# PROBABILISTIC APPROACH SPACES

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*Abstract.* We study a probabilistic generalization of Lowen's approach spaces. Such a probabilistic approach space is defined in terms of a probabilistic distance which assigns to a point and a subset a distance distribution function. We give a suitable axiom scheme and show that the resulting category is isomorphic to the category of left-continuous probabilistic topological convergence spaces and hence is a topological category. We further show that the category of Lowen's approach spaces is isomorphic to a simultaneously bireflective and bicoreflective subcategory and that the category of probabilistic quasi-metric spaces is isomorphic to a bicoreflective subcategory of the category of probabilistic approach spaces.

*Keywords*: approach space; probabilistic approach space; probabilistic convergence space; probabilistic metric space

*MSC 2010*: 54A20, 54E70, 54E99

## 1. Introduction

The starting point for this paper are two different generalizations of metric spaces. The first one is based on the idea that in reality it is often an over-idealization to assign to two points $p$, $q$ of a space $S$ a distance $d(p,q)$ and that it seems more appropriate to assign to two points $p,q \in S$ a distribution function $F(p,q)$ where the value $F(p,q)(x)$ is interpreted as the probability that the distance between $p$ and $q$ is less than $x$. This theory of probabilistic metric spaces has its origins in the works of Menger [12] and Wald [21], cf. also [18], and reached a certain maturity in the famous textbook by Schweizer and Sklar [19].

The other generalization is more recent and has one of its origins in the deficiency of metric spaces, not allowing the metrization of arbitrary topological product spaces where the factors are metric spaces. Lowen's generalization, which he termed approach spaces [9], assigns to a point $p \in S$ and a subset $A \subseteq S$ a number $d(p,A) \in [0,\infty]$ which has the interpretation of the distance between $p$ and $A$. The

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resulting category is topological and hence allows initial constructions, in particular arbitrary products, but, in contrast to the category of topological spaces, retains numerical information about distances. This theory also has reached a certain stage of maturity as is manifested in the textbooks [10] and the more recent [11].

Looking at approach spaces it becomes clear that also here one can argue that it is more reasonable to assign to a point $p \in S$ and a subset $A \subseteq S$ a distribution function $\delta(p, A)$, whose value at $x$, $\delta(p, A)(x)$ is then interpreted as the probability that the distance between $p$ and $A$ is less than $x$. This is the idea that we develop in this paper. We give a suitable set of axioms which generalize the axioms of a distance of an approach space. We show, using the notion of a probabilistic convergence space as introduced recently [7], that our category of probabilistic approach spaces is topological. Furthermore, it embeds the category of approach spaces both bireflectively and bicoreflectively in a natural way. Like the category of quasi-metric spaces embeds into the category of approach spaces as a coreflective subcategory, the category of probabilistic quasi-metric spaces embeds into the category of probabilistic approach spaces as a coreflective subcategory in a natural way.

The paper is organised as follows. We collect the necessary theory and notations in a preliminary section. Section 3 then defines probabilistic approach spaces. Section 4 is devoted to the category of probabilistic topological convergence spaces and Section 5 then shows that this category is isomorphic to the category of probabilistic approach spaces. Sections 6 and 7 treat the subcategories of approach spaces and of probabilistic metric spaces, respectively. Finally, we draw some conclusions.

## 2. Preliminaries

For an ordered set $(A, \leqslant)$ we denote, in the case of existence, by $\bigwedge_{i\in I} \alpha_i$ the infimum and by $\bigvee_{i\in I} \alpha_i$ the supremum of $\{\alpha_i \colon i \in I\} \subseteq A$. In the case of a two-point set $\{\alpha, \beta\}$ we write $\alpha \wedge \beta$ and $\alpha \vee \beta$, respectively.

For a set $S$ we denote its power set by $P(S)$ and the set of all filters $\mathcal{F}, \mathcal{G}, \dots$ on $S$ by $\mathbb{F}(S)$. The set $\mathbb{F}(S)$ is ordered by set inclusion and maximal elements of $\mathbb{F}(S)$ in this order are called *ultrafilters*. The set of all ultrafilters on $S$ is denoted by $\mathbb{U}(S)$. In particular, for each $p \in S$, the point filter $[p] = \{A \subseteq S \colon p \in A\} \in \mathbb{F}(S)$ is an ultrafilter. For $\mathcal{G} \in \mathbb{F}(J)$ and $\mathcal{F}_j \in \mathbb{F}(S)$ for each $j \in J$, we denote $\kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J}) = \bigvee_{G\in\mathcal{G}} \bigwedge_{j\in G} \mathcal{F}_j \in \mathbb{F}(S)$ the *diagonal filter* [8].

We assume some familiarity with category theory and refer to the textbooks [2] and [13] for more details and notation. A *construct* is a category $\mathcal{C}$ with a faithful functor $U \colon \mathcal{C} \to \mathbf{SET}$, from $\mathcal{C}$ to the category of sets. We always consider a construct as a category whose objects are structured sets $(S, \xi)$ and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i\colon\ S \to (S_i,\xi_i))_{i\in I}$ there is a unique structure $\xi$ on $S$, such that a mapping $g\colon\ (T,\eta)\to(S,\xi)$ is a morphism if and only if for each $i\in I$ the composition $f_i\circ g\colon\ (T,\eta)\to(S_i,\xi_i)$ is a morphism. A topological construct allows *final constructions*, i.e. for each sink $(f_i\colon\ (S_i,\xi_i)\to S)_{i\in I}$ there is a unique structure $\xi$ on $S$ such that a mapping $g\colon\ (S,\xi)\to(T,\eta)$ is a morphism if and only if for each $i\in I$, the composition $g\circ f_i : (S_i,\xi_i)\to(T,\eta)$ is a morphism.

A function $\varphi\colon\ [0,\infty]\to[0,1]$, which is non-decreasing, left-continuous on $(0,\infty)$ (in the sense that for all $x\in(0,\infty)$ we have $\varphi(x)=\bigvee\limits_{y<x}\varphi(y))$ and satisfies $\varphi(0)=0$ and $\varphi(\infty)=1$, is called a *distance distribution function* [19]. The set of all distance distribution functions is denoted by $\Delta^+$. For example, for each $0\leqslant a<\infty$ the functions

$$\varepsilon_a(x)=\begin{cases}0 & \text{if } 0\leqslant x\leqslant a,\\ 1 & \text{if } a<x\leqslant\infty\end{cases}\qquad\text{and}\qquad \varepsilon_\infty(x)=\begin{cases}0 & \text{if } 0\leqslant x<\infty,\\ 1 & \text{if } x=\infty\end{cases}$$

are in $\Delta^+$. The set $\Delta^+$ is ordered pointwise, i.e. for $\varphi,\psi\in\Delta^+$ we define $\varphi\leqslant\psi$ if for all $x\geqslant 0$ we have $\varphi(x)\leqslant\psi(x)$. The smallest element of $\Delta^+$ is then $\varepsilon_\infty$ and the largest element is $\varepsilon_0$. The following result is mentioned in Schweizer and Sklar [19].

**Lemma 2.1.**
(1) *If $\varphi,\psi\in\Delta^+$, then also $\varphi\wedge\psi\in\Delta^+$.*
(2) *If $\varphi_i\in\Delta^+$ for all $i\in I$, then also $\bigvee\limits_{i\in I}\varphi_i\in\Delta^+$.*

Here, $\varphi\wedge\psi$ denotes the pointwise minimum of $\varphi$ and $\psi$ in $(\Delta^+,\leqslant)$ and $\bigvee\limits_{i\in I}\varphi_i$ denotes the pointwise supremum of the family $\{\varphi_i\colon\ i\in I\}$ in $(\Delta^+,\leqslant)$. The set $\Delta^+$ with this order then becomes a complete lattice. We note that $\bigwedge\limits_{i\in I}\varphi_i$ is in general not the pointwise infimum. It is shown in [5] that this lattice is even completely distributive, i.e. it satisfies the following distributive laws.

$$\text{(CD1)}\qquad \bigvee_{j\in J}\Big(\bigwedge_{i\in I_j}\varphi_{ji}\Big)=\bigwedge_{f\in\prod\limits_{j\in J}I_j}\Big(\bigvee_{j\in J}\varphi_{jf(j)}\Big)$$

$$\text{(CD2)}\qquad \bigwedge_{j\in J}\Big(\bigvee_{i\in I_j}\varphi_{ji}\Big)=\bigvee_{f\in\prod\limits_{j\in J}I_j}\Big(\bigwedge_{j\in J}\varphi_{jf(j)}\Big)$$

It is well known that in any complete lattice $L$ (CD1) and (CD2) are equivalent. In any complete lattice $L$ we can define the *wedge-below relation* $\alpha\lhd\beta$ which holds if for all subsets $D\subseteq L$ such that $\beta\leqslant\bigvee D$ there is $\delta\in D$ such that $\alpha\leqslant\delta$. Then $\alpha\leqslant\beta$

whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j\in J} \beta_j$ if and only if $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee\{\beta\colon\ \beta \lhd \alpha\}$ for any $\alpha \in L$, see e.g. Theorem 7.2.3 in [1]. For more results on lattices we refer to [6].

A binary operation $\tau\colon\ \Delta^+ \times \Delta^+ \to \Delta^+$, which is commutative, associative (i.e. $\tau(\varphi, \tau(\psi, \eta)) = \tau(\tau(\varphi, \psi), \eta)$ for all $\varphi, \psi, \eta \in \Delta^+$), non-decreasing in each place and which satisfies the boundary condition $\tau(\varphi, \varepsilon_0) = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function* [19]. For a good survey on triangle functions see e.g. [16], [17]. A triangle function is called *sup-continuous* [19], [20], if $\tau\Bigl(\bigvee_{i\in I} \varphi_i, \psi\Bigr) = \bigvee_{i\in I} \tau(\varphi_i, \psi)$ for all $\varphi_i, \psi \in \Delta^+$, $i \in I$.

