assumptions of Proposition 4.4. Then in view of Proposition 4.4 the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) + |u|^{p_0(x)-2}u = h(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.34}$$

has at least one weak solution $u \in W^{1,\vec{p}(\cdot)}(\Omega)$.

(b) Now, we set $f(x,t) \equiv \alpha(x) f_1(t)$, with $\alpha(\cdot) \in L^1(\Omega)$ being a positive function such that $\|\alpha(\cdot)\|_{L^1(\Omega)} \neq 0$, and let $f_1(\cdot)$ be a continuous function such that $f_1(t) = F_1'(t)$ and $F_1(-t) = F_1(t)$. We have $F(x,t) = \alpha(x)F_1(t)$. Choose two positive sequences $\{y_n\}$ and $\{z_n\}$ such that $y_1 \geqslant 1$, $z_n^{\underline{p}} = n y_n^{p_0^+}$, and $y_{n+1} > z_n$ for every n. Define $F_1(y_n) = y_n^{p_0^+ + 1}$ and $F_1(z_n)$ such that

$$\begin{aligned} F_1(y_n) &< F_1(z_n) \\ &< \frac{1}{\|\alpha\|_{L^1(\Omega)}} \Big(\frac{1}{(N+1)^{\underline{p}-1}} \min\Big\{\frac{\alpha}{\beta}, \frac{1}{p_0^+}\Big\} \Big(\frac{z_n}{C_0}\Big)^{\underline{p}} - \frac{|\Omega|}{\underline{p}} (|y_n|^{p_0^+} + 1) \Big) + F_1(y_n). \end{aligned} \tag{4.35}$$

Take $r_n = (N+1)^{1-\underline{p}} \min\{\alpha/\beta, 1/p_0^+\}(z_n/C_0)^{\underline{p}}$ and $\xi_n = y_n$. Since

$$\int_\Omega \frac{1}{p_0(x)} |y_n|^{p_0(x)}\,\mathrm{d}x - \int_\Omega \alpha(x) F_1(y_n)\,\mathrm{d}x \leqslant \frac{|\Omega|}{\underline{p}} |y_n|^{p_0^+} - \|\alpha(\cdot)\|_{L^1(\Omega)} y_n^{p_0^+ + 1} \to -\infty$$

as $n \to \infty$, conditions (4.2) and (4.4) hold true. Taking $h(x) = \alpha(x)$, and in view of (4.35) we can conclude conditions (4.5) and (4.6).

The hypotheses of Theorem 4.1 are satisfied, so the problem (4.34) admits a sequence of weak solutions $(u_n)_n$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ such that $\lim\limits_{n\to\infty} \|u_n\|_{1,\vec{p}(\cdot)} = \infty$.

(c) Take $f(x,t) \equiv \alpha(x) f_1(t)$ defined as in (b), and choose two positive sequences $\{y_n\}$ and $\{z_n\}$ such that $y_n^{p_0^-} = n^{-1} z_n^{\overline{p}^+}$ and $z_{n+1} < y_n$. We define the function $F_1(\cdot)$ such that $F_1(0) = 0$, $F_1(y_n) = y_n^{p_0^- + 1}$ and

$$\begin{aligned} F_1(y_n) &< F_1(z_n) \\ &< \frac{1}{\|\alpha\|_{L^1(\Omega)}} \Big(\frac{1}{(N+1)^{\underline{p}-1}} \min\Big\{\frac{\alpha}{\beta}, \frac{1}{p_0^+}\Big\} \Big(\frac{z_n}{C_0}\Big)^{\overline{p}^+} - d_3 |y_n|^{p_0^-} \Big) + F_1(y_n). \end{aligned} \tag{4.36}$$

By taking $r_n = (N+1)^{1-\underline{p}} \min\{\alpha/\beta, 1/p_0^+\}(z_n/C_0)^{\underline{p}}$ and $\xi_n = y_n$, we have

$$\int_\Omega F(x,y_n)\,\mathrm{d}x - \int_\Omega \frac{1}{p_0(x)} |y_n|^{\underline{p}}\,\mathrm{d}x > \int_\Omega \alpha(x) y_n^{p_0^- + 1}\,\mathrm{d}x - \frac{|\Omega|}{p_0^+} |y_n|^{p_0^-} \to 0 \quad \text{as } n \to \infty,$$

therefore (4.7)–(4.10) hold true. By taking $h(x) = \alpha(x)$ and using (4.36) we conclude conditions (4.11) and (4.12). Thus, all the assumptions of Theorem 4.2 are satisfied, so the problem (4.34) admits a sequence of weak solutions $(u_n)_n$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ such that $\lim\limits_{n\to\infty} \|u_n\|_{1,\vec{p}(\cdot)} = 0$.

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Authors' addresses: *Ahmed Ahmed*, *Taghi Ahmedatt*, *Abdelfattah Touzani*, Department of Mathematics, Laboratory LAMA, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, BP 1796 Atlas, Fez, Morocco, e-mail: ahmedmath2001@gmail.com, taghi-med@hotmail.fr, atouzani07@gmail.com; *Hassane Hjiaj*, Department of Mathematics, Faculty of Sciences, Tetouan University Abdelmalek Essaadi, Quartier M'haneche II, Avenue Palestine, BP 2121, Tetouan 93000, Morocco, e-mail: hjiajhassane@yahoo.fr.