A *t-norm* $*\colon\ [0,1]\times[0,1] \to [0,1]$ is a binary operation on $[0,1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. A t-norm is called continuous if it is continuous as a mapping from $[0,1] \times [0,1] \to [0,1]$. It is shown e.g. in [19] that for a continuous t-norm $*$, the mapping $\tau_*$ defined by $\tau_*(\varphi, \psi)(x) = \bigvee_{u+v=x} \varphi(u) * \psi(v)$ for $\varphi, \psi \in \Delta^+$ is a triangle function. Typical examples for continuous t-norms are the minimum t-norm $\alpha * \beta = \alpha \wedge \beta$, the product t-norm $\alpha * \beta = \alpha\beta$ and the Lukasiewicz t-norm $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

## 3. The category of probabilistic approach spaces

**Definition 3.1.** A pair $(S, \delta)$ with a set $S$ and $\delta : S \times P(S) \to \Delta^+$ is called a *probabilistic approach space* (*under the triangle function* $\tau$) if for all $p \in S$, $A, B \subseteq S$ the following axioms are satisfied.

(PD1) $\delta(p, \{p\}) = \varepsilon_0$;
(PD2) $\delta(p, \emptyset) = \varepsilon_\infty$;
(PD3) $\delta(p, A) \vee \delta(p, B) = \delta(p, A \cup B)$ for all $A, B \subseteq S$;
(PD4) $\delta(p, A) \geqslant \tau(\delta(p, \overline{A}^{\varphi}), \varphi)$ for all $\varphi \in \Delta^+$, where $\overline{A}^{\varphi} = \{p \in S\colon\ \delta(p, A) \geqslant \varphi\}$.

A mapping $f\colon\ (S, \delta) \to (S', \delta')$ is called a *contraction* if $\delta(p, A) \leqslant \delta'(f(p), f(A))$ for all $p \in S$, $A \subseteq S$. The category with objects being the probabilistic approach spaces under the triangle function $\tau$ and morphisms being the contractions is denoted by $\mathsf{ProbAp}_\tau$.

The value $\delta(p, A)(x)$ can be interpreted as the probability that the distance between $p$ and $A$ is less than $x$.

**Lemma 3.2.** *Let* $(S, \delta) \in |\mathsf{ProbAp}_\tau|$.

(1) *If* $A \subseteq B$, *then* $\delta(p, A) \leqslant \delta(p, B)$.
(2) *Axiom* (PD4) *is equivalent to each of the following axioms.*
(PD4′) $\delta(p, A) \geqslant \tau(\varphi, \psi)$ *whenever* $\delta(p, \overline{A}^{\varphi}) \geqslant \psi$.
(PD4″) $\overline{\overline{A}^{\varphi}}^{\psi} \subseteq \overline{A}^{\tau(\varphi,\psi)}$.

P r o o f. (1) If $A \subseteq B$, then $A \cup B = B$ and hence by (PD3) $\delta(p,A) \leqslant \delta(p,A) \vee \delta(p,B) = \delta(p,B)$.

(2) Let (PD4) be true and let $\delta(p,\overline{A}^{\varphi}) \geqslant \psi$. Using (1) we obtain $\delta(p,A) \geqslant \tau(\delta(p,\overline{A}^{\varphi}),\varphi) \geqslant \tau(\psi,\varphi)$ and (PD4′) is satisfied. Let now (PD4′) be true and let $p \in \overline{\overline{A}^{\varphi}}^{\psi}$. Then $\delta(p,\overline{A}^{\varphi}) \geqslant \psi$ and hence $\delta(p,A) \geqslant \tau(\varphi,\psi)$. But this means $p \in \overline{A}^{\tau(\varphi,\psi)}$ and (PD4″) is satisfied. If (PD4″) is true, then $p \in \overline{\overline{A}^{\varphi}}^{\delta(p,\overline{A}^{\varphi})} \subseteq \overline{A}^{\tau(\delta(p,\overline{A}^{\varphi}),\varphi)}$, i.e. $\delta(p,A) \geqslant \tau(\delta(p,\overline{A}^{\varphi}),\varphi)$ and (PD4) is valid. $\square$

**Theorem 3.3.** *The category* $\mathsf{ProbAp}_{\tau}$ *is topological.*

In order to prove this theorem, we make a detour, which is also interesting in its own right, in the next two sections.

R e m a r k 3.4. Recently a paper with a similar definition appeared [15]. The authors of [15] call a pair $(S,F)$ with a mapping $F\colon\ S \times P(S) \times [0,\infty] \to [0,1]$ a *fuzzy approach space* if certain axioms are satisfied. If we define $\delta(p,A)(x) = F(p,A,x)$, then we obtain a distance distribution function which satisfies (PD1)–(PD3) and vice-versa a probabilistic approach space in our definition generates a fuzzy approach space in this way. The major difference of the approach in [15] is their axiom (FA6), which corresponds to our axiom (PD4). In [15] axiom (FA6) is, expressed in our terminology,

$$\text{(FA6)} \qquad \delta(p,A)(t+s) \geqslant \delta(p,A^{(r)})(t) \quad \text{for all } r \in [0,s).$$

The set $A^{(r)}$ is nothing else than our $\overline{A}^{\varepsilon_r}$. If we use a triangle function $\tau_*$ induced by a continuous t-norm $*$, then we have with (PD4)

$$\begin{aligned}\delta(p,A)(x) &\geqslant \tau_*(\delta(p,\overline{A}^{\varepsilon_s}),\varepsilon_s)(x)\\ &= \bigvee_{u+v=x} \delta(p,\overline{A}^{\varepsilon_s})(u) * \varepsilon_s(v) = \bigvee_{v>s} \delta(p,\overline{A}^{\varepsilon_s})(x-v) = \delta(p,\overline{A}^{\varepsilon_s})(x-s).\end{aligned}$$

Letting $x = t+s$ and noting that for $0 \leqslant r < s$ we have $A^{(r)} \subseteq A^{(s)}$ we obtain (FA6). Hence, for triangle functions induced by continuous t-norms, an axiom stronger than (FA6) is recovered.

There are many similarities between [15] and this paper, however also notable differences. The major difference is that in [15] morphisms are not considered, so no categorical properties are stated at all. In particular, the important Theorem 3.3 above is not there. Moreover, our approach is explicitely based on a triangle function and can easily be generalized to lattices different from $\Delta^+$. We will point to the similarities and differences at the appropriate places later.

# 4. Probabilistic topological convergence spaces

Let $S$ be a set. A family of mappings $\overline{c} = (c_\varphi \colon \mathbb{F}(S) \to P(S))_{\varphi \in \Delta^+}$ which satisfies the axioms

(PC1) $p \in c_\varphi([p])$ for all $p \in S$, $\varphi \in \Delta^+$;
(PC2) $c_\varphi(\mathcal{F}) \subseteq c_\varphi(\mathcal{G})$ whenever $\mathcal{F} \leqslant \mathcal{G}$;
(PC3) $c_\psi(\mathcal{F}) \subseteq c_\varphi(\mathcal{F})$ whenever $\varphi \leqslant \psi$;
(PC4) $p \in c_{\varepsilon_\infty}(\mathcal{F})$ for all $p \in S$, $\mathcal{F} \in \mathbb{F}(S)$;

is called a *probabilistic convergence structure on* $S$. The pair $(S, \overline{c})$ is called a *probabilistic convergence space* [7]. A mapping $f \colon S \to S'$, where $(S, \overline{c})$ and $(S', \overline{c'})$ are probabilistic convergence spaces, is called *continuous* if $f(p) \in c'_\varphi(f(\mathcal{F}))$ whenever $p \in c_\varphi(\mathcal{F})$ for all $p \in S$, all $\mathcal{F} \in \mathbb{F}(S)$ and all $\varphi \in \Delta^+$. The category of probabilistic convergence spaces continuous maps is denoted by $\mathsf{ProbConv}$. This category is topological, Cartesian closed and extensional, see [7].

A probabilistic convergence space $(S, \overline{c})$ is called *pretopological* if the axiom

$$\text{(PCPT)} \qquad \bigcap_{i \in I} c_\varphi(\mathcal{F}_i) \subseteq c_\varphi\left(\bigwedge_{i \in I} \mathcal{F}_i\right) \quad \text{whenever } \varphi \in \Delta^+ \text{ and } (\mathcal{F}_i)_{i \in I} \in \mathbb{F}(S)^I$$

is satisfied. It is called *left-continuous* if for all subsets $A \subseteq \Delta^+$ we have

$$\text{(PCL)} \qquad p \in c_{\bigvee A}(\mathcal{F}) \quad \text{whenever } p \in c_\alpha(\mathcal{F}) \text{ for all } \alpha \in A.$$

It is called *topological* (under the triangle function $\tau$) if it is left-continuous, pretopological and satisfies moreover the following *diagonal axiom*:

($\tau$-PK) for all $\mathcal{G}, \mathcal{F}_q \in \mathbb{F}(S)$, $q \in S$ we have that $p \in c_{\tau(\varphi,\psi)}(\kappa(\mathcal{G}, (\mathcal{F}_q)_{q \in S}))$ whenever $p \in c_\psi(\mathcal{G})$ and $q \in c_\varphi(\mathcal{F}_q)$ for all $q \in S$.

The category of probabilistic topological convergence spaces (under the triangle function $\tau$) is denoted by $\mathsf{ProbTConv}_\tau$.

For a probabilistic convergence space $(S, \overline{c})$, $\varphi \in \Delta^+$ and $p \in S$ let $\mathcal{U}_p^\varphi := \bigwedge_{p \in c_\varphi(\mathcal{F})} \mathcal{F}$. For $A \subseteq S$ we define the *$\varphi$-interior of* $A$, $\underline{A}_\varphi$, by

$$p \in \underline{A}_\varphi \iff A \in \mathcal{U}_p^\varphi.$$

In [7] we showed the following result.

**Lemma 4.1** ([7]). *Let $(S, \overline{c})$ be a probabilistic pretopological space and let $\tau$ be a triangle function. Then axiom ($\tau$-PK) is equivalent to $\underline{A}_{\tau(\varphi,\psi)} \subseteq \underline{\underline{A}_\varphi}_\psi$ for all $\varphi, \psi \in \Delta^+$ and all $A \subseteq S$.*

We will now introduce a generalization of a diagonal axiom attributed to Fischer. We say that $(S, \overline{c}) \in |\mathsf{ProbConv}|$ satisfies axiom ($\tau$-PF) if

($\tau$-PF) for all sets $J$, all $\mathcal{G} \in \mathbb{F}(J)$, all $h\colon J \to S$ and all $\mathcal{F}_j \in \mathbb{F}(S)$, $j \in J$ we have $p \in c_{\tau(\varphi,\psi)}(\kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J}))$ whenever $p \in c_\psi(h(\mathcal{G}))$ and $h(j) \in c_\varphi(\mathcal{F}_j)$ for all $j \in J$.

If we take $J = S$ and $h = \mathrm{id}_S$, then we see that ($\tau$-PF) implies ($\tau$-PK).

**Lemma 4.2.** *Let $(S, \overline{c}) \in |\mathsf{ProbConv}|$ satisfy axiom ($\tau$-PF). Then also axiom* (PCPT) *is satisfied.*

Proof. Let $\mathcal{F}_i \in \mathbb{F}(S)$ for all $i \in J$. We define $h(i) = p$ and $\mathcal{G} = [J]$. Then $h(\mathcal{G}) = [p]$ and $\kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J}) = \bigwedge\limits_{i\in J} \mathcal{F}_i$. If $p \in c_\varphi(\mathcal{F}_i)$ for all $i \in J$, then because $p \in c_{\varepsilon_0}([p])$ and $h(i) = p \in c_\varphi(\mathcal{F}_i)$, we have by ($\tau$-PF) $p \in c_{\tau(\varepsilon_0,\varphi)}\Big(\bigwedge\limits_{i\in J} \mathcal{F}_i\Big) = c_\varphi\Big(\bigwedge\limits_{i\in J} \mathcal{F}_i\Big)$. □

Hence, axiom ($\tau$-PF) implies both ($\tau$-PK) and (PCPT). The converse is also true.

**Lemma 4.3.** *Let $(S, \overline{c}) \in |\mathsf{ProbConv}|$. If* (PCPT) *and for all $A \subseteq S$ and $\varphi, \psi \in \Delta^+$ we have $\underline{A}_{\tau(\varphi,\psi)} \subseteq \underline{\underline{A}_{\varphi}}_{\psi}$, then* ($\tau$-PF) *is true.*

Proof. Let $J$ be a set, $h\colon J \to S$, $\mathcal{G} \in \mathbb{F}(J)$ and for all $i \in J$ let $\mathcal{F}_i \in \mathbb{F}(S)$. If $p \in c_\psi(h(\mathcal{G}))$ and $h(j) \in c_\varphi(\mathcal{F}_j)$ for all $j \in J$, then by (PCPT) $h(\mathcal{G}) \geqslant \mathcal{U}_p^\psi$ and for all $j \in J$ we have $\mathcal{F}_j \geqslant \mathcal{U}_{h(j)}^\varphi$. Let $A \in \mathcal{U}_p^{\tau(\varphi,\psi)}$. Then $p \in \underline{A}_{\tau(\varphi,\psi)} \subseteq \underline{\underline{A}_{\varphi}}_{\psi}$ and hence $\underline{A}_\varphi \in \mathcal{U}_p^\psi \leqslant h(\mathcal{G})$. Thus, there is $G \in \mathcal{G}$ such that $h(G) \subseteq \underline{A}_\varphi$, i.e. for all $j \in G$ we have $h(j) \in \underline{A}_\varphi$. This means that $A \in \mathcal{U}_{h(j)}^\varphi \leqslant \mathcal{F}_j$ for all $j \in G$. Consequently, for all $j \in G$ we have $A \in \mathcal{F}_j$, i.e. $A \in \bigwedge\limits_{j\in G} \mathcal{F}_j \leqslant \kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J})$. Hence, we have shown that $\mathcal{U}_p^{\tau(\varphi,\psi)} \leqslant \kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J})$. From axiom (PCPT) we then conclude that $p \in c_{\tau(\varphi,\psi)}(\kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J}))$ and ($\tau$-PF) is satisfied. □

We finally show that the category $\mathsf{ProbTConv}_\tau$ is a topological category. To this end, we show that axioms (PLC) and ($\tau$-PF) are preserved under initial constructions in $\mathsf{ProbConv}$. For a source $(f_i\colon S \to (S_i, \overline{c^i}))_{i\in I}$ we define the initial probabilistic convergence structure on $S$ by $p \in c_\varphi(\mathcal{F})$ if and only if for all $i \in I$ we have $f_i(p) \in c_\varphi^i(f_i(\mathcal{F}))$, see [7].

**Lemma 4.4.** *Axiom* (PLC) *is preserved under initial constructions.*

P r o o f. Let $(f_i\colon S \to (S_i, \overline{c^i}))_{i\in I}$ be a source and let $\overline{c}$ be the initial probabilistic convergence structure on $S$. Let $A \subseteq \Delta^+$. Let $p \in c_\varphi(\mathcal{F})$ for all $\varphi \in A$. Then $f_i(p) \in c^i_\varphi(f_i(\mathcal{F}))$ for all $\varphi \in A$ and all $i \in I$. Hence, for all $i \in I$, $f_i(p) \in c^i_{\bigvee A}(f_i(\mathcal{F}))$ and this means $p \in c_{\bigvee A}(\mathcal{F})$. The converse is always true by (PC3). $\square$

**Lemma 4.5.** *Axiom ($\tau$-PF) is preserved under initial constructions.*

P r o o f. Let again $(f_i\colon S \to (S_i, \overline{c^i}))_{i\in I}$ be a source and let $\overline{c}$ be the initial probabilistic convergence structure on $S$. Let $J$ be a set, $h\colon J \to S$, $\mathcal{G} \in \mathbb{F}(J)$ and for all $j \in J$ let $\mathcal{F}_j \in \mathbb{F}(S)$. If $p \in c_\varphi(h(\mathcal{G}))$ and for all $j \in J$, $h(j) \in c_\psi(\mathcal{F}_j)$, then for all $i \in I$ we have $f_i(p) \in c^i_\varphi(f_i(h(\mathcal{G})))$ and $f_i(h(j)) \in c^i_\psi(f_i(\mathcal{F}_j))$ for all $j \in J$. We denote $k_i = f_i \circ h\colon J \to S_i$ for all $i \in I$. Then $f_i(p) \in c^i_{\tau(\varphi,\psi)}(\kappa(\mathcal{G}, (f_i(\mathcal{F}_j))_{j\in J}))$ for all $i \in I$. It is not difficult to show that $\kappa(\mathcal{G}, (f_i(\mathcal{F}_j))_{j\in J}) = f_i(\kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J}))$. Hence, $f_i(p) \in c^i_{\tau(\varphi,\psi)}(f_i(\kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J})))$ for all $i \in I$, i.e. $p \in c_{\tau(\varphi,\psi)}(\kappa(\mathcal{G}, (\mathcal{F}_j)_{j\in J}))$. $\square$

We collect these results in the following theorem.

**Theorem 4.6.** *The category* $\mathsf{ProbTConv}_\tau$ *is topological.*

R e m a r k 4.7. In [14] a related category of probabilistic convergence spaces was introduced. The major difference to our approach here is that Richardson and Kent use the unit interval $[0,1]$ as an index set whereas we use here $\Delta^+$ as an index set. In [7] it was shown that Richardson and Kent's category can be reflectively embedded into our category. To this end, we define for a probabilistic convergence space in Richardson and Kent's sense, $(S, \overline{q})$ with $\overline{q} = (q_\alpha\colon \mathbb{F}(S) \to P(S))_{\alpha\in[0,1]}$ for $\varphi \in \Delta^+$, the right-hand limit of $\varphi$ at 0 $\varphi(0+) = \lim\limits_{x\to 0+} \varphi(x)$ and then define $p \in c^{\overline{q}}_\varphi(\mathcal{F})$ if $p \in q_{\varphi(0+)}(\mathcal{F})$. This yields an embedding functor of Richardson and Kent's category into our category $\mathsf{ProbConv}$. For more details see [7]. It is possible in Richardson and Kent's category to characterize metric spaces, see [3]. An advantage of using $\Delta^+$ as an index set is that we can characterize probabilistic metric spaces in our category, see [7].

## 5. The isomorphy of $\mathsf{ProbAp}_\tau$ and $\mathsf{ProbTConv}_\tau$

For $A \subseteq S$ and $\mathcal{F} \in \mathbb{F}(S)$ we say that $A$ *meshes with* $\mathbb{F}$, $A\sharp\mathcal{F}$, if $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

**Lemma 5.1.** *Let $A \subseteq S$, $\mathcal{F}, \mathcal{G}, \mathcal{F}_i \in \mathbb{F}(S)$ for $i \in J$, $f : S \to S'$ and $W \subseteq S'$. Then*

(1) *$\mathcal{F} \leqslant \mathcal{G}$ and $A\sharp\mathcal{G}$ implies $A\sharp\mathcal{F}$;*

(2) $A\sharp \bigwedge_{i\in J} \mathcal{F}_i$ *if and only if* $A\sharp\mathcal{F}_i$ *for some* $i \in J$*;*

(3) $W\sharp f(\mathcal{F})$ *implies* $f^{-1}(W)\sharp\mathcal{F}$*;*

(4) *If* $\mathcal{U}$ *is an ultrafilter, then* $A\sharp\mathcal{U}$ *if and only if* $A \in \mathcal{U}$*.*

Proof. (1) and (4) are clear. We prove (2). Let $A\sharp \bigwedge_{i\in J} \mathcal{F}_i$ and assume that for all $j \in J$ there is $F_j \in \mathcal{F}_j$ such that $A \cap F_j = \emptyset$. Then $\bigcup_{j\in J} F_j \in \bigwedge_{j\in J} \mathcal{F}_j$ and $A \cap \bigcup_{j\in J} F_j = \bigcup_{j\in J} (A \cap F_j) = \emptyset$, a contradiction. The other direction follows from (1). For (3) we notice that $W \cap f(F) \neq \emptyset$ if and only if $F \cap f^{-1}(W) \neq \emptyset$. □

Let $(S,\delta) \in |\mathsf{ProbAP}_\tau|$. We define for $\varphi \in \Delta^+$, $p \in S$ and $\mathcal{F} \in \mathbb{F}(S)$,

$$p \in c^\delta_\varphi(\mathcal{F}) \iff \bigwedge_{A\sharp\mathcal{F}} \delta(p,A) \geqslant \varphi.$$

Clearly then for $\mathcal{U} \in \mathbb{F}(S)$ an ultrafilter we have $p \in c^\delta_\varphi(\mathcal{U})$ if and only if $\bigwedge_{A\in\mathcal{U}} \delta(p,A) \geqslant \varphi$.

**Lemma 5.2.** *Let* $(S,\delta) \in |\mathsf{ProbAp}_\tau|$*. Then* $(S,\overline{c^\delta}) \in |\mathsf{ProbTConv}_\tau|$*.*

Proof. (PC1) We have $\varphi \leqslant \varepsilon_0 = \delta(p,\{p\}) = \bigwedge_{U\in[p]} \delta(p,U)$ for all $\varphi \in \Delta^+$ and hence $p \in c^\delta_\varphi([p])$.

(PC2) Let $\mathcal{F} \leqslant \mathcal{G}$ and $p \in c^\delta_\varphi(\mathcal{F})$. Then $\bigwedge_{A\sharp\mathcal{G}} \delta(p,A) \geqslant \bigwedge_{A\sharp\mathcal{F}} \delta(p,A) \geqslant \varphi$ and hence $p \in c^\delta_\varphi(\mathcal{G})$.

(PC3) Let $\varphi \leqslant \psi$ and $p \in c^\delta_\psi(\mathcal{F})$. Then $\varphi \leqslant \psi \leqslant \bigwedge_{A\sharp\mathcal{F}} \delta(p,A)$ and hence $p \in c^\delta_\varphi(\mathcal{F})$.

(PC4) We have $\bigwedge_{A\sharp\mathcal{F}} \delta(p,A) \geqslant \varepsilon_\infty$ for all $p \in S$ and hence $c^\delta_{\varepsilon_\infty}(\mathcal{F}) = S$.

(PCPT) Let $p \in c^\delta_\varphi(\mathcal{F}_j)$ for all $j \in J$. Then $\bigwedge_{A\sharp\mathcal{F}_j} \delta(p,A) \geqslant \varphi$ for all $j \in J$ and hence $\bigwedge_{A\sharp \bigwedge_{j\in J} \mathcal{F}_j} \delta(p,A) = \bigwedge_{A\sharp\mathcal{F}_j \text{ for some } j\in J} \delta(p,A) \geqslant \varphi$. This implies $p \in c^\delta_\varphi\Big(\bigwedge_{j\in J} \mathcal{F}_j\Big)$.

(PCL) If $p \in c^\delta_\alpha(\mathcal{F})$ for all $\alpha \in \Gamma \subseteq \Delta^+$, then $\bigwedge_{A\sharp\mathcal{F}} \delta(p,A) \geqslant \alpha$ for all $\alpha \in \Gamma \subseteq \Delta^+$ and hence $\bigwedge_{A\sharp\mathcal{F}} \delta(p,A) \geqslant \bigvee\Gamma$, i.e. $p \in c^\delta_{\bigvee\Gamma}(\mathcal{F})$.

($\tau$-PK) Let $p \in c^\delta_\psi(\mathcal{G})$ and $q \in c^\delta_\varphi(\mathcal{F}_q)$ for all $q \in S$. Consider first the case that all filters are ultrafilters and let $D \in \kappa(\mathcal{G},(\mathcal{F}_q)_{q\in S}) = \bigcup_{G\in\mathcal{G}} \bigwedge_{q\in G} \mathcal{F}_q$. Then there is $G \in \mathcal{G}$ such that $D \in \mathcal{F}_q$ for all $q \in G$. Hence, $\delta(q,D) \geqslant \bigwedge_{F\in\mathcal{F}_q} \delta(q,F) \geqslant \varphi$ for all $q \in G$, i.e. $q \in \overline{D}^{\varphi}$ for all $q \in G$. But this means $G \subseteq \overline{D}^{\varphi}$ and hence $\overline{D}^{\varphi} \in \mathcal{G}$. We conclude with axiom (PD4)

$$\delta(p, D) \geqslant \tau(\delta(p, \overline{D}^{\varphi}), \varphi) \geqslant \tau\Big(\bigwedge_{G \in \mathcal{G}} \delta(p, G), \varphi\Big) \geqslant \tau(\psi, \varphi).$$

Hence, $\bigwedge\limits_{D \in \kappa(\mathcal{G}, (\mathcal{F}_q)_{q \in S})} \delta(p, A) \geqslant \tau(\varphi, \psi)$ and we have $p \in c^{\delta}_{\tau(\varphi,\psi)}(\kappa(\mathcal{G}, (\mathcal{F}_q)_{q \in S}))$.

If $\mathcal{F}_q$ and $\mathcal{G}$ are arbitrary, we consider ultrafilters $\mathcal{U}_q \geqslant \mathcal{F}_q$ and $\mathcal{V} \geqslant \mathcal{G}$. According to [10], Proposition 1.2.3, we have

$$\kappa(\mathcal{G}, (\mathcal{F}_q)_{q \in S}) = \bigwedge_{\mathcal{V} \geqslant \mathcal{G} \text{ ultra}} \; \bigwedge_{\mathcal{U}_q \geqslant \mathcal{F}_q \text{ ultra}} \kappa(\mathcal{V}, (\mathcal{U}_q)_{q \in S})$$

and from above we have $p \in c^{\delta}_{\tau(\varphi,\psi)}(\kappa(\mathcal{V}, (\mathcal{U}_q)_{q \in S}))$ and hence by (PCPT) we obtain $p \in c^{\delta}_{\tau(\varphi,\psi)}(\kappa(\mathcal{G}, (\mathcal{F}_q)_{q \in S}))$. $\square$

**Lemma 5.3.** *Let $f\colon (S, \delta) \to (S', \delta')$ be a contraction. Then $f\colon (S, \overline{c^{\delta}}) \to (S', \overline{c^{\delta'}})$ is continuous.*

Proof. Let $p \in c^{\delta}_{\varphi}(\mathcal{F})$. Then $\bigwedge\limits_{A \sharp \mathcal{F}} \delta(p, A) \geqslant \varphi$ and we conclude

$$\begin{aligned}\bigwedge_{W \sharp f(\mathcal{F})} \delta'(f(p), W) &\geqslant \bigwedge_{f^{-1}(W) \sharp \mathcal{F}} \delta'(f(p), W) \geqslant \bigwedge_{f^{-1}(W) \sharp \mathcal{F}} \delta'(f(p), f(f^{-1}(W))) \\ &\geqslant \bigwedge_{f^{-1}(W) \sharp \mathcal{F}} \delta(p, f^{-1}(W)) \geqslant \bigwedge_{A \sharp \mathcal{F}} \delta(p, A) \geqslant \varphi.\end{aligned}$$

Hence $f(p) \in c^{\delta'}(f(\mathcal{F}))$. $\square$

As a consequence we can define a functor

$$H\colon \begin{cases} \mathsf{ProbAp}_{\tau} \longrightarrow \mathsf{ProbTConv}_{\tau}, \\ (S, \delta) \longmapsto (S, \overline{c^{\delta}}), \\ f \longmapsto f. \end{cases}$$

Let now $(S, \overline{c}) \in |\mathsf{ProbTConv}_{\tau}|$. We define for $p \in S$ and $A \subseteq S$ the distance distribution function

$$\delta^{c}(p, A) = \bigvee_{\mathcal{U} \in \mathbb{U}(S), A \in \mathcal{U}} \; \bigvee_{\varphi\colon p \in c_{\varphi}(\mathcal{U})} \varphi.$$

**Lemma 5.4.** *Let $(S, \overline{c}) \in |\mathsf{ProbTConv}_{\tau}|$ and let $\mathcal{U} \in \mathbb{F}(S)$ be an ultrafilter. Then we have $p \in c_{\varphi}(\mathcal{U})$ if and only if $\bigwedge\limits_{B \in \mathcal{U}} \delta^{c}(p, B) \geqslant \varphi$.*

P r o o f. Let $p \in c_\varphi(\mathcal{U})$. Then $\bigwedge\limits_{B\in\mathcal{U}} \delta^c(p,B) \geqslant \bigwedge\limits_{B\in\mathcal{U}} \bigvee\limits_{p\in c_\psi(\mathcal{U})} \psi \geqslant \varphi$. For the converse we use the complete distributivity of $\Delta^+$. Let $\eta \lhd \varphi$ and let $\bigwedge\limits_{B\in\mathcal{U}} \delta^c(p,B) \geqslant \varphi \rhd \eta$. Then for all $B \in \mathcal{U}$ there is an ultrafilter $\mathcal{V}^B$ with $B \in \mathcal{V}^B$ and $\psi \in \Delta^+$ with $p \in c_\psi(\mathcal{V}^B)$ such that $\psi \rhd \eta$. By (PC3), for all $B \in \mathcal{U}$ there is an ultrafilter $\mathcal{V}^B$ with $B \in \mathcal{V}^B$ such that $p \in c_\eta(\mathcal{V}^B)$. As $(S,\overline{c})$ is pretopological, we conclude $p \in \bigcap\limits_{B\in\mathcal{U}} c_\eta(\mathcal{V}^B) = c_\eta\Big(\bigwedge\limits_{B\in\mathcal{U}} \mathcal{V}^B\Big)$. If $F \in \bigwedge\limits_{B\in\mathcal{U}} \mathcal{V}^B$, then for all $B \in \mathcal{U}$ we have $F \in \mathcal{V}^B$ and hence $F \cap B \neq \emptyset$. Hence, $F \in \mathcal{U}$, i.e. we have $\bigwedge\limits_{B\in\mathcal{U}} \mathcal{V}^B \leqslant \mathcal{U}$. Consequently, $p \in c_\eta(\mathcal{U})$ for all $\eta \lhd \varphi$ and from the left-continuity we conclude $p \in c_{\bigvee\{\eta\in\Delta^+ \colon \eta\lhd\varphi\}}(\mathcal{U}) = c_\varphi(\mathcal{U})$. □

**Lemma 5.5.** *Let $(S,\overline{c}) \in |\mathsf{ProbTConv}_\tau|$ and let $p \in S$, $A \subseteq S$ and $\varphi \in \Delta^+$. Then $\delta^c(p,A) \geqslant \varphi$ if and only if there is an ultrafilter $\mathcal{U}$ with $A \in \mathcal{U}$ such that $p \in c_\varphi(\mathcal{U})$.*

P r o o f. The one direction is obvious. For the other direction we again make use of the complete distributivity of $\Delta^+$. Let $\delta^c(p,A) \geqslant \varphi$ and assume that for all ultrafilters $\mathcal{U}$ with $A \in \mathcal{U}$ we have $p \notin c_\varphi(\mathcal{U})$. Hence, for all ultrafilters $\mathcal{U}$ with $A \in \mathcal{U}$ we have $\bigwedge\limits_{B\in\mathcal{U}} \delta^c(p,B) \ngeqslant \varphi$, i.e. for all ultrafilters $\mathcal{U}$ with $A \in \mathcal{U}$ there is $\eta_{\mathcal{U}} \lhd \varphi$ such that $\bigwedge\limits_{B\in\mathcal{U}} \delta^c(p,B) \ntrianglerighteq \eta_{\mathbb{U}}$. This implies that for each ultrafilter $\mathcal{U}$ with $A \in \mathcal{U}$ there is $\eta_{\mathcal{U}} \lhd \varphi$ and $B_{\mathcal{U}} \in \mathcal{U}$ such that $\delta^c(p,B_{\mathcal{U}}) \ntrianglerighteq \eta_{\mathcal{U}}$. By Proposition 1.2.2 in [10], there are ultrafilters $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ with $A, B_{\mathcal{U}_i} \in \mathcal{U}_i$ for $i = 1,2,\ldots,n$ and $A \subseteq B_{\mathcal{U}_1} \cup \ldots \cup B_{\mathcal{U}_n}$. It is not difficult to show that $\delta^c$ satisfies axiom (PD3) and hence $\delta^c(p,A) \leqslant \delta^c(p,B_{\mathcal{U}_1}) \vee \ldots \vee \delta^c(p,B_{\mathcal{U}_n})$. We have $\bigvee\limits_{i=1}^{n} \eta_{\mathcal{U}_i} \lhd \varphi \leqslant \delta^c(p,A)$, and hence $\eta_{\mathcal{U}_i} \leqslant \bigvee\limits_{i=1}^{n} \eta_{\mathcal{U}_i} \lhd \delta^c(p,B_{\mathcal{U}_i})$ for some $i \in \{1,2,\ldots,n\}$, a contradiction. □

**Theorem 5.6.** *Let $(S,\overline{c}) \in |\mathsf{ProbTConv}_\tau|$. Then $(S,\delta^c) \in |\mathsf{ProbAp}|$.*

P r o o f. (PD1) follows from (PC1), (PD2) is a consequence of $\bigvee\emptyset = \varepsilon_\infty$ in $\Delta^+$. (PD3) follows as $A, B \in \mathcal{U}$ is equivalent to $A \in \mathcal{U}$ or $B \in \mathcal{U}$ for ultrafilters $\mathcal{U}$. In order to show (PD4) we prove that the $\varphi$-closure of $A$ in $(S,\overline{c})$ coincides with $\overline{A}^\varphi$ in $(S,\delta^c)$. We have that $p$ is in the $\varphi$-closure of $A \subseteq S$ in $(S,\overline{c})$ if and only if there is an ultrafilter $\mathcal{U}$ with $A \in \mathcal{U}$ and $p \in c_\varphi(\mathcal{U})$. Then clearly $\delta^c(p,A) \geqslant \varphi$. Conversely, if $p \in \overline{A}^\varphi$ in $(S,\delta^c)$, then $\delta^c(p,A) \geqslant \varphi$ and by the previous lemma $p$ belongs to the $\varphi$-closure of $A$ in $(S,\overline{c})$. Hence, $(S,\delta^c)$ satisfies (PD4$''$). □

**Lemma 5.7.** *Let $f\colon (S,\overline{c}) \to (S',\overline{c'})$ be continuous. Then $f\colon (S,\delta^c) \to (S',\delta^{c'})$ is a contraction.*

P r o o f. If $A \in \mathcal{U}$ and $p \in c_\varphi(\mathcal{U})$ for an ultrafilter $\mathcal{U} \in \mathbb{U}(S)$, then $f(A) \in f(\mathcal{U})$ and $f(p) \in c'_\varphi(f(\mathcal{U}))$ and $f(\mathcal{U})$ is an ultrafilter. Hence,

$$\delta^c(p,A) = \bigvee_{A\in\mathcal{U}\in\mathbb{U}(S)} \bigvee_{p\in c_\varphi(\mathcal{U})} \varphi \leqslant \bigvee_{f(A)\in\mathcal{V}\in\mathbb{U}(S')} \bigvee_{f(p)\in c'(\varphi(\mathcal{V}))} \varphi = \delta^{c'}(f(p), f(A)).$$

$\square$

As a consequence, we can define a functor

$$K : \begin{cases} \mathsf{ProbTConv}_\tau \longrightarrow \mathsf{ProbAp}_\tau, \\ (S,\overline{c}) \longmapsto (S,\delta^c), \\ f \longmapsto f. \end{cases}$$

We will show that $H$ and $K$ are isomorphisms.

**Proposition 5.8.** *Let $(S,\overline{c}) \in |\mathsf{ProbTConv}_\tau|$. Then $\overline{c^{(\delta^c)}} = \overline{c}$.*

P r o o f. Let $\mathcal{U} \in \mathbb{F}(S)$ be an ultrafilter and let $\varphi \in \Delta^+$. By definition, $p \in c_\varphi^{(\delta^c)}(\mathcal{U})$ if and only if $\bigwedge\limits_{A\in\mathcal{U}} \delta^c(p,A) \geqslant \varphi$. From Lemma 5.5 we see that this is equivalent to $p \in c_\varphi(\mathcal{U})$. As both $(S,\overline{c})$ and $(S,\overline{c^{(\delta^c)}})$ are pretopological, the claim follows. $\square$

**Proposition 5.9.** *Let $(S,\delta) \in |\mathsf{ProbAp}_\tau|$. Then $\delta^{(c^\delta)} = \delta$.*

P r o o f. Let $p \in S$ and $A \subseteq S$. If $\mathcal{U} \in \mathbb{F}(S)$ is an ultrafilter with $A \in \mathcal{U}$, then by definition we have $\delta(p,A) \geqslant \varphi$ whenever $p \in c^\delta(\mathcal{U})$. Hence,

$$\bigvee_{p\in c_\varphi^\delta(\mathcal{U})} \varphi \leqslant \bigvee_{\varphi\leqslant\delta(p,A)} \varphi = \delta(p,A)$$

and we conclude

$$\delta^{(c^\delta)}(p,A) = \bigvee_{A\in\mathcal{U}\in\mathbb{U}(S)} \bigvee_{p\in c_\varphi^\delta(\mathcal{U})} \varphi \leqslant \delta(p,A).$$

To prove the other inequality, we use the complete distributivity of $\Delta^+$. We first note that we have $\bigvee\{\varphi\colon\ p \in c_\varphi^\delta(\mathcal{U})\} = \bigwedge\limits_{B\in\mathcal{U}} \delta(p,B)$. With the notation $J = \{\mathcal{U} \in \mathbb{U}(S)\colon A \in \mathcal{U}\}$ and $I_\mathcal{U} = \mathcal{U}$ for $\mathcal{U} \in J$, we then have

$$\delta^{(c^\delta)}(p,A) = \bigvee_{\mathcal{U}\in J} \bigwedge_{B\in I_\mathcal{U}} \delta(p,B) = \bigwedge_{\Theta\in\prod\limits_{\mathcal{U}\in J} I_\mathcal{U}} \bigvee_{\mathcal{U}\in J} \delta(p,\Theta(\mathcal{U})),$$

where $\Theta\colon J \to \bigcup_{\mathcal{U}\in J} I_{\mathcal{U}}$, $\Theta(\mathcal{U}) \in I_{\mathcal{U}} = \mathcal{U}$. With $\mathcal{F} = [A]$ we have $\mathcal{U} \in J$ if and only if $\mathcal{U} \geqslant \mathcal{F}$ and as $\Theta(\mathcal{U}) \in \mathcal{U}$ for all $\mathcal{U} \in J$, there are finitely many $\mathcal{U}_1, \ldots, \mathcal{U}_n \in J$ with $\bigcup_{i=1}^{n} \Theta(\mathcal{U}_i) \in \mathcal{F}$, i.e. $A \subseteq \bigcup_{i=1}^{n} \Theta(\mathcal{U}_i)$. Hence, we conclude

$$\delta^{(c^{\delta})}(p, A) \geqslant \bigwedge_{\Theta \in \prod_{\mathcal{U}\in J} I_{\mathcal{U}}} \bigvee_{i=1}^{n} \delta(p, \Theta(\mathcal{U}_i)) = \bigwedge_{\Theta \in \prod_{\mathcal{U}\in J} I_{\mathcal{U}}} \delta\Bigl(p, \bigcup_{i=1}^{n} \Theta(\mathcal{U}_i)\Bigr) \geqslant \delta(p, A),$$

and the proof is complete. $\square$

We collect the results of Propositions 5.8 and 5.9 in the following theorem.

**Theorem 5.10.** *The categories* $\mathsf{ProbAp}_\tau$ *and* $\mathsf{ProbTConv}_\tau$ *are isomorphic.*

In particular, we obtain the following result.

**Corollary 5.11.** *The category* $\mathsf{ProbAp}_\tau$ *is topological.*

## 6. Lowen's approach spaces as probabilistic approach spaces

A pair $(S, d)$ with a *distance* $d : S \times P(S) \to [0, \infty]$ is an *approach space* [9], [10], [11] if

(D1) $d(p, \{p\}) = 0$ for all $p \in S$,
(D2) $d(p, \emptyset) = \infty$ for all $p \in S$,
(D3) $d(p, A \cup B) = d(p, A) \wedge d(p, B)$ for all $p \in S$, $A, B \subseteq S$,
(D4) $d(p, A) \leqslant d(p, A^{(\varepsilon)}) + \varepsilon$ for all $p \in S, A \subseteq S$, $\varepsilon \in [0, \infty]$.

Here $A^{(\varepsilon)} = \{p \in S\colon\ d(p, A) \leqslant \varepsilon\}$. A mapping between two approach spaces $f\colon (S, d) \to (S', d')$ is called a *contraction* if $d'(f(p), f(A)) \leqslant d(p, A)$ for all $p \in S$, $A \subseteq S$. The category of approach spaces with contractions as morphisms is denoted by $\mathsf{Ap}$, see [10].

For $(S, d) \in |\mathsf{Ap}|$ we define $\delta^d(p, A) = \varepsilon_{d(p,A)}$. We then have $p \in \overline{A}^{\varphi}$ if and only if $\varepsilon_{d(p,A)} \geqslant \varphi$, which holds if and only if $x \leqslant d(p, A)$ implies $\varphi(x) = 0$. This is equivalent to $\bigwedge\{x\colon\ \varphi(x) > 0\} \geqslant d(p, A)$, i.e. to $p \in A^{(\bigwedge\{x\colon\ \varphi(x)>0\})}$.

**Proposition 6.1.** *Let* $(S, d) \in |\mathsf{Ap}|$. *Then* $(S, \delta^d) \in |\mathsf{ProbAp}_{\tau_*}|$ *under any triangle function of the form* $\tau_*$, *where* $*$ *is a continuous t-norm.*

P r o o f. (PD1), (PD2) and (PD3) are easy and left for the reader. We prove (PD4). Let $\delta^d(p,A)(z)=\varepsilon_{d(p,A)}(z)=0$. This is equivalent to $z\leqslant d(p,A)$. We show that then $\tau_*(\delta^d(p,\overline{A}^{\varphi}),\varphi)(z)=0$. Let $x+y=z$ and consider $\delta^d(p,\overline{A}^{\varphi})(x)*\varphi(y)$. If $\varphi(y)=0$, then $\delta^d(p,\overline{A}^{\varphi})(x)*\varphi(y)=0$. If $\varphi(y)>0$, then $y\geqslant\bigwedge\{x\colon\ \varphi(x)>0\}$. Assume that $\varepsilon_{d(p,\overline{A}^{\varphi})}(x)=\varepsilon_{d(p,\overline{A}^{\varphi})}(z-y)=1$. Then $z-y>d(p,\overline{A}^{\varphi})$ and hence $z>d(p,A^{(\bigwedge\{x\colon\ \varphi(x)>0\})})+y\geqslant d(p,A^{(\bigwedge\{x\colon\ \varphi(x)>0\})})+\bigwedge\{x\colon\ \varphi(x)>0\}\geqslant d(p,A)$, by (D4). This contradicts $\delta^d(p,A)(z)=0$. Hence, $\varepsilon_{d(p,\overline{A}^{\varphi})}(z-y)=0$ and we get $\delta^d(p,\overline{A}^{\varphi})(x)*\varphi(y)=0$ whenever $x+y=z$. As $x+y=z$ was arbitrary, $\tau_*(\delta^d(p,\overline{A}^{\varphi}),\varphi)(z)=0$ and we have $\tau_*(\delta^d(p,\overline{A}^{\varphi}),\varphi)\leqslant\delta^d(p,A)$. □

**Proposition 6.2.** *Let $(S,d),(S',d')\in|\mathsf{Ap}|$. Then $f\colon\ (S,d)\to(S',d')$ is a contraction if and only if $f\colon\ (S,\delta^d)\to(S',\delta^{d'})$ is a contraction.*

P r o o f. If $f\colon\ (S,d)\to(S',d')$ is a contraction, then we have for $p\in S$, $A\subseteq S$ that $d'(f(p),f(A))\leqslant d(p,A)$. Hence $\delta^d(p,A)=\varepsilon_{d(p,A)}\leqslant\varepsilon_{d'(f(p),f(A))}=\delta^{d'}(f(p),f(A))$. Conversely, if $f\colon\ (S,\delta^d)\to(S',\delta^{d'})$ is a contraction, then $\varepsilon_{d(p,A)}=\delta^d(p,A)\leqslant\delta^{d'}(f(p),f(A))=\varepsilon_{d'(f(p),f(A))}$ and hence $d'(f(p),f(A))\leqslant d(p,A)$. □

Hence we have a functor

$$G:\begin{cases}\mathsf{Ap}\longrightarrow\mathsf{ProbAp}_{\tau_*},\\ (S,d)\longmapsto(S,\delta^d),\\ f\longmapsto f.\end{cases}$$

This functor is clearly injective on objects and hence an embedding functor.

Let now $(S,\delta)\in|\mathsf{ProbAp}_{\tau}|$ and let $\alpha>0$. We define

$$d^{\delta}_{\alpha}(p,A)=\bigwedge\{x\colon\ \delta(p,A)\geqslant\alpha\}=\bigvee\{x\colon\ \delta(p,A)(x)<\alpha\}.$$

Then $p\in A^{(\gamma)}$ if and only if $\delta(p,A)(x)<\alpha$ implies $x\leqslant\gamma$. This is equivalent to $\delta(p,A)(x)\geqslant\alpha$ whenever $x>\gamma$, i.e. to $\varphi_{\alpha,\gamma}\leqslant\delta(p,A)$, i.e. to $p\in\overline{A}^{\varphi_{\alpha,\gamma}}$. Here $\varphi_{\alpha,\gamma}(x)=0$ if $x\leqslant\gamma$ and $\varphi_{\alpha,\gamma}(x)=\alpha$ if $x>\gamma$.

**Proposition 6.3.** *Let $(S,\delta)\in|\mathsf{ProbAp}_{\tau_\wedge}|$ where $\wedge$ is the minimum t-norm. Then $(S,d^{\delta}_{\alpha})\in|\mathsf{Ap}|$.*

P r o o f. (D1), (D2) and (D3) are easy and are left for the reader. We prove (D4). We have $\tau_*(\delta(p, \overline{A}^{\varphi_{\alpha,\gamma}}), \varphi_{\alpha,\gamma})(x) \leqslant \delta(p, A)(x)$ for all $x \in [0, \infty]$ and hence

$$\begin{aligned}
d^\delta_\alpha(p, A) &\leqslant \bigwedge\{x\colon \tau_\wedge(\delta(p, \overline{A}^{\varphi_{\alpha,\gamma}}), \varphi_{\alpha,\gamma})(x) \geqslant \alpha\} \\
&= \bigwedge\Big\{x\colon \bigvee_{u+v=x} \delta(p, A^{(\gamma)})(u) \wedge \varphi_{\alpha,\gamma}(v) \geqslant \alpha\Big\} \\
&= \bigwedge\Big\{x\colon \bigvee_{v>\gamma} \delta(p, A^{(\gamma)})(x - v) \wedge \alpha \geqslant \alpha\Big\} \\
&= \bigwedge\{x\colon \delta(p, A^{(\gamma)})(x - \gamma) \wedge \alpha \geqslant \alpha\} \\
&\leqslant \bigwedge\{x\colon \delta(p, A^{(\gamma)})(x - \gamma) \geqslant \alpha\} \\
&= \bigwedge\{u + \gamma\colon \delta(p, A^{(\gamma)})(u) \geqslant \alpha\} \\
&= \bigwedge\{u\colon \delta(p, A^{(\gamma)})(u) \geqslant \alpha\} + \gamma = d^\delta_\alpha(p, A^{(\gamma)}) + \gamma.
\end{aligned}$$

□

**Proposition 6.4.** *Let $(S, \delta), (S', \delta') \in |\mathsf{ProbAp}_{\tau_\wedge}|$ where $\wedge$ is the minimum t-norm. If $f\colon (S, \delta) \to (S', \delta')$ is a contraction, then also $f\colon (S, d^\delta_\alpha) \to (S', d^{\delta'}_\alpha)$ is a contraction.*

P r o o f. We have for $p \in S$, $A \subseteq S$,

$$\begin{aligned}
d^{\delta'}(f(p), f(A)) &= \bigwedge\{x\colon \delta'(f(p), f(A))(x) \geqslant \alpha\} \\
&\leqslant \bigwedge\{x\colon \delta(p, A)(x) \geqslant \alpha\} = d^\delta(p, A).
\end{aligned}$$

□

We consider first the case $\alpha = 1$ and denote $d^\delta = d^\delta_1$.

**Proposition 6.5.** *Let $(S, d) \in |\mathsf{Ap}|$ and let $p \in S$, $A \subseteq S$. Then $d^{\delta^d}(p, A) = d(p, A)$.*

P r o o f. We have $d^{\delta^d}(p, A) = \bigwedge\{x\colon \delta^d(p, A)(x) = 1\} = \bigwedge\{x\colon \varepsilon_{d(p,A)}(x) = 1\} = d(p, A)$. □

**Proposition 6.6.** *Let $(S, \delta) \in |\mathsf{ProbAp}_{\tau_\wedge}|$ and let $p \in S$, $A \subseteq S$. Then $\delta^{d^\delta}(p, A) \leqslant \delta(p, A)$.*

P r o o f. If $\delta^{d^\delta}(p, A)(x) = \varepsilon_{d^\delta(p,A)}(x) = 1$, then $\bigwedge\{z\colon \delta(p, A)(z) = 1\} < x$ and hence $\delta(p, A)(x) = 1$. □

We summarize these results in the following theorem.

**Theorem 6.7.** *The category $\mathsf{Ap}$ can be embedded into the category $\mathsf{ProbAp}_{\tau_\wedge}$ as a bicoreflective subcategory.*

We are now going to show that $\mathsf{Ap}$ can also be embedded as a bireflective subcategory. To this end we note that $\mathsf{Ap}$ is a topological category [9] and hence the fibres form a complete lattice. In particular, $d_\delta = \bigwedge\limits_{\alpha>0} d_\alpha^\delta$ is a distance on $S$. This distance is the final structure on $S$ with respect to the sink $(\mathrm{id}_S\colon (S, d_\alpha^\delta) \to S)_{\alpha>0}$.

**Proposition 6.8.** *Let $(S,\delta),(S',\delta') \in |\mathsf{ProbAp}_{\tau_\wedge}|$. If $f\colon (S,\delta) \to (S',\delta')$ is a contraction, then also $f\colon (S,d_\delta) \to (S',d_{\delta'})$ is a contraction.*

Proof. By Proposition 6.4, we have that $f\colon (S,d_\alpha^\delta) \to (S',d_{\delta'})$ is a contraction for every $\alpha > 0$. Hence, as final construction, also $f\colon (S,d_\delta) \to (S',d_{\delta'})$ is a contraction. □

**Proposition 6.9.** *Let $(S,d) \in |\mathsf{Ap}|$ and let $p \in S$, $A \subseteq S$. Then $d_{\delta^d}(p,A) = d(p,A)$.*

Proof. We have for the pointwise infimum

$$\inf\{d_\alpha^{\delta^d}(p,A)\colon \alpha > 0\} = \inf_{\alpha>0} \bigwedge\{x\colon \varepsilon_{d(p,A)} \geqslant \alpha\} = d(p,A).$$

Hence, in this case $e(p,A) = \inf\{d_\alpha^{\delta^d}(p,A)\colon \alpha > 0\}$ defines a distance on $S$ and we have $e(p,A) \leqslant d_\alpha^{\delta^d}(p,A)$ for all $\alpha > 0$ and all $p \in S$, $A \subseteq S$, whence $d = e \leqslant d_{\delta^d}$. On the other hand, $d_{\delta^d}(p,A) \leqslant d_\alpha^{\delta^d}(p,A) = \bigwedge\{x\colon \varepsilon_{d(p,A)}(x) \geqslant \alpha\} = d(p,A)$ and hence $d_{\delta^d} \leqslant d$. □

**Proposition 6.10.** *Let $(S,\delta) \in |\mathsf{ProbAp}_{\tau_\wedge}|$ and let $p \in S$, $A \subseteq S$. Then $\delta^{d_\delta}(p,A) \geqslant \delta(p,A)$.*

Proof. We have for $p \in S$, $A \subseteq S$ that $\delta^{(d_\delta)}(p,A) = \varepsilon_{d_\delta(p,A)} \geqslant \varepsilon_{d_\alpha^\delta(p,A)}$ for all $\alpha > 0$ and hence $\delta^{(d_\delta)}(p,A) \geqslant \bigvee\limits_{\alpha>0} \varepsilon_{d_\alpha^\delta(p,A)}$. Let now $\delta(p,A)(x) = \alpha > 0$. Then $d_\alpha^\delta(p,A) \leqslant x$. If $d_\alpha^\delta(p,A) < x$, then $\varepsilon_{d_\alpha^\delta(p,A)}(x) = 1 \geqslant \delta(p,A)(x)$. If $d_\alpha^\delta(p,A) = x$, then $\inf\{y\colon \delta(p,A)(y) \geqslant \frac{1}{2}\alpha\} < x$ because otherwise we would have $\delta(p,A)(y) < \frac{1}{2}\alpha$ whenever $y < x$, i.e. $\delta(p,A)(x) = \sup\limits_{y<x} \delta(p,A)(y) \leqslant \frac{1}{2}\alpha$, a contradiction. Hence, in this case $\varepsilon_{d_{\alpha/2}^\delta}(x) = 1 \geqslant \delta(p,A)$. □

We summarize these results in the following theorem.

**Theorem 6.11.** *The category $\mathsf{Ap}$ can be embedded into the category $\mathsf{ProbAp}_{\tau_\wedge}$ as a bireflective subcategory.*

Remark 6.12. The definition of $\delta^d$ is already in [15]. However, it is not shown as an embedding of the categories and also neither reflectiveness nor coreflectiveness of $AP$ in $\mathsf{ProbAP}_\tau$ is considered in [15].

R e m a r k 6.13. For an order-reversing homeomorphism $S\colon [0,1] \to [0,\infty]$ it is shown in [4] that $\mathsf{Ap}$ and a subcategory of the category of probabilistic convergence spaces in the sense of Richardson and Kent [14] are isomorphic. In this sense, Richardson and Kent's probabilistic convergence spaces can be used to characterize approach spaces. If we start with an approach space $(S,d) \in |\mathsf{Ap}|$, then for $p \in S$ and $\mathcal{F} \in \mathbb{F}(S)$ the probabilistic convergence in Richardson and Kent's sense is defined by

$$p \in q_\alpha^{S,d}(\mathcal{F}) \iff \bigvee_{A \sharp \mathcal{F}} d(p,A) \geqslant S(\alpha).$$

From Example 3.3 in [7] then results the following embedding of $\mathsf{Ap}$ into $\mathsf{ProbConv}$:

$$p \in c_\varphi^{S,d}(\mathcal{F}) \iff \bigvee_{A \sharp \mathcal{F}} d(p,A) \geqslant S(\varphi(0+)).$$

In contrast, the embedding of $\mathsf{Ap}$ into $\mathsf{ProbConv}$ that results from the approaches in this paper is given by

$$p \in c_\varphi^{\delta^d}(\mathcal{F}) \iff \bigwedge_{A \sharp \mathcal{F}} \varepsilon_{d(p,A)} \geqslant \varphi \iff \bigvee_{A \sharp \mathcal{F}} d(p,A) \leqslant \bigwedge\{x \in [0,\infty]\colon\ \varphi(x) > 0\}.$$

Therefore the embeddings are different.

## 7. Probabilistic metric spaces as probabilistic approach spaces

A mapping $F\colon\ S \times S \to \Delta^+$ is called a *probabilistic quasi-metric under the triangle function* $\tau$ if for all $p,q,r \in S$, $F_{pp} = \varepsilon_0$ and $F_{pq} \geqslant \tau(F_{pr}, F_{rq})$. It is called a *probabilistic pseudo-metric under* $\tau$ if it is a probabilistic quasi-metric which is symmetric, i.e. for all $p,q \in S$ we have $F_{pq} = F_{qp}$ and it is called a *probabilistic metric under* $\tau$ if it is a probabilistic pseudo-metric which is non-degenerate, i.e. $p = q$ whenever $F_{pq} = \varepsilon_0$. A pair $(S,F)$ is then called a *probabilistic quasi-metric space*, a *probabilistic pseudo-metric space* or a *probabilistic metric space*, under the triangle function $\tau$, respectively [19]. A mapping $f\colon\ (S,F) \to (S',F')$ between two probabilistic quasi-metric spaces is called *non-expansive* if $F'_{f(p)f(q)} \geqslant F_{pq}$ for all $p,q \in S$. The category of probabilistic quasi-metric spaces with non-expansive mappings as morphisms is denoted by $\mathsf{ProbQMet}_\tau$, the subcategories of probabilistic pseudo-metric spaces and of probabilistic metric spaces by $\mathsf{ProbPMet}_\tau$ and $\mathsf{ProbMet}_\tau$, respectively.

Let now $(S,F) \in |\mathsf{ProbQMet}_\tau|$ and let the triangle function $\tau$ be sup-continuous. We define $\delta^F\colon\ S \times P(S) \to \Delta^+$ by

$$\delta^F(p,A) = \bigvee_{q \in A} F_{pq}.$$

**Proposition 7.1.** *Let $(S,F) \in |\mathsf{ProbQMet}_\tau|$ and let the triangle function $\tau$ be sup-continuous. Then $(S,\delta^F) \in |\mathsf{ProbAp}_\tau|$.*

P r o o f. (PD1) We have $\delta^F(p,\{p\}) = F_{pp} = \varepsilon_0$.

(PD2) follows from $\bigvee \emptyset = \varepsilon_\infty$ in $\Delta^+$.

(PD3) We have first $\bigvee\limits_{a\in A} F_{pa} \vee \bigvee\limits_{b\in B} F_{pb} \leqslant \bigvee\limits_{c\in A\cup B} F_{pc}$. On the other hand,

$$\bigvee_{a\in A} F_{pa} \vee \bigvee_{b\in B} F_{pb} = \bigvee_{a\in A, b\in B} F_{pa} \vee F_{pb} \geqslant F_{pq}$$

for all $q \in A\cup B$ and hence

$$\bigvee_{a\in A} F_{pa} \vee \bigvee_{b\in B} F_{pb} \geqslant \bigvee_{c\in A\cup B} F_{pc}.$$

(PD4) For $q \in \overline{A}^{\varphi}$ we have $\bigvee\limits_{r\in A} F_{qr} = \delta^F(q,A) \geqslant \varphi$. Hence we conclude

$$\tau(\delta^F(p,\overline{A}^{\varphi}),\varphi) = \bigvee_{q\in\overline{A}^{\varphi}} \tau(F_{pq},\varphi) \leqslant \bigvee_{q\in\overline{A}^{\varphi}} \bigvee_{r\in A} \tau(F_{pq},F_{qr}) \leqslant \bigvee_{q\in\overline{A}^{\varphi}} \bigvee_{r\in A} F_{pr} = \delta^F(p,A).$$

□

**Proposition 7.2.** *Let $(S,F),(S',F') \in |\mathsf{ProbQMet}_\tau|$. Then $f\colon (S,F)\to(S',F')$ is non-expansive if and only if $f\colon (S,\delta^F)\to(S',\delta^{F'})$ is a contraction.*

P r o o f. Let first $f\colon (S,F)\to(S',F')$ be non-expansive and let $p\in S$ and $A\subseteq S$. Then

$$\delta^F(p,A) = \bigvee_{q\in A} F_{pq} \leqslant \bigvee_{q\in A} F'_{f(p)f(q)} \leqslant \bigvee_{s\in f(A)} F'_{f(p)s} = \delta^{F'}(f(p),f(A)).$$

For the converse, let $f\colon (S,\delta^F)\to(S',\delta^{F'})$ be a contraction and let $p,q\in S$. Then

$$F_{pq} = \delta^F(p,\{q\}) \leqslant \delta^{F'}(f(p),\{f(q)\}) = F'_{f(p)f(q)}.$$

□

We note that if $(S,F)\neq(S,F')$, then there are $p,q\in S$ such that $\delta^F(p,\{q\}) = F_{pq} \neq F'_{pq} = \delta^{F'}(p,\{q\})$. Hence we have the following result.

**Corollary 7.3.** *If $\tau$ is a sup-continuous triangle function, then*

$$M\colon \begin{cases} \mathsf{ProbQMet}_\tau \longrightarrow \mathsf{ProbAp}_\tau, \\ (S,F) \longmapsto (S,\delta^F), \\ f \longmapsto f. \end{cases}$$

*is an embedding functor, i.e. $\mathsf{ProbQMet}_\tau$ is isomorphic to a subcategory of $\mathsf{ProbAp}_\tau$.*

We call $(S,\delta) \in |\mathsf{ProbAp}_\tau|$ *probabilistic quasi-metric* or *probabilistic pseudo-metric* or *probabilistic metric* if there is a probabilistic quasi-metric or a probabilistic pseudo-metric or a probabilistic metric, respectively, $F$ such that $\delta = \delta^F$.

**Proposition 7.4.** *Let $(S,\delta) \in |\mathsf{ProbAp}_\tau|$. Then*
(1) *$(S,\delta)$ is probabilistic quasi-metric if and only if for all $p \in S$ and all $A \subseteq S$ we have $\delta(p,A) = \bigvee_{a\in A} \delta(p,\{a\})$.*
(2) *$(S,\delta)$ is probabilistic pseudo-metric if and only if for all $p \in S$ and all $A \subseteq S$ we have $\delta(p,A) = \bigvee_{a\in A} \delta(p,\{a\})$ and for all $p,q \in S$, $\delta(p,\{q\}) = \delta(q,\{p\})$.*
(3) *$(S,\delta)$ is probabilistic metric if and only if for all $p \in S$ and all $A \subseteq S$ we have $\delta(p,A) = \bigvee_{a\in A} \delta(p,\{a\})$ and for all $p,q \in S$, $\delta(p,\{q\}) = \delta(q,\{p\})$ and $p = q$ whenever $\delta(p,\{q\}) = \varepsilon_0$.*

P r o o f. (1) If $\delta = \delta^F$ with a probabilistic quasi-metric, then $\delta(p,A) = \bigvee_{a\in A} F_{pa} = \bigvee_{a\in A} \delta(p,\{a\})$. Conversely, we define $F_{pq} = \delta(p,\{q\})$. Then $F_{pp} = \varepsilon_0$ and $F_{pq} \geqslant \tau(F_{pr},F_{rq})$. Moreover, $\delta^F(p,A) = \bigvee_{a\in A} F_{pa} = \bigvee_{a\in A} \delta(p,\{a\}) = \delta(p,A)$, i.e. $(S,\delta)$ is a probabilistic quasi-metric space.

(2) If $\delta = \delta^F$ with a probabilistic pseudo-metric, then we can copy the proof of (1), except that we have additionally $F_{pq} = \delta(p,\{q\}) = \delta(q,\{p\}) = F_{qp}$. Conversely, we define $F_{pq} = \delta(p,\{q\}) \wedge \delta(q,\{p\})$. Then $F$ is a probabilistic pseudo-metric and we have $\delta^F(p,A) = \bigvee_{a\in A} (\delta(p,\{a\}) \wedge \delta(a,\{p\})) = \bigvee_{a\in A} \delta(p,\{a\}) = \delta(p,A)$.

(3) We can copy the proof of (2). The last condition ensures that $F_{pq} = \varepsilon_0$ implies $p = q$. $\square$

We are now going to show that the embedding of $\mathsf{ProbQMet}_\tau$ into $\mathsf{ProbAp}_\tau$ is coreflective. Let $(S,\delta) \in |\mathsf{ProbAP}_\tau|$. We define $F^\delta_{pq} = \delta(p,\{q\})$.

**Proposition 7.5.** *Let $(S,\delta) \in |\mathsf{ProbAp}_\tau|$. Then $(S,F^\delta) \in |\mathsf{ProbQMet}_\tau|$.*

P r o o f. We have $F^\delta_{pp} = \delta(p,\{p\}) = \varepsilon_0$. Furthermore, by definition we have $q \in \overline{\{r\}}^{\delta(q,\{r\})}$ and hence $\{q\} \subseteq \overline{\{r\}}^{\delta(q,\{r\})}$ and we conclude

$$\begin{aligned}\tau(F^\delta_{pq},F^\delta_{qr}) = \tau(\delta(p,\{q\}),\delta(q,\{r\})) &\leqslant \tau(\delta(p,\overline{\{r\}}^{\delta(q,\{r\})}),\delta(q,\{r\}))\\ &\leqslant \delta(p,\{r\}) = F^\delta_{pr}.\end{aligned}$$

$\square$

**Proposition 7.6.** *If $f\colon (S,\delta) \to (S',\delta')$ is a contraction, then $f\colon (S,F^\delta) \to (S',F^{\delta'})$ is non-expansive.*

P r o o f. We have $F^{\delta}_{pq} = \delta(p,\{q\}) \leqslant \delta'(f(p),\{f(q)\}) = F^{\delta'}_{f(p),f(q)}$. $\square$

**Proposition 7.7.** *Let* $(S,F) \in |\mathsf{ProbQMet}_\tau|$. *Then* $F^{(\delta^F)} = F$.

P r o o f. We have $F^{(\delta^F)}_{pq} = \delta^F(p,\{q\}) = F_{pq}$ for all $p,q \in S$. $\square$

**Proposition 7.8.** *Let* $(S,\delta) \in |\mathsf{ProbAp}_\tau|$. *Then for all* $p \in S$ *and* $A \subseteq S$ *we have* $\delta^{(F^\delta)}(p,A) \leqslant \delta(p,A)$.

P r o o f. We have $\delta^{(F^\delta)}(p,A) = \bigvee\limits_{a\in A} F^{\delta}_{pa} = \bigvee\limits_{a\in A} \delta(p,\{a\}) \leqslant \delta(p,A)$. $\square$

We collect these results in the following theorem.

**Theorem 7.9.** *If* $\tau$ *is a sup-continuous triangle function, then the category* $\mathsf{ProbQMet}_\tau$ *can be embedded into* $\mathsf{ProbAp}_\tau$ *as a bicoreflective subcategory.*

If we define for $(S,\delta) \in |\mathsf{ProbAp}_\tau|$ the probabilistic pseudo-metric $F^{\delta}_{pq} = \delta(p,\{q\}) \wedge \delta(q,\{p\})$, then we can repeat the foregoing results and proofs almost word-by-word and obtain the following theorem.

**Theorem 7.10.** *If* $\tau$ *is a sup-continuous triangle function, then the category* $\mathsf{ProbPMet}_\tau$ *can be embedded into* $\mathsf{ProbAp}_\tau$ *as a bicoreflective subcategory.*

In order to embed the category $\mathsf{ProbMet}_\tau$ in this way, we need to consider a subcategory of $\mathsf{ProbAp}_\tau$. A space $(S,\delta) \in |\mathsf{ProbAp}_\tau|$ is called *non-degenerate* if $p = q$ whenever $\delta(p,\{q\}) = \varepsilon_0$. The subcategory of $\mathsf{ProbAp}_\tau$ with objects being the non-degenerate probabilistic approach spaces is denoted by $\mathsf{ProbNDAp}_\tau$. It is clear that for a probabilistic metric space $(S,F)$, the space $(S,\delta^F)$ is a non-degenerate probabilistic approach space. In a similar way as above we then obtain the following theorem.

**Theorem 7.11.** *If* $\tau$ *is a sup-continuous triangle function, then the category* $\mathsf{ProbMet}_\tau$ *can be embedded into* $\mathsf{ProbNDAp}_\tau$ *as a bicoreflective subcategory.*

R e m a r k 7.12. The definition of $\delta^F$ is already in [15]. However, they consider only Menger spaces $(S,F,*)$ for a continuous t-norm $*$. As for a continuous t-norm $*$ the induced triangle function $\tau_*$ is sup-continuous, this case is also included here. Moreover, it is not shown in [15] that $\mathsf{ProbQMet}_\tau$ is embedded into the category $\mathsf{ProbAp}_\tau$ and also coreflectiveness of $\mathsf{ProbQMet}_\tau$ in $\mathsf{ProbAp}_\tau$ is not considered.

## 8. Conclusions

We introduced a probabilistic version of Lowen's approach spaces [10], [11]. We showed that the resulting category of probabilistic approach spaces is isomorphic to the category of probabilistic topological convergence spaces in the sense of [7] and hence is a topological category. Furthermore, we showed that Lowen's category of approach spaces can be embedded as a simultaneously bireflective and bicoreflective subcategory. Also the category of probabilistic quasi-metric spaces is isomorphic to a bicoreflective subcategory of our category of probabilistic approach spaces. Compared with similar approaches in the literature, our approach offers the characterization of probabilistic approach spaces by probabilistic convergence spaces, whereas, e.g. in [4], approach spaces are characterized by probabilistic convergence spaces in Richardson and Kent's sense [14].

There is a natural generalization of our probabilistic approach spaces to the lattice-valued case. If $L$ is a completely distributive lattice with a quantale operation $*\colon L \times L \to L$, we call $(S, \delta)$ with $\delta\colon S \times P(S) \to L$ an *$L$-approach space* if the axioms

(LD1) $\delta(p, \{p\}) = \top_L$,
(LD2) $\delta(p, \emptyset) = \bot_L$,
(LD3) $\delta(p, A \cup B) = \delta(p, A) \vee \delta(p, B)$ and
(LD4) $\delta(p, A) \geqslant \delta(p, \overline{A}^{\alpha}) * \alpha$ for all $\alpha \in L$, where $\overline{A}^{\alpha} = \{p \in S\colon \delta(p, A) \geqslant \alpha\}$,

are satisfied. For $L = [0, \infty]$ with the dual order and $* = +$ the addition, we obtain Lowen's approach spaces. For $L = \Delta^+$ and $*$ a sup-continuous triangle function, we obtain the case considered in this paper. A theory of such spaces can be developed along the same lines as the theory developed in this paper and has close connections to the theory of continuity spaces [5], a generalization of both metric spaces and probabilistic metric spaces. We shall look into this in our future work.

In the theory of approach spaces, there are many equivalent ways for defining an approach space. While some of them, e.g. towers, have a natural probabilistic generalization, it is an interesting question to find suitable probabilistic generalizations for others. Also, the extension of what R. Lowen calls "index analysis" [11] to the probabilistic case could lead to interesting results. Lastly, it might be beneficial to describe initial structures directly in terms of probabilistic distance functions. This could lead to further insights into the question of defining products for probabilistic metric spaces, see e.g. [20].

